The solution of the Cauchy problem with large data for a model of a mixture of gases

Helge Holden
Department of Mathematical Sciences, Norwegian University of Science and Technology, Norway
and *Centre of Mathematics for Applications, University of Oslo, Norway*
holden@math.ntnu.no

Nils Henrik Risebro
Centre of Mathematics for Applications, University of Oslo, Norway
nilshr@math.uio.no

Hilde Sande*
Department of Mathematical Sciences, Norwegian University of Science and Technology, Norway
hildes@math.ntnu.no

We consider a one dimensional model for the flow of a mixture of isentropic gases. The different gases are initially separated, and the pressure is for all gases given by a γ-law. The different gases cannot mix, thus, the flow of these gases is in Lagrangian coordinates described for $x \in \mathbb{R}$ and $t \in (0, \infty)$ by the system

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v,\gamma)_x &= 0, \\ \gamma_t &= 0, \end{aligned} \tag{1}$$

where $v = 1/\rho$ is the specific volume, u is the velocity, and $p(v, \gamma) = v^{-\gamma}$ is the pressure function. We assume $\gamma(x, t) > 1$. This 3×3 system of hyperbolic conservation laws is strictly hyperbolic. The first and third family are genuinely nonlinear and the second family is linearly degenerate. The pressure, p, and the velocity, u, are constant along any contact discontinuity of the second family, while γ is constant along all waves of the first and third family. The system does not possess a coordinate system of Riemann invariants, thus, we work with the variables in which the Riemann problem is easiest described, that is, p, u and γ.

Glimm [3] proved global existence of a weak solution of the Cauchy problem with small initial data for strictly hyperbolic systems where each family is either genuinely nonlinear or linearly degenerate, thus including the present system. This solution is found by using the Glimm scheme [3] or by using front tracking [2, 4] by which one can prove stability of the Cauchy problem. Here we extend the existence result to large initial data for (1).

We show that the Cauchy problem for system (1) has a global, weak solution if $(\sup(\gamma) - 1)T.V.(p(\cdot, 0), u(\cdot, 0))$ and $T.V.(\gamma(\cdot, 0))$ are sufficiently small. Thus, by reducing the total variation and the supremum of γ, we can allow large variation in p and u. We show this both using the Glimm scheme and using front tracking. The key point for both methods is choosing a suitable Glimm functional and show that it is decreasing in time. This requires detailed analysis of all possible interactions.

System (1) is an extension of the p-system describing flow of one isentropic gas, thus, with a constant γ. The Cauchy problem with large data for the p-system is solved using the Glimm scheme by Nishida and Smoller [5] and using front tracking by Asakura [1]. Since the interactions for system (1) with no contact discontinuity are similar to the interactions for the p-system, we make use of the estimates obtained in [5].

Interactions where γ changes, that is, interactions involving a contact discontinuity, are substantially more complex and each interaction has several possible outcomes. Estimating the difference of two waves with different γ's is central in obtaining the needed interaction estimates. For interactions involving more than two waves we present a technique of splitting the interaction into simpler interactions in order to obtain the needed estimates. This technique applies to interactions in the Glimm scheme as well as interactions appearing in front tracking.

For the front tracking approach we are able to show that the number of fronts remain finite in finite time. Therefore we do not need to remove weak fronts or introduce non-physical fronts in the front tracking algorithm.

We show numerical examples for both the Glimm scheme and front tracking.

References

[1] Asakura F., *Wave-front tracking for the equation of isentropic gas dynamics*, Quart. Appl. Math., 63(1):20–33, 2005.

[2] Bressan A., *Hyperbolic Systems of Conservation Laws*, Oxford University Press, 2000.

[3] Glimm J., *Solution in the large for nonlinear hyperbolic systems of equations*, Comm. Pure Appl. Math., 18:697–715, 1965.

[4] Holden H., Risebro N. H., *Front Tracking for Hyperbolic Conservation Laws*, Springer-Verlag, 2002.

[5] Nishida T., Smoller J. A., *Solution in the large for some nonlinear hyperbolic conservation laws*, Comm. Pure Appl. Math., 26:183–200, 1973.