Control Problems for Hyperbolic Equations

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12th International Conference on Hyperbolic Problems
University of Maryland, College Park, June 9–13, 2008

Outline

1. Introduction
   - General setting
   - Physical Motivations
   - Main problems
2. Controllability & Stabilizability
   - Exact controllability
   - Asymptotic stabilizability
3. Optimal control problems
   - Generalized tangent vectors
   - Linearized evolution equations
4. Pontryagin Maximum Principle for Temple systems
   - Temple systems
   - Evolution of first order variations
   - Pontryagin Maximum Principle

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General setting

$$\begin{cases} \partial_t u + \partial_x f(u) = h(x, u, z), \\ u(0, x) = \bar{u}(x), \\ \text{b.c. at} \quad x = \psi^0(t), \qquad t \geq 0, \quad \psi^0(t) < x < \psi^1(t) \\ \qquad\qquad x = \psi^1(t), \\ \text{with bdr data} \quad \alpha^0, \ \alpha^1 \end{cases}$$

- $u = u(t, x) \in \mathbb{R}^n$ conserved quantities
- $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ smooth flux
- $h : \mathbb{R} \times \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ smooth source
- $z = z(t, x) \in Z \subset \mathbb{R}^m$ distributed control
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General Assumptions

- Strictly Hyperbolic System

$$\partial_t u + \partial_x f(u) = h(x, u, z)$$

$$Df(u) r_i(u) = \lambda_i(u) r_i(u) \qquad i = 1, \ldots, n$$

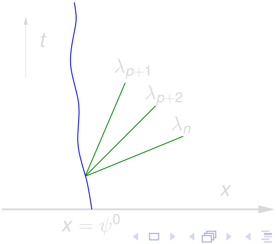
$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u)$$

- Weaker Formulation of B.C.

Dirichlet b.c. not fulfilled pointwise

If $\lambda_p(u) < \dot{\psi}^0(t) < \lambda_{p+1}(u)$

$n - p$ cond's at $x = \psi^0$



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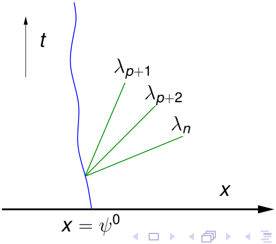

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Isentropic gas dynamic (p-system)

Gas in a clinder with moving piston (in Lagrangian coord.)

$$\begin{cases} \partial_t v - \partial_x u = 0 \\ \partial_t u + \partial_x p(v) = 0 \end{cases} \qquad x \in]0, h[$$

v specific volume, u speed, p pressure

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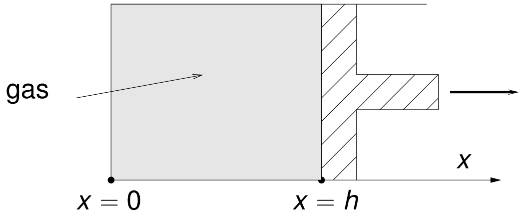

Stabilization problem for gas dynamic

- a control acting only on speed u at $x = h$:

$$u(t, h) = \alpha(t).$$

- a reflection condition at $x = 0$:

$$u(t, 0) = 0.$$

Pb: given

$$v(0, x) = \bar{v}(x), \quad u(0, x) = \bar{u}(x) \qquad x \in]0, h[\,,$$

Stabilize the system at an equilibrium

$$(v, u) = (v^*, 0).$$

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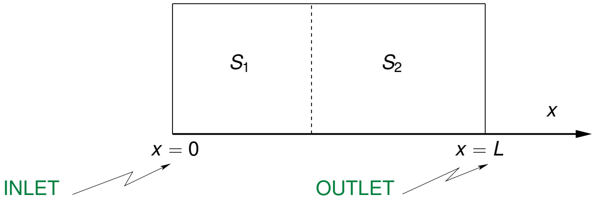
Multicomponent chromatoghraphy

Separate two chemical species in a fluid by selective absorption on a solid medium



$$\begin{cases} \partial_x c_1 + \partial_t\left(\frac{\gamma c_1}{1+c_1+c_2}\right) = 0 \\ \partial_x c_2 + \partial_t\left(\frac{c_2}{1+c_1+c_2}\right) = 0 \end{cases} \qquad x \in]0, L[$$

c_i concentration solute S_i $\quad (\gamma \in]0,1])$

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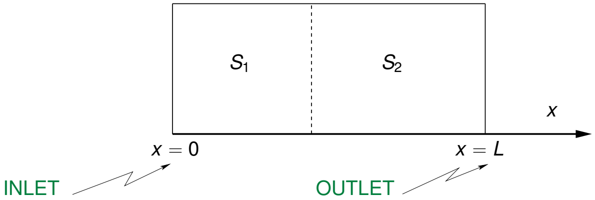

