Nonexistence of smooth solutions to the compressible Navier-Stokes equations with finite energy and finite moment of mass

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We consider the motion of compressible viscous, heat-conductive, Newtonian polytropic fluid in $\mathbb{R} \times \mathbb{R}^n$, $n \geq 1$, is governed by the compressible Navier-Stokes (NS) equations

$$\partial_t \rho + \text{div}_x (\rho u) = 0, \quad \partial_t (\rho u) + \text{div}_x (\rho u \otimes u) + \nabla_x p = \text{Div} T,$$

(*)

$$\partial_t \left( \frac{1}{2} \rho |u|^2 + \rho e \right) + \text{div}_x \left( \frac{1}{2} \rho |u|^2 + \rho e + p \right) u = \text{div}(Tu) + k \Delta_x \theta,$$

where $\rho$, $u = (u_1, \ldots, u_n)$, $p$, $e$, $\theta$ denote the density, velocity, pressure, internal energy and absolute temperature, respectively, $T$ is the stress tensor given by the Newton law $T = T_{ij} = \mu (\partial_i u_j + \partial_j u_i) + \lambda \text{div} \delta_{ij}$, the constants $\mu$ and $\lambda$ are the coefficient of viscosity and the second coefficient of viscosity, $k \geq 0$ is the coefficient of heat conduction, $\mu \geq 0$, $\lambda + \frac{2}{n} \mu \geq 0$. The state equations have the form $p = R \rho \theta$, $e = c \theta$, $p = A \exp \left( \frac{S}{c} \right) \rho^\gamma$. Here $A > 0$ is a constant, $R$ is the universal gas constant, $\gamma > 1$ is the specific heat ratio, $S = \log e - (\gamma - 1) \log \rho$ is the specific entropy, $c = \frac{R}{\gamma - 1}$. The state equations imply $p = (\gamma - 1) \rho e$, which allows us to consider (NS) as a system for the unknown $\rho$, $u$, $p$.

(NS) is supplemented with the initial data

$$\left( \begin{array}{ccc} \rho, u, p \end{array} \right) \bigg|_{t=0} = (\rho_0(x), u_0(x), p_0(x)) \in H^m(\mathbb{R}^n), \ m > [n/2] + 2. \quad (IC)$$

We consider also the isentropic case where the fluid obeys equations (*) and $p = A \rho^\gamma$, we call this system (NSI) for short.

In the absence of vacuum, the local existence of classical solutions is known (Nash; Volpert and Khudiaev). The uniqueness of the solution was proved by Serrin. The existence and uniqueness of local strong solutions in the case where the initial density need not be positive and may vanish in an open set are proved recently by Cho and Kim.

At the same time there exists a major open problem: to prove or disprove that a smooth solution to (NS) (in higher space dimensions) exists globally in time. There are partial results concerning the Cauchy problem for (NS) away from a vacuum (e.g. Matsamura and Nishida; Kazhikhov).

However if the initial density $\rho_0$ is compact, then in arbitrary space dimensions no solution to (NS) from $C^1([0, \infty), H^m(\mathbb{R}^n))$ exists (Xin). This blowup result depends crucially on the assumption about compactness of support of the initial density. It does not seem to solve in a negative way the question of regularity for (NS). Indeed, (NS) is a model of non-dilute fluids where the density is bounded below away from zero, and therefore
it is natural to expect the problem to be ill-posed when vacuum regions are present at the initial time. At the same time, the conservation of mass in the whole space require a decrease of the density down to zero.

Let us introduce the functionals $P = \int_{\mathbb{R}^n} pu\,dx$ (momentum) and $F(t) = \int_{\mathbb{R}^n} (u, x)\rho\,dx$.

For technical reasons we impose the following conditions of decay on the solution components as $|x| \to \infty$ at every fixed $t \in \mathbb{R}_+$: $\rho = O\left(|x|^{-(n+2+\varepsilon)}\right)$, $p = O\left(|x|^{-(n+\varepsilon)}\right)$, $\varepsilon > 0$, $|u| = o\left(|x|^{1-n}\right)$, $|Du| = o\left(|x|^{-n}\right)$. If $k \neq 0$, we require additionally $|D\theta| = o\left(|x|^{1-n}\right)$, $|x| \to \infty$, $t \in \mathbb{R}_+$.

We impose no restriction on the solution support, however the decay of the solution at $|x| \to \infty$ in the class considered is greater than it is necessary for belonging to $C^1([0,T), H^n(\mathbb{R}^n))$.

We will say that a solution $(\rho, u, p)$ to the Cauchy problem (NS), (ID) belongs to the class $\mathcal{K}$ if it has the following properties for all $t \geq 0$:

(i) the solution is classical;

(ii) the solution decays at infinity as prescribed above;

(iii) $\rho(t, x) \geq 0$,

(iv) $\frac{dS(t, x)}{dt} = \sigma(t, x) \geq 0$, $\sigma(t, x) = o(t^n)$, $t \to \infty$, where $\alpha = \frac{(\gamma-1)n^2+n-2}{n}$ if $\gamma \leq 1 + \frac{1}{n}$, and $\alpha = \frac{3n-2}{n}$, otherwise.

For (NSI) condition (iv) holds trivially.

**Theorem 1** Let $n \geq 3$, $\gamma \geq \frac{2n}{n+2}$, the momentum $P \neq 0$. If $\inf_{x \in \mathbb{R}^n} S(0, x) > -\infty$, then there exists no global in time solution to (NS) from the class $\mathcal{K}$.

For the isentropic case we have the following version of this theorem.

**Theorem 2** Let $n \geq 3$, $\gamma \geq \frac{2n}{n+2}$, the momentum $P \neq 0$. If initial data $(\rho_0(x), u_0(x))$ are such that $F(0) > 0$, then the solution to (NSI) from the class $\mathcal{K}$ cannot exist for all $t > 0$. If $\gamma \leq 1 + \frac{1}{n}$, then the respective solution cannot exist for all initial data such that $P \neq 0$.

Analyzing condition (iv) we can see that the nondecreasing of entropy along trajectories seems natural, whereas the upper bound on the growth of entropy can be considered as unreasonable. So, we can re-formulate Theorem 1 as follows:

**Theorem 3** Let us assume $n \geq 3$, $\gamma \geq \frac{2n}{n+2}$ and $P \neq 0$. Then any solution to (NS) with properties (i, ii, iii) such that the entropy non-decreases along the particles trajectories, blowups in a finite or infinite time. If the solution keeps smoothness for all $t > 0$, then $\|S\|_{L^{\infty}(\mathbb{R}^n)}$, $\|p\|_{L^{\infty}(\mathbb{R}^n)}$, $\|	ext{div}u\|_{L^{\infty}(\mathbb{R}^n)}$ rise as $t \to \infty$ at least as $O(t^{\alpha+1})$ (the constant $\alpha > 0$ is indicated in condition (iv)).

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