The Maxwell-Dirac system: Null structure and almost optimal local well-posedness

Piero D'Ancona
University of Rome (La Sapienza), Italy
dancona@mat.uniroma1.it

Damiano Foschi
University of Ferrara, Italy
damiano.foschi@unife.it

Sigmund Selberg*
NTNU, Norway
sigmund.selberg@math.ntnu.no

The Maxwell-Dirac system is the fundamental PDE of classical quantum electrodynamics, describing the motion of an electron in a self-induced electromagnetic field. We show that the system has some special structural properties (so-called "null" structure) which improve the regularity of solutions. For the resulting multilinear forms we prove frequency-localized estimates, in Bourgain spaces adapted to the Dirac equation, at the scale invariant regularity up to a logarithmic loss. Using these estimates we are then able to prove almost optimal local well-posedness of the system by iteration. In other words, we can get arbitrarily close the scale invariant regularity, which corresponds to the L^2 norm for the Dirac spinor.

The null structure that we have found is not of the usual bilinear type which can be seen in each component equation of the system. Rather, it depends on the structure of the system as a whole. In this respect it is analogous to the structure that Machedon and Sterbenz found for the Maxwell-Klein-Gordon system.

The Maxwell-Dirac system is obtained by coupling Maxwell's equations,

$$\nabla \cdot \mathbf{E} = \rho, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0, \quad \nabla \times \mathbf{B} - \partial_t \mathbf{E} = \mathbf{J},$$

with the Dirac equation

$$(\boldsymbol{\alpha}^\mu D_\mu + m\boldsymbol{\beta})\,\psi = 0.$$

Here the unknowns are the electric and magnetic fields $\mathbf{E}, \mathbf{B}$ and the Dirac 4-spinor ψ, which are functions of $t \in \mathbf{R}$ and $x \in \mathbf{R}^3$; $m \geq 0$ is the rest mass of the electron, and $\boldsymbol{\alpha}^\mu$, $\mu = 0, 1, 2, 3$ and $\boldsymbol{\beta}$ are 4×4 Dirac matrices. We represent $\mathbf{E}, \mathbf{B}$ by a four-potential $A_\mu = A_\mu(t, x) \in \mathbf{R}$, $\mu = 0, 1, 2, 3$, such that

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = \nabla A_0 - \partial_t \mathbf{A} \qquad (\mathbf{A} = (A_1, A_2, A_3)).$$

In the absence of an electromagnetic field, the operator D_μ in the Dirac equation would just be $-i\partial_\mu$, but in the presence of an electromagnetic field $\mathbf{E}, \mathbf{B}$ represented by the potential A_μ, this must be modified by the minimal coupling transformation, so that D_μ becomes the gauge covariant derivative

$$D_\mu = D_\mu^{(A)} = -i\partial_\mu - A_\mu.$$

To complete the coupling we plug into the Maxwell equation the Dirac four-current density

$$J^\mu = \langle \boldsymbol{\alpha}^\mu \psi, \psi \rangle \qquad (\mu = 0, 1, 2, 3), \tag{1}$$

which splits into the charge density $\rho = J^0 = |\psi|^2$ and the three-current density $\mathbf{J} = (J^1, J^2, J^3)$. Here $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbf{C}^4$.

By gauge invariance, we are free to choose an additional gauge condition on A_μ. We choose the Lorenz gauge condition $\partial_t A_0 = \nabla \cdot \mathbf{A}$, which has the obvious advantage that the Maxwell equations written in terms of the A_μ are just wave equations. In fact, with this gauge condition the full Maxwell-Dirac system becomes

$$(-i\boldsymbol{\alpha}^\mu \partial_\mu + m\boldsymbol{\beta})\,\psi = A_\mu \boldsymbol{\alpha}^\mu \psi, \qquad (\partial_t^2 - \Delta) A_\mu = \langle \boldsymbol{\alpha}^\mu \psi, \psi \rangle, \qquad \partial_t A_0 = \nabla \cdot \mathbf{A}.$$

The last equation however, is automatically satisfied by a solution of the first two equations provided certain constraints on the initial data are satisfied.

We consider the initial value problem with data

$$\psi(0, x) = \psi_0(x), \qquad \mathbf{E}(0, x) = \mathbf{E}_0(x), \qquad \mathbf{B}(0, x) = \mathbf{B}_0(x),$$

which must satisfy the constraints $\nabla \cdot \mathbf{E}_0 = |\psi_0|^2$ and $\nabla \cdot \mathbf{B}_0 = 0$. The initial data for the four-potential A_μ, which we denote by $A_\mu(0, x) = a_\mu(x)$, $\partial_t A_\mu(0, x) = \dot{a}_\mu(x)$, must be constructed from the observable data $\mathbf{E}_0, \mathbf{B}_0$. We write $\mathbf{a} = (a_1, a_2, a_3)$ and $\dot{\mathbf{a}} = (\dot{a}_1, \dot{a}_2, \dot{a}_3)$. From the Lorenz condition and the defining relationship between $\mathbf{E}, \mathbf{B}$ and the A_μ, we get the constraints

$$\dot{a}_0 = \nabla \cdot \mathbf{a}. \qquad \mathbf{B}_0 = \nabla \times \mathbf{a}, \qquad \mathbf{E}_0 = \nabla a_0 - \dot{\mathbf{a}},$$

which determine $\mathbf{a}, \dot{\mathbf{a}}$, for arbitrary $a_0, \dot{a}_0$. The simplest choice is of course $a_0 = \dot{a}_0 = 0$.

Next we split A_μ into its homogeneous and inhomogeneous parts: $A_\mu = A_\mu^{\text{hom.}} + A_\mu^{\text{inh.}}$. Thus, we reduce the Maxwell-Dirac system to a single nonlinear Dirac equation:

$$(-i\boldsymbol{\alpha}^\mu \partial_\mu + m\boldsymbol{\beta})\,\psi = A_\mu^{\text{hom.}} \boldsymbol{\alpha}^\mu \psi - \mathcal{N}(\psi, \psi, \psi), \qquad \mathcal{N}(\psi_1, \psi_2, \psi_3) \equiv \left(\square^{-1} \langle \boldsymbol{\alpha}_\mu \psi_1, \psi_2 \rangle \right) \boldsymbol{\alpha}^\mu \psi_3.$$

Here we use the notation $\square^{-1} F$ for the solution of the inhomogeneous wave equation $\square u = F$ with vanishing data at time $t = 0$.

The scale invariant data space would be $(\psi_0, \mathbf{E}_0, \mathbf{B}_0) \in L^2 \times \dot{H}^{-1/2} \times \dot{H}^{-1/2}$, and one does not expect well-posedness with any less regularity than this. Our first main result is that local-well posedness holds for the above nonlinear Dirac equation, with only slightly more regularity: $\psi_0 \in H^s$, $\mathbf{E}_0, \mathbf{B}_0 \in H^{s-1/2}$ for any $s > 0$. Given such data, we first construct the data for A_μ as above, which then define $A_\mu^{\text{hom.}}$. Then we prove by iteration in Bourgain spaces that the above nonlinear Dirac equation is locally well-posed. To prove closed estimates for the iterates, we rely heavily on the special structure of the system, which is seen from a certain quadrilinear integral form for spinors.

Once we have solved the above nonlinear Dirac equation, we can also reconstruct $\mathbf{E}, \mathbf{B}$, and our second main result is that these fields are as regular as their initial data, throughout the time interval of existence. The field $A_\mu^{\text{inh.}}$ on the other hand, appears to lose a lot of regularity compared to the data for A_μ; but since we are ultimately only interested in the observables, this is not a problem.