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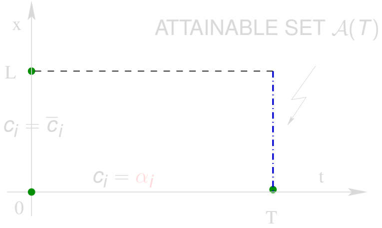
Multicomponent chromatoghraphy

- Temple system with GNL characteristic fields
- Ree, Aris & Amundson (1986, 1989)
- control concentration solute S_i entering the tube at $x = 0$:

$$c_i(t, 0) = \alpha_i(t).$$



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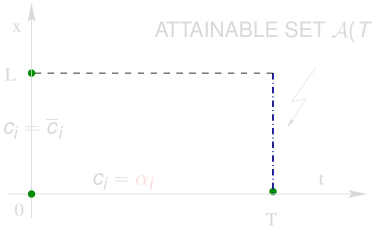

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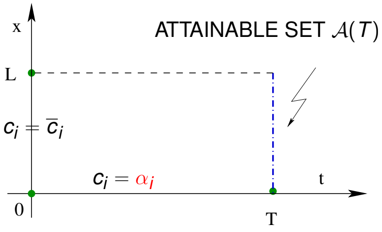

Optimization problem for chromatography

Maximize separation of solutes at time T

$$\max_{x,\alpha}\left\{\int_0^x (c_1(T,\xi)-c_2(T,\xi))\,d\xi + \right.$$
$$\left. + \int_x^L (c_2(T,\xi)-c_1(T,\xi))\,d\xi\right\}$$

$$\begin{cases} \partial_x c_1 + \partial_t\left(\frac{\gamma c_1}{1+c_1+c_2}\right) = 0\,, \\ \partial_x c_2 + \partial_t\left(\frac{c_2}{1+c_1+c_2}\right) = 0\,, \\ c_i(0,x) = \bar{c}_i\,, \\ c_i(t,0) = \alpha_i(t)\,. \end{cases} \qquad x \in]0,L[\,,$$

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Two Classes of Problems

1. Controllability & Stabilizability
2. Optimal control problems

(Mostly boundary controls will be considered)

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Boundary Controllability & Stabilizability

Boundary conditions (non characteristic boundary)

$$b^j(u(t,\psi^j(t))) = g^j(\alpha^j(t)) \qquad (j = 0, 1)$$

Given:

- initial datum $\overline{u}$
- desired terminal profile Φ (e.g. a constant state $\Phi(x) \equiv u^*$)

Do exist:

boundary controls α^j at $x = \psi^j$ so that solution $u_\alpha(t,x)$ of corresponding IBVP satisfies:

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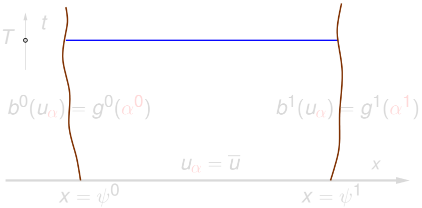
$$u_\alpha(T,\cdot) = \Phi$$

(finite time exact controllability)



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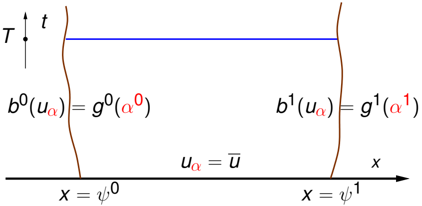

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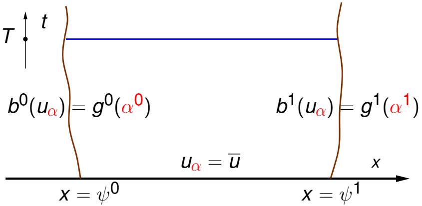

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Optimization problem

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$$\mathcal{J}(u,z,\alpha) = \int_0^T \int_0^{+\infty} L(x,u,z)\; dxdt + \int_0^{+\infty} \Phi\big(x, u(T,x)\big)\; dx + \\ + \int_0^T \Psi\big(u(t,0), \alpha(t)\big)\; dt$$

