THE SOLUTIONS OF THE RIEMANN PROBLEM FOR
TWO-COMPONENT TWO-PHASE FILTRATION SYSTEM IN CASE OF
THE COMPRESSIBILITY OF BOTH PHASES

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We consider the following system of PDEs that represents the simplest system of equations governing the filtration process in porous media when one has two phases presented and also both phases are considered as compressible

\[
\begin{align*}
&\frac{\partial}{\partial t} \left( x_G(P) \varrho_G(P) \cdot s + x_L(P) \varrho_L(P) \cdot (1-s) \right) + \frac{\partial}{\partial x} \left[ \left( x_G(P) \varrho_G(P) \frac{k_G(s)}{\mu_G} + x_L(P) \varrho_L(P) \frac{k_L(1-s)}{\mu_L} \right) Q \right] = 0 \\
&\frac{\partial}{\partial t} \left( \varrho_G(P) \cdot s + \varrho_L(P) \cdot (1-s) \right) + \frac{\partial}{\partial x} \left[ \left( \varrho_G(P) \frac{k_G(s)}{\mu_G} + \varrho_L(P) \frac{k_L(1-s)}{\mu_L} \right) Q \right] = 0 \\
&\quad - K \frac{\partial P}{\partial x} = Q.
\end{align*}
\]

Here $x_G, x_L$ are equilibrium values of phases; $\varrho_G, \varrho_L$ are phases’ densities. All these functions are given as functions of pressure $P$ and are determined by the relevant Equation of State (EoS). Functions $k_G, k_L$ are the relative permeabilities and are also given as functions of gas saturation $s$ or liquid saturation $1-s$ respectively, they are determined by the physics of filtration process. The values $\mu_G, \mu_L$ represent the viscosities of the phases and here are assumed to be constants. The value $K$ is absolute permeability of solid matrix and the value $\Phi$ is the porosity. They are also assumed to be constants.

Thus in the system (1) functions $s(t, x)$ and $P(t, x)$ are the only unknown functions. The system (1) can be considered as some kind of degenerate parabolic system and as a consequence it exhibits the hyperbolic properties as well as parabolic ones.

We consider the following Cauchy problem for (1)

\[
\begin{align*}
s_{t=0}(x) &= \begin{cases} 
 s_{\text{left}} = s_0^- \equiv \text{const}, & x < 0 \\
 s_{\text{right}} = s_0^+ \equiv \text{const}, & x > 0
\end{cases} \\
P_{t=0}(x) &= \begin{cases} 
 P_{\text{left}} = P_0^- \equiv \text{const}, & x < 0 \\
 P_{\text{right}} = P_0^+ \equiv \text{const}, & x > 0
\end{cases},
\end{align*}
\]

i.e. the classical Riemann problem with piece-wise constant initial data.

We will seek the solution of the problem (1), (2) in the class of self-similar functions with self-similar variable $\xi = x/\sqrt{2t}$. The solutions can include discontinuities that obey to the analog of Hugoniot conditions in the theory of conservation laws (we show these conditions below).

Consider first the continuous self-similar solutions. After substitution of self-similar form of unknown functions in (1) one gets the following system of ODEs with respect to variable $\xi$
\[ \begin{align*}
\Phi' &= \frac{Q'}{Q} Q(GP_0 - FR_0 - \xi(GP_0 - FR_0 - GX_0 - FR_0)) \\
\varphi' &= \frac{Q'}{Q} Q(GP_0 - FR_0 - GX_0 - FR_0 - XG_0 - FR_0) + \xi^2(RF_0 - XFR_0) \\
\end{align*} \]

where

\[ R = \varphi G(P) \cdot s + \varphi L(P) \cdot (1 - s) \quad ; \quad X = x G(P) \varphi G(P) \cdot s + x L(P) \varphi L(P) \cdot (1 - s) \]

\[ G = \varphi G(P) \frac{k_G(s)}{\mu_G} + \varphi L(P) \frac{k_L(1 - s)}{\mu_L} \quad ; \quad F = x G(P) \varphi G(P) \frac{k_G(s)}{\mu_G} + x L(P) \varphi L(P) \frac{k_L(1 - s)}{\mu_L} \]

Using the conservation form of the equations in (1) and after some algebra one infers the analogs of Hugoniot relations for shocks

\[ \begin{align*}
Q^+ \left( \frac{k_G(s)}{\mu_G} + \frac{k_L}{\mu_L} \right) &= Q^- \left( \frac{k_G(s)}{\mu_G} + \frac{k_L}{\mu_L} \right) = D \\
D(\varphi G(s^+) - \varphi G(s^-)) &= \xi^*(s^+ - s^-) \quad ; \quad P^+ = P^- 
\end{align*} \]

where \( \varphi G(s) = \frac{k_G(s)}{k_G(s) + k_L(1 - s)/\mu_L} \) is some nonconvex function, \( \xi^* \) is the location of the shock in self-similar variables and the notation \( A^\pm \) means \( A(t, x \pm 0) \).

In the conventional theory of conservation laws it is well known that Hugoniot conditions are not enough in order to elicit unique discontinuous solution. The stability or entropy conditions are necessary. Following [1] we introduce the stability conditions in the form

\[ Q^+ \theta^+ \leq \xi^* \leq Q^- \theta^- \]

where \( Q \theta(s) \) is the speed of propagation of weak singularities for system (1), see for example [2], and \( \theta(s) = \varphi_G'(s) \cdot (k_G/\mu_G + k_L/\mu_L) \).

**THEOREM.** Suppose \( |s^+ - s^-| \) and \( |P^+ - P^-| \) are sufficiently small and some additional technical requirements hold. Suppose that at shocks (4) and (5) also hold. Then the solution of the Riemann problem (1), (2) exists and is unique in the class of self-similar functions.

The proof of the Theorem is based on accurate estimates of solutions of (3) and on topological methods used for finding such trajectories of (3) that relations (4) and (5) are satisfied at some point of these trajectories.

**References**
