We are concerned with the system

\[ E(u)u_t + A(u, u_x)u_x = B(u)u_{xx}. \]  

(1)

The function \( u \) takes values into \( \mathbb{R}^N \) and depends on the two scalar variables \( t \) and \( x \). The matrices \( E, A \) and \( B \) have all dimension \( N \times N \). A conservative system can in particular be written in the form (1). We focus on travelling waves and boundary layers solutions of (1). They satisfy the equations

\[
\begin{align*}
[A(U, U')] - \sigma E(U) U' &= B(U) U'' \quad \text{and} \\
A(U, U') U' &= B(U) U''
\end{align*}
\]

(2)

respectively. The constant \( \sigma \) in (2) is the speed of the wave. In the following, we restrict to the case of travelling waves and boundary layers having small enough total variation. The matrix \( B \) in (1) is non invertible in the case of several examples interesting from the physical point of view, like the compressible Navier Stokes equation in one space variable. In the following, we assume that system (1) satisfies the hypotheses introduced by Kawashima and Shizuta. These hypotheses are satisfied by the equations of the hydrodynamic. In particular, we assume that the matrix \( B(u) \) satisfies the decomposition (3) below. The block \( b(u) \) belongs to \( \mathbb{M}^{r \times r} \) and there exists a constant \( c_b > 0 \) such that \( \langle b(u) \vec{\xi}, \vec{\xi} \rangle \geq c_b |\vec{\xi}|^2 \) for every \( \vec{\xi} \in \mathbb{R}^r \). The block decomposition of \( B \) and the corresponding block decomposition of \( A \) are

\[
B(u) = \begin{pmatrix} 0 & 0 \\ 0 & b(u) \end{pmatrix} \quad A(u, u_x) = \begin{pmatrix} A_{11}(u) & A_{21}^T(u) \\ A_{21}(u) & A_{22}(u, u_x) \end{pmatrix}
\]

(3)

Writing \( U' \) as a column vector \( (w, z)^T \), with \( w \in \mathbb{R}^{N-r} \) and \( z \in \mathbb{R}^r \), the second equation in (2) becomes

\[
\begin{align*}
A_{11}(u) w + A_{21}^T(u) z &= 0 \\
A_{21}(u) w + A_{22}(u, u_x) z &= b z'
\end{align*}
\]

(4)

The situation in the case of travelling waves is similar, just the notations are more complicated. If the block \( A_{11} \) is invertible, then (4) can be rewritten as

\[
w = -A_{11}^{-1} A_{21}^T(u) z \quad z' = b^{-1} [A_{22}(u, u_x) - A_{21} A_{11}^{-1} A_{21}^T] z.
\]

However, in general, the block \( A_{11} \) is not invertible. In a previous work with Stefano Bianchini we studied system (4) under a new condition of block linear degeneracy. We also exhibited some examples that show that, if this condition is violated, then there might be solutions of (2) that are not continuously differentiable: more precisely, the first derivative may either be discontinuous or blow up. However, as Frédéric Rousset pointed out, the block
linear degeneracy is satisfied by the viscous profiles of the compressible Navier Stokes equation when this equation is written in Lagrangian coordinates, but it is violated by the viscous profiles of the same equation written in Eulerian coordinates. Note that, in the case of the Navier Stokes equation in one space variable, the dimension of the unknown $u$ is 3 and the rank of $B$ is 2, thus the block $A_{11}$ is a real valued function. The problem with the Eulerian coordinates is that $A_{11}(u)$ is invertible (i.e different from 0) at some $u$, but is equal to 0 at other points. As a consequence, one can show that the viscous profiles satisfy a singular ordinary differential equation in the form

$$dV/dx = \phi_s(V)/\zeta(V) + \phi_{ns}(V).$$

(5)

The unknown $V$ is vector valued and has the same dimension as the functions $\phi_s$ and $\phi_{ns}$. The function $\zeta$ is real valued and the singularity of the equation comes from the fact that $\zeta$ can attain the value 0. We are concerned with the solutions of (5) that belong to a sufficiently small neighbourhood of a point $\bar{V}$ such that $\phi_s(\bar{V}) = \phi_{ns}(\bar{V}) = 0$ and $\zeta(\bar{V}) = 0$. A difficulty in the study of (5) comes from the fact that the singularity $\zeta$ depends on the solution $V$ itself. Thus, $d\zeta/dt \neq 0$ in general and hence it may happen that $\zeta$ is different from 0 at $x = 0$ but attains the value 0 for a finite value of $x$. When this happens, the derivative $dV/dx$ may either blow up or become discontinuous. From the point of view of the applications to the study of viscous profiles with small total variation, it is interesting to consider the solutions of (5) that lay on suitable invariant manifolds, namely the center and the stable manifold. It is possible to define suitable conditions, satisfied by the viscous profiles of the Navier Stokes equation written in Eulerian coordinates, which ensure that the solutions of (5) laying either on the stable or on the center manifold enjoy the following property: if $\zeta(V(0)) \neq 0$, then $\zeta(V(x)) \neq 0$ for every $x > 0$. In the following we refer to this property as to (P). As a consequence of (P), the losses of regularity mentioned before are ruled out.

To define the conditions under which property (P) holds, one has first to introduce the notions of fast and slow dynamic. To do this, let us consider a toy model: assume that $\zeta$ is just a parameter, $d\zeta/dx = 0$ and that equation (5) is linear,

$$
\begin{pmatrix}
\frac{dv_1}{dx} \\
\frac{dv_2}{dx}
\end{pmatrix}
= \frac{1}{\zeta}
\begin{pmatrix}
-1 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 \\
0 & -2
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix}
$$

Then the component $v_1$ behaves like $e^{-x/\zeta}$ and hence can be regarded as a fast dynamic, in the sense that the speed of exponential decay at $+\infty$ gets faster and faster as $\zeta \to 0^+$. On the other side, the component $v_2$ goes like $e^{-2x}$ and thus can be regarded as a slow dynamic, because its equation does not get singular when $\zeta \to 0^+$. These definitions can be extended to the general case in which the system is non linear and $d\zeta/dx \neq 0$. Apart from some non-degeneracy conditions, the key hypothesis which guarantee that property (P) is satisfied is the following: the set $\{V : \zeta(V) = 0\}$ is invariant with respect to both the fast and the slow dynamics.

**Note:** if accepted, the talk will be based on the following works: A connection between viscous profiles and singular ODEs and Invariant manifolds for a singular ordinary differential equation. They are both available either at http://arxiv.org/ or at http://www.math.northwestern.edu/~spinolo/.