- single boundary $\psi^0 \equiv 0$
- L, Φ, Ψ smooth
- $\mathcal{A} \subset L^\infty(0,T)$ admissible boundary controls at $x = 0$
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Finite time exact controllability to constant states u^*

1. Quasilinear systems

$$\partial_t u + A(u)\,\partial_x u = h(u) \qquad x \in \,]a, b[\,,$$

with suff. small $\mathcal{C}^1$ initial data $\overline{u}$

(Cirinà, 1969; T.Li, B. Rao & co, 2002-2008; M.Gugat & G. Leugering, 2003)

Finite time exact controllability to constant states u^*

2. Nonlinear scalar convex con laws and GNL Temple systems

$$\partial_t u + \partial_x (f(u)) = 0 \qquad x \in]a, b[\,,$$

with initial data $\overline{u} \in \mathbf{L}^\infty\ (\mathbf{L}^1)$ (discontinuous entropy weak solutions)

(F.A., A.Marson, 1998; T. Horsin, 1998;
F.A. & G.M. Coclite, 2005)

Finite time exact controllability to constant states u^*

3. Isentropic gas dynamic (in Eulerian coord.)

$$\begin{cases} \partial_t \rho + \partial_x(\rho\, u) = 0 \\ \partial_t(\rho\, u) + \partial_x \left(\rho\, u^2 + K\rho^\gamma\right) = 0 \end{cases}$$

with T.V.$\{$bdr controls$\} \gg \|u^* - \overline{u}\|_\infty$
(strong perturbation of the solution)

(O. Glass, 2006)

NO exact controllability to constant states u^*

4. Isentropic gas dynamic for a polytropic gas (in Eulerian coord.)

$$\begin{cases} \partial_t \rho + \partial_x(\rho\, u) = 0 \\ \partial_t u + \partial_x \left(\dfrac{u^2}{2} + \dfrac{K}{\gamma - 1} \rho^{\gamma-1} \right) = 0 \end{cases}$$

$\exists$ initial datum so that corresponding sol. has dense set of discontinuities, whatever bdr controls are prescribed

(A.Bressan & G.M.Coclite, 2002)

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- Evolution of first order variations
- Pontryagin Maximum Principle

1. Stabilizability with total control on both boundaries

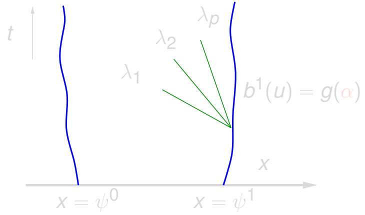
Asymptotic stabilizability around a constant state
with exponential rate

(A.Bressan & G.M.Coclite, 2002)

2. Stabilizability with total control on single boundary

$$\begin{cases} b^0(u(t,\psi^0(t))) = 0\,, \\ b^1(u(t,\psi^1(t))) = g(\alpha(t)) \end{cases}$$

- Assume $Dg(\alpha)$ has full rank
  $\Rightarrow$ full control on waves entering the domain from $x = \psi^1$



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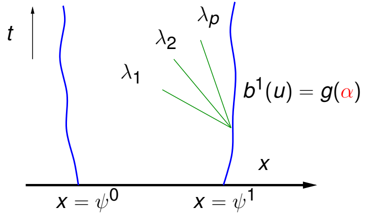
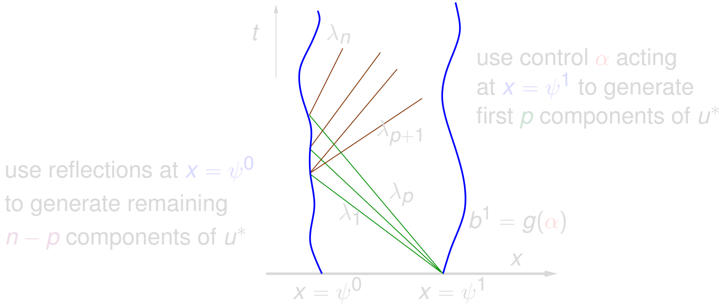


2. Stabilizability with total control on single boundary

- Assume $p \geq n - p$ and $Db^0(u)$ with maximum rank

$$rk\left[Db_0 \cdot r_1(u), \dots, Db_0 \cdot r_p(u)\right] = n - p$$



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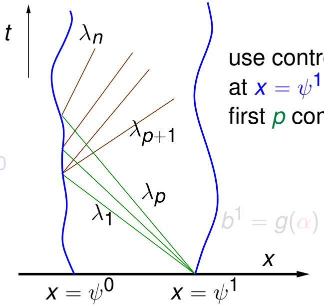

