

High Order Well-Balanced Finite Volume Schemes for Systems of Balance Laws

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In this paper we propose a new family of high-order well-balanced schemes for the numerical solution of systems of balance laws of the form $\partial_t u + \partial_x f(u, x) = g(u, x)$.

The schemes are well balanced in the sense that they will preserve discretized version of the equilibrium solution u^e . They make use of two sets of variables, conservative (denoted by u) and equilibrium variables (denoted by v). The latter are defined in such a way that they are constant at equilibrium. For example, in the case of the shallow water equations,

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hw) = 0, \quad \frac{\partial hu}{\partial t} + \frac{\partial}{\partial x}\left(hw^2 + \frac{1}{2}\tilde{g}h^2\right) = -\tilde{g}hB_x, \quad (1)$$

the conservative variables are $h, q = hw$, while the equilibrium variables are $q, \eta = w^2/2 + \tilde{g}(h + B)$.

A suitable mapping is defined between the two sets of variables u and v . We shall assume that the mapping is one to one (in some cases this assumption has to be taken with care), i.e. $u = U(v, x), \quad v = V(u, x)$.

The idea of the scheme is to discretize the equation in the conservative variables, while the equilibrium variables will be used in the reconstruction. The procedure can be used to obtain both well balanced staggered and non staggered schemes, as well as well balanced ADER schemes.

To show the simplicity of the approach, we show here how to construct a first order well balanced scheme on staggered grids. We start from the evolution equation for the cell average

$$\frac{d\bar{u}_{j+1/2}}{dt} = -\frac{1}{\Delta x}(f(u(x_{j+1}, t)) - f(u(x_j, t))) + \langle g \rangle_{j+1/2}(t) \quad \text{with} \quad \langle g \rangle_{j+1/2}(t) \equiv \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} g(u(x, t), x) dx,$$

The first order scheme in time is given by

$$\bar{u}_{j+1/2}^{n+1} = \bar{u}_{j+1/2}^n - \frac{\Delta t}{\Delta x}(f(u(x_{j+1}, t^n)) - f(u(x_j, t^n))) + \langle g \rangle_{j+1/2}(t^n). \quad (2)$$

Three quantities have to be defined: given $\{\bar{u}_j^n\}$, compute staggered cell values $\{\bar{u}_{j+1/2}^n\}$, pointwise values $u(x_j, t^n)$, cell average of the source $\langle g \rangle_{j+1/2}(t^n)$. A conservative and well balanced scheme is obtained using a piecewise

constant approximation of the solution. Let us define the cell average \bar{v}_j of equilibrium variables as the numbers such that the average of the conservative variables, expressed in terms of \bar{v} , is the correct one:

$$\frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} U(\bar{v}_j, x) dx = \bar{u}_j.$$

Then the above quantities (staggered cell averages, pointwise values, and source cell averages) are defined as:

$$\bar{u}_{j+1/2}^n = \frac{1}{\Delta x} \left(\int_{x_j}^{x_{j+1/2}} U(\bar{v}_j, x) dx + \int_{x_{j+1/2}}^{x_{j+1}} U(\bar{v}_{j+1}, x) dx \right), \quad u_j = U(\bar{v}_j, x_j)$$

$$\langle g \rangle_{j+1/2} (t^n) = \frac{1}{\Delta x} \left(\int_{x_j}^{x_{j+1/2}} g(U(\bar{v}_j, x)) dx + \int_{x_{j+1/2}}^{x_{j+1}} g(U(\bar{v}_{j+1}, x)) dx \right).$$

To verify the well balanced property, let $u^e(x)$ denote the equilibrium solution, and let v^e be the corresponding constant value for the equilibrium variable. Therefore one has $u^e(x) = U(v^e, x)$.

Now let us integrate in space the balance equation for u over staggered cell $I_{j+1/2} = [x_j, x_{j+1}]$, in the stationary case. One has

$$f(u^e(x_{j+1})) - f(u^e(x_j)) = \int_{x_j}^{x_{j+1}} g(u^e(x)) dx.$$

Dividing by Δx , and making use of the fact that $u^e(x) = U(v^e, x)$, one observes that the sum of all the terms to the right of $\bar{u}_{j+1/2}^n$ in equation (2) is zero, and therefore $\bar{u}_{j+1/2}^{n+1} = \bar{u}_{j+1/2}^n$.

Higher order in space is obtained by using a piecewise reconstruction for the equilibrium variables. We use a WENO reconstruction on v , which preserves the cell averages of the conservative variables. Higher order in time obtained by using Central Runge-Kutta time differentiation [1] and by the crucial observation that the evolution equation for the stage values in the equilibrium variables reduces simply to $\partial u / \partial t + A(\partial U / \partial v)(\partial v / \partial x) = 0$, where $A = \partial f / \partial u$ is the Jacobian matrix of the system. Such relation immediately shows that stage values are constant if the equilibrium variables are constant.

Several numerical examples on shallow water equation and on nozzle flow illustrate the high accuracy and well-balanced properties of the proposed schemes, for solutions near static and moving equilibria.

References

- [1] Pareschi L., Puppo G., Russo G., *Central Runge-Kutta Schemes for Conservation Laws*, SIAM J. Sci. Comput. 26 (2005), no. 3, 979–999.