Ill-posedness for bounded admbissible solutions of the 2-dimensional p-system

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Consider the p–system of isentropic gas dynamics in Eulerian coordinates. The unknowns of the system, which consists of $n+1$ equations, are the density ρ and the velocity v of the gas:

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho v) \; = \; 0 \\ \partial_t(\rho v) + \mathrm{div}_x(\rho v \otimes v) + \nabla[p(\rho)] \; = \; 0 \\ \rho(0, \cdot) \; = \; \rho^0 \\ v(0, \cdot) \; = \; v^0 \end{cases} \tag{1}$$

The pressure p is a function of ρ, which is determined from the constitutive thermodynamic relations of the gas in question and satisfies the assumption $p' > 0$. A typical example is $p(\rho) = k\rho^\gamma$, with constants $k > 0$ and $\gamma > 1$, which gives the

As usual, with "admissible solutions" we understand bounded distributional solutions of (1) which satisfy an additional constraint. Consider the internal energy $\varepsilon : \mathbf{R}^+ \to \mathbf{R}$ given through the law $p(r) = r^2 \varepsilon'(r)$. Then a weak solution is admissible if satisfies the inequality

$$\partial_t \left[\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2}\right] + \mathrm{div}_x \left[\left(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2} + p(\rho)\right) v\right] \; \leq \; 0 \,. \tag{2}$$

Definition 1 *A weak solution of (1) is admissible if the following inequality holds for every nonnegative* $\psi \in C_c^\infty(\mathbf{R}^n \times \mathbf{R})$*:*

$$\int_{\mathbf{R}^n \times \mathbf{R}^+} \left[\left(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2}\right) \partial_t \psi + \left(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2} + p(\rho)\right) \cdot \nabla_x \psi\right]$$

$$+ \quad \int_{\mathbf{R}^n} \left(\rho^0 \varepsilon(\rho^0) + \frac{\rho^0|v^0|^2}{2}\right) \psi(\cdot, 0) \; \geq \; 0 \,. \tag{3}$$

In a recent joint work with László Székelyhidi, we prove the following result.

Theorem 1 *Let $n \geq 2$. Then, for any given function p, there exist bounded initial data (ρ^0, v^0) with $\rho^0 \geq c > 0$ for which there are infinitely many bounded admissible solutions (ρ, v) of (1) with $\rho \geq c > 0$.*

The result is based on a previous work in which we treat the incompressible Euler equations as a differential inclusion and construct very irregular weak solutions with the so-called "Baire Category argument" (or using the method of "Convex Integration"). In the case at hand we extend the approach to the p-system and we enhance the techniques in oder to construct admissible weak solutions.