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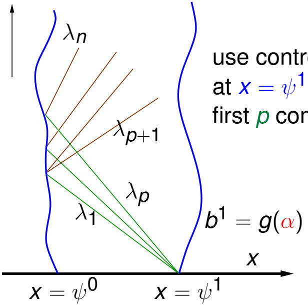

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2. Stabilizability with total control on single boundary

- Nonlinear system $\Rightarrow$ waves produced by bndr control interact with each other generating new waves (2nd generation waves)

$\exists \tau$, bdr control α s.t.

$$\text{T.V.} u_\alpha(\tau, \cdot) = \mathcal{O}(1) \cdot |\bar{u} - u^*|^2$$
$$\|u_\alpha(\tau, \cdot) - u^*\|_\infty = \mathcal{O}(1) \cdot |\bar{u} - u^*|^2$$

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Asymptotic stabilization to equilibrium u^* $\quad (b^0(u^*) = 0)$

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Optimization problem

$$\max_{z\in\mathcal{Z},\,\alpha\in\mathcal{A}} \int_0^T\int_0^{+\infty} L(x,u,z)\,dxdt + \int_0^{+\infty}\Phi\big(x,u(T,x)\big)\,dx + \int_0^T \Psi\big(u(t,0),\alpha(t)\big)\,dt$$

- $u = u_{z,\alpha}(t,x)$ solution to $(\psi^0 \equiv 0)$:

$$\begin{cases} \partial_t u + \partial_x f(u) = h(x,u,z)\,, \\ u(0,x) = \bar{u}(x)\,, \\ b(u(t,0)) = \alpha(t) \end{cases}$$

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Goals

1. Establish existence of optimal solutions
2. Seek necessary conditions for optimality of controls $\widehat{z}, \widehat{\alpha}$
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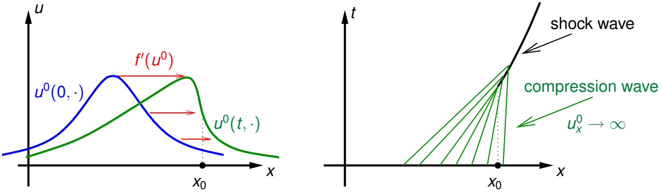
Main difficulties

- Lack of regularity of sol'ns to cons. laws



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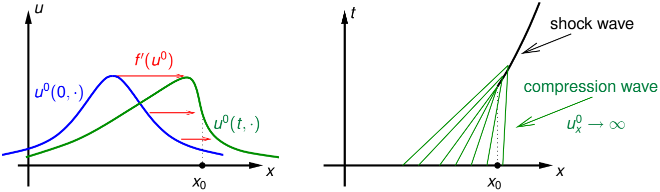

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Non differentiability

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0 \,, \qquad u(0,x) = \bar{u}^\theta(x) \doteq (1+\theta)x \cdot \chi_{[0,1]}(x) \tag{1}$$

Sol. to (1):

$$u^\theta(t,x) = \frac{(1+\theta)x}{1+(1+\theta)t} \cdot \chi_{[0,\sqrt{1+(1+\theta)t}]}(x)$$

Notice:

- $\bar{u}^\theta$ is differentiable in $\mathbf{L}^1$ at $\theta = 0$

$$\lim_{\theta \to 0} \frac{\| \bar{u}^\theta - \bar{u}^0 - \theta\, \bar{u}^0 \|_{L^1}}{\theta} = 0$$

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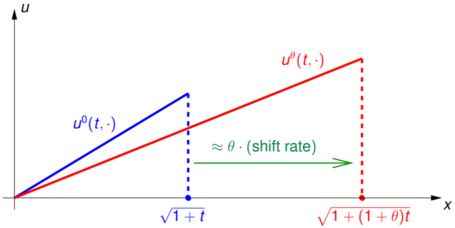

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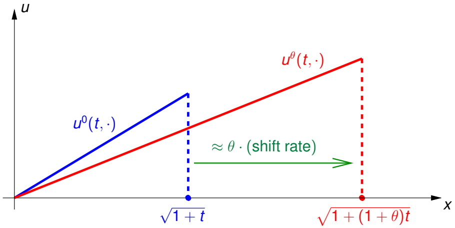

