In this work we present a family of relaxation schemes for non linear convection diffusion problems, which can tackle also the cases of degenerate diffusion and of convection dominated regimes. The schemes proposed can achieve any order of accuracy, give non-oscillatory solutions even in the presence of singularities and their structure depends only weakly on the particular PDE being integrated.

Relaxation approximations to non-linear PDE’s are based on the replacement of the original PDE with a semi-linear hyperbolic system of equations, with a stiff source term, tuned by a relaxation parameter $\varepsilon$. When $\varepsilon \to 0$, the system relaxes on the original PDE. A consistent discretization of the relaxation system for $\varepsilon = 0$ yields a consistent discretization of the original PDE, see for instance [3] and [1]. The advantage of this procedure is that the numerical scheme obtained in this fashion does not need approximate Riemann solvers for the convective term, still enjoying the robustness of upwind discretizations. The hyperbolic part of the relaxation system in fact is linear, with constant coefficients. Thus it can be diagonalized explicitly once for all, and written in characteristic form, where upwinding can be immediately applied.

In this work, we consider convection-diffusion equations of the form:

$$\frac{\partial u}{\partial t} + \text{div} f(u) = \Delta(p(u)), \quad (1)$$

where $f$ is a differential function of the form $f(u) = [f_1, \ldots, f_d]$, with $d$ denoting the number of space dimensions. We will suppose that all eigenvalues of $f'$ will be bounded by a positive constant $\alpha$. The diffusion term $p(u)$ is a non-negative, non-decreasing, Lipschitz continuous function with Lipschitz constant $\delta$. The equation is called degenerate when $p(0) = 0$, in which case, even the parabolic term generates travelling fronts.

Several strategies are possible to introduce relaxation systems for equation (1), see also [1]. In this work, we will propose a wide class of relaxation systems, from which several numerical approaches can be derived for the integration of (1).
We are particularly interested in the numerical schemes obtained in the relaxed case, that is $\varepsilon = 0$. In fact, when $\varepsilon = 0$, the complexity introduced replacing the original PDE with a stiff system of equations is only apparent, because one actually recovers a scheme where only the original unknown is updated. Thus the numerical solution of the full relaxation system is not needed. In other words, the relaxation system, in this approach, can be seen as a tool to produce a consistent numerical approximation of the original PDE. It is interesting to note that the numerical schemes obtained with this technique inherit interesting properties from the relaxation system, such as automatic stabilizing terms which contribute to avoid the onset of spurious oscillations.

As a matter of fact, we need only that the relaxed numerical schemes obtained through a given relaxation system are consistent with the original PDE to be discretized. We study a set of conditions that ensures this requirement. In particular we are able to employ relaxation systems that are slightly more general than those studied in [1].

We study also the numerical properties of some of the relaxed schemes thus obtained, especially their efficiency and numerical diffusion. Special care will be given to the construction of stability conditions for our relaxation schemes. A straightforward analysis recovers the subcharacteristic stability restriction of [3], coupled with a CFL condition for the time step that allows the automatic adaption to the convection-dominated and diffusion-dominated regimes. A more detailed analysis carried out in the linear case shows that the subcharacteristic condition can be eased, if diffusion is non zero, with an important decrease in artificial diffusion.

References