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Non differentiability

$$\lim_{\theta\to 0} \frac{u^\theta(t,\cdot) - u^0(t,\cdot)}{\theta}$$

yields a measure μ_t with a nonzero singular part located at the point of jump $x(t) = \sqrt{1+t}$ of $u^0(t,\cdot)$

$$(\mu_t)^s = \underbrace{\Delta u^0(t,x(t))}_{\text{size of the jump}} \cdot \underbrace{\frac{d}{d\theta}\sqrt{1+(1+\theta)t}\,\Big|_{\theta=0}}_{\text{shift rate}} \cdot \delta_{x(t)}$$

$$= \frac{t}{2(1+t)} \cdot \delta_{x(t)}$$

$$\left(\Delta u^0(t,x(t)) = u^0(t,x(t)-) - u^0(t,x(t)+) = \frac{1}{\sqrt{1+t}}\right)$$

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1 Introduction
- General setting
- Physical Motivations
- Main problems

2 Controllability & Stabilizability
- Exact controllability
- Asymptotic stabilizability

3 Optimal control problems
- Generalized tangent vectors
- Linearized evolution equations

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- Temple systems
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Generalized tangent vectors

A **generalized tangent vector** generated by a family of solutions $\{u^\theta\}$, with $\dfrac{u^\theta(t) - u^0(t)}{\theta} \rightharpoonup \mu_t$, is an element

$$(v, \xi) \in L^1(\mathbb{R}) \times \mathbb{R}^\sharp$$ jumps in u

- v (vertical displacement) takes into account of the absolutely continuous part of μ_t
- ξ (horizontal displacement) takes into account of the singular part of μ_t

(no Cantor part in μ_t)

(A.Bressan & A.Marson, 1995)

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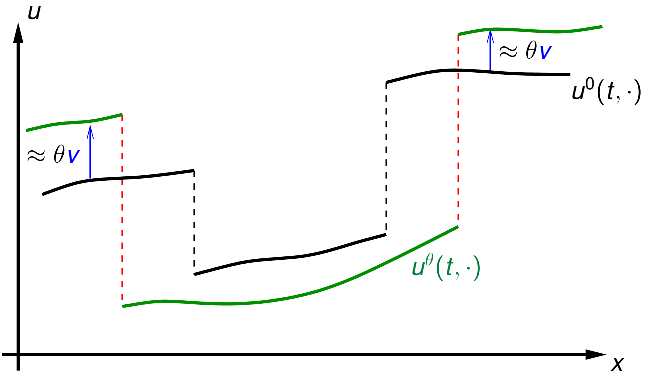

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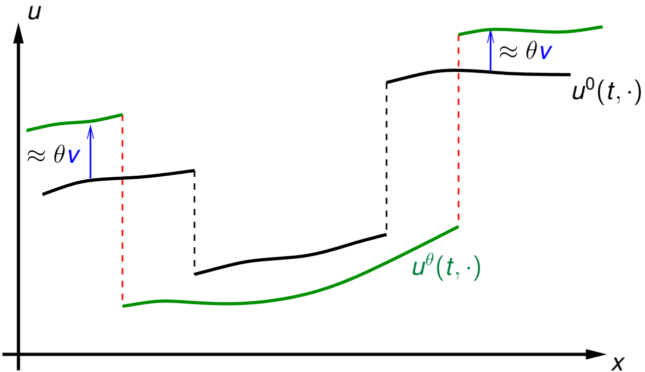

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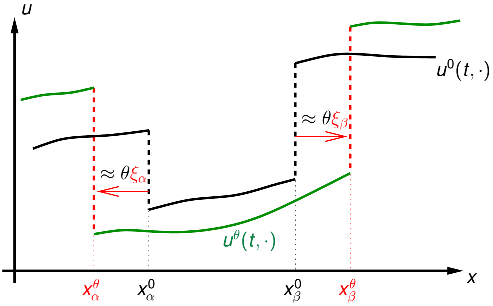

$$\xi_\alpha(t) = \lim_{\theta\to 0} \frac{x_\alpha^\theta(t) - x_\alpha^0(t)}{\theta}$$

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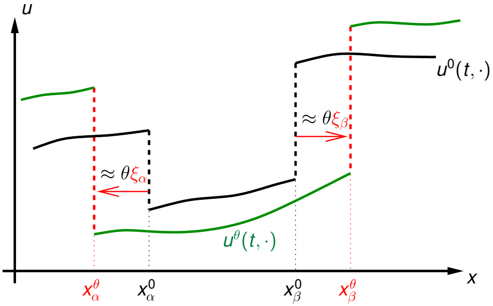

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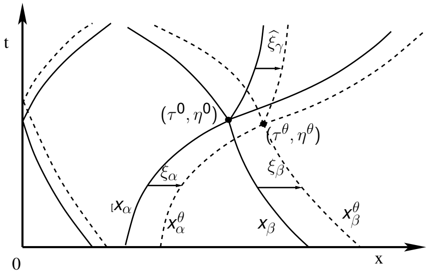
Admissible variations



$$u^\theta(t) \approx u^0(t) + \theta v(t) + \sum_{\xi_\alpha<0} \Delta u^0(t, x_\alpha(t)) \cdot \chi_{[x^0(t)+\theta\xi_\alpha(t), x^0(t)]}$$

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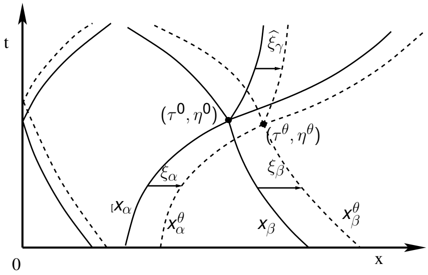

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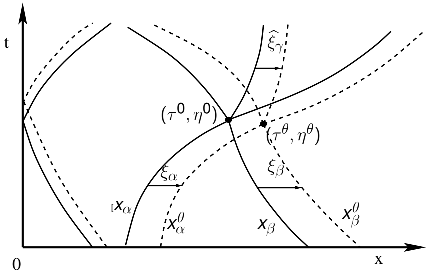

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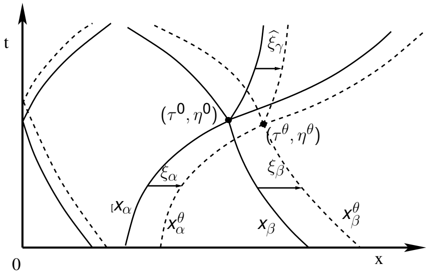

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Evolution of generalized tangent vectors

If

- $u^\theta(\bar{t}, \cdot)$ generates a generalized tangent vector
- discontinuities of u^0 interact at most two at the time
- u^θ is piecewise Lipschitz with uniform in θ Lipschitz constant outside the discontinuities

Then

- $u^\theta(t, \cdot)$ generates a generalized tangent vector $(v(t, \cdot), \xi(t))$ for $t > \bar{t}$

(A.Bressan & A.Marson, 1995)

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Moreover

- $v(t,x)$ is a broad solution of

$$\partial_t v + Df(u)\partial_x v + \left[D^2 f(u)\cdot v\right]\partial_x u = D_u h(x,u,z)\cdot v$$

- $\xi_\alpha(t)$ satisfies an ODE along the α-th discontinuity $x = x_\alpha(t)$
- explicit restarting conditions at the interaction of two discontinuities

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Necessary conditions for optimality

Necessary conditions for optimality obtained by means of generalized cotangent vectors (v^*, ξ^*) satisfying

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Extend variational calculus on generalized tangent and cotangent vectors to first order variations u^θ that do not satisfy

- structural stability assumption on wave structure of reference solution u^0
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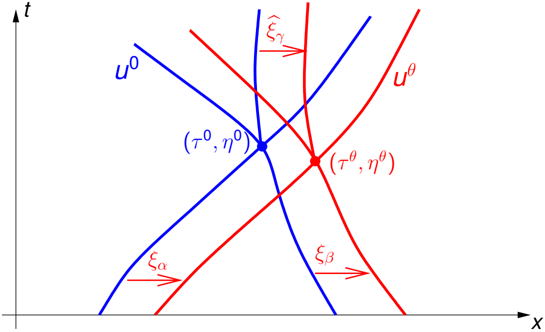
Shock interactions

- the discontinuities of u^0 interact at most two at time



Stability of outgoing wave structure $\Rightarrow$ existence of outgoing tangent vectors

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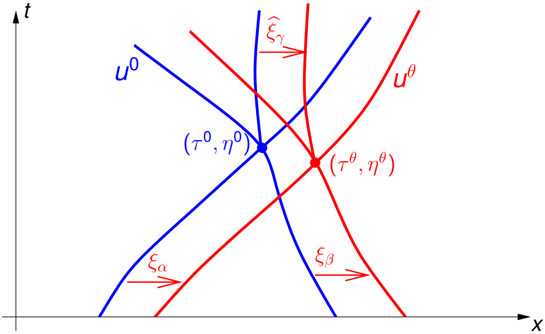

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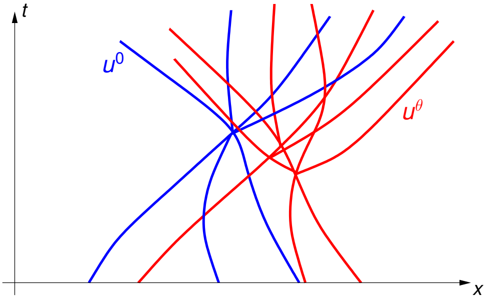

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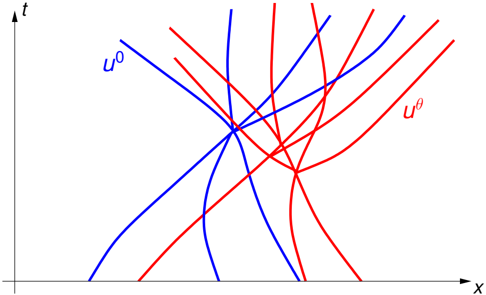

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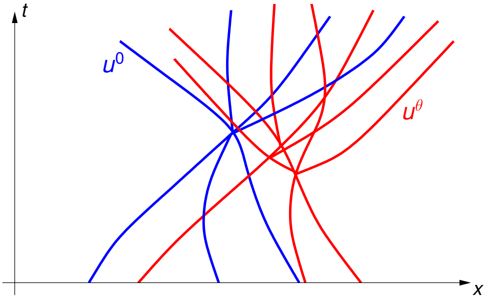

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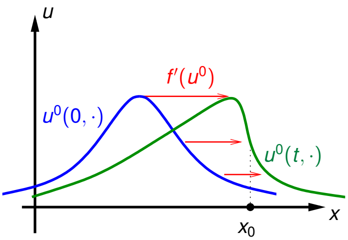
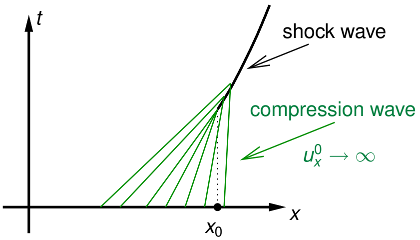
Shock generation

- u^θ is piecewise Lipschitz with uniform in θ Lipschitz constant outside the discontinuities

$\Rightarrow$ no gradient catastrophe in u^0





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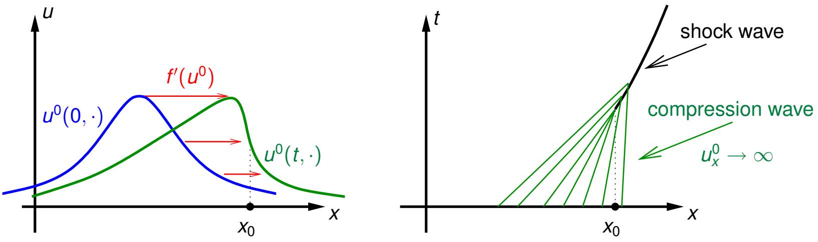

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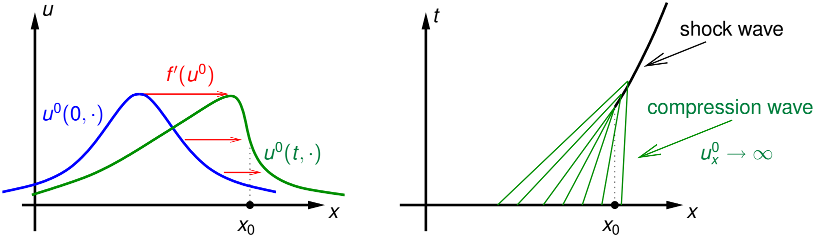

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   - General setting
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   - Main problems
2. Controllability & Stabilizability
   - Exact controllability
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   - Temple systems
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A first step ... towards the goal

Provide necessary conditions for optimality of piecewise Lipschitz solutions with finite number of discontinuities, that may contain compression waves

- Extend variational calculus on generalized tangent and cotangent vectors for a particular class of hyperbolic systems (Temple systems)
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What a Temple system is

Exists a system of coordinates $w = (w_1, \ldots, w_n)$ consisting of Riemann invariants so that

$$\partial_t w_i + \lambda_i(w)\partial_x w_i = \widetilde{h}(x, w, z)\,, \qquad i = 1, \ldots n$$

and the level sets

$$\{u : w_i(u) = \text{const}\}\,, \qquad i = 1, \ldots n$$

are hyperplanes $\Rightarrow$ Hugoniot curves $\equiv$ integral curves of characteristic fields and are straight lines.

Models: chromatography, traffic flow

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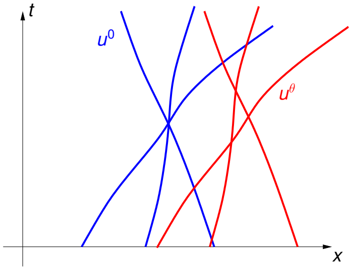

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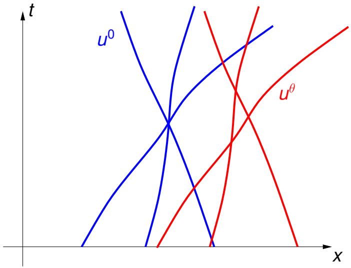

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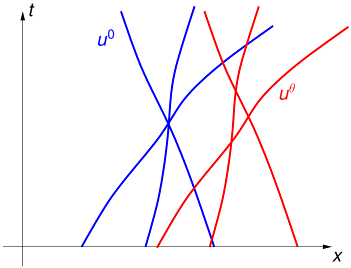

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A PDE for first order variations

Key point: consider a perturbation u^θ that generates a generalized tangent vector (v, ξ) on the domain $[0, T] \times \mathbb{R}$. Then the limit Radon measure

$$\frac{u^\theta(t) - u^0(t)}{\theta} \rightharpoonup \mu_t = \mu^{ac} + \mu^s$$

$$(\mu^s = \sum_\alpha \Delta_\alpha u^0 \, \xi_\alpha \, \delta_{x_\alpha})$$

is a (measure valued) solution of

$$\mu_t + \left(Df(u^0)\mu^{ac}_x\right) + \sum_\alpha \left(\Delta_\alpha u^0 \, \xi_\alpha \, \lambda_{k_\alpha}(u^{0,-}_\alpha, u^{0,+}_\alpha) \, \delta_{x_\alpha}\right)_x = 0$$

$(\lambda_{k_\alpha}(u^{0,-}_\alpha, u^{0,+}_\alpha)$ is shock speed of jump $\Delta_\alpha u^0)$

- if a new shock of u^0 is generated at $\bar{t}$, apply divergence theorem for measure valued solutions to obtain $\mu(\bar{t}, \cdot)$, relying on $\mu(t, \cdot)$ for $t < \bar{t}$
- in time intervals where no new shock is generated evolution of μ is determined by the linearized equation for generalized tangent vectors and the corresponding ODE along discontinuities of u^0

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The Maximum Principle

Assume

- $(\widehat{z}, \widehat{w}) =$ (optimal control–optimal trajectory) be a solution to the optimal control problem
- $\widehat{w}$ with a finite number of discontinuities
- cotangent vector $(v^*(t,x), \xi^*(t))$ be a backward solution of

$$\partial_t v^* + \partial_x v^* \cdot \Lambda(\widehat{w}) + v^* \widetilde{D\Lambda}(\widehat{w}) \cdot \partial_x(\widehat{w}) =$$

$$= -v^* D_w \widetilde{h}(x, \widehat{w}, \widehat{z}) - D_w L(x, \widehat{w}, \widehat{z}), \quad \Lambda(\widehat{w}) = \mathrm{diag}\big(\lambda_i(\widehat{w})\big)$$

$$v^*(T,x) = D_w \Phi\big(x, \widehat{w}(T,x)\big)$$

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Then

at every point of continuity of $\widehat{w}(t,x)$ and $v^*(t,x)$ there holds

$$v^*(t,x)\cdot h(x,\widehat{w},\widehat{z}) + L(x,\widehat{w},\widehat{z}) = \\ = \max_{z\in Z}\left\{v^*(t,x)\cdot h(x,\widehat{w},z) + L(x,\widehat{w},z)\right\}$$

Future directions

- Consider feedback controls $z = z(u)$ which yield regular solutions of balance law

$$\partial_t u + \partial_x f(u) = h(u, z)$$

- Study the optimization problem within a class of (more regular) approximate solutions, e.g.

$$\begin{cases} \partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = h(x, u^\varepsilon, z) + \varepsilon\, \partial_x^2 u^\varepsilon \\ u^\varepsilon(0, x) = \overline{u}(x)\,, \\ u^\varepsilon(t, 0) = g(\alpha(t)) \end{cases} \qquad \varepsilon \longrightarrow 0^+$$

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Thank you for your attention!!