Control Problems for Hyperbolic Equations

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Outline

1. Introduction
   - General setting
     - Physical Motivations
     - Main problems

2. Controllability & Stabilizability
   - Exact controllability
   - Asymptotic stabilizability

3. Optimal control problems
   - Generalized tangent vectors
   - Linearized evolution equations

4. Pontryagin Maximum Principle for Temple systems
   - Temple systems
   - Evolution of first order variations
   - Pontryagin Maximum Principle
General setting

\[ \partial_t u + \partial_x f(u) = h(x, u, z), \]

\[ u(0, x) = \bar{u}(x), \]

b.c. at \( x = \psi^0(t), \quad t \geq 0, \quad \psi^0(t) < x < \psi^1(t) \)

\[ x = \psi^1(t), \]

with bdr data \( \alpha^0, \alpha^1 \)

- \( u = u(t, x) \in \mathbb{R}^n \) conserved quantities
- \( f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n \) smooth flux
- \( h : \mathbb{R} \times \Omega \times \mathbb{R}^m \to \mathbb{R}^n \) smooth source
- \( z = z(t, x) \in Z \subseteq \mathbb{R}^m \) distributed control
- \( \alpha^j = \alpha^j(t) \in \mathbb{R}^{p_j} \) boundary control
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General Assumptions

- **Strictly Hyperbolic System**

  \[
  \partial_t u + \partial_x f(u) = h(x, u, z)
  \]

  \[
  Df(u)r_i(u) = \lambda_i(u)r_i(u) \quad i = 1, \ldots, n
  \]

  \[
  \lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u)
  \]

- **Weaker Formulation of B.C.**

  Dirichlet b.c. not fulfilled pointwise

  If \( \lambda_p(u) < \dot{\psi}^0(t) < \lambda_{p+1}(u) \)

  \( n - p \) cond’s at \( x = \psi^0 \)
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   • Evolution of first order variations
   • Pontryagin Maximum Principle
Isentropic gas dynamic (p-system)

Gas in a clinder with moving piston (in Lagrangian coord.)

\[
\begin{align*}
\partial_t v - \partial_x u &= 0 \\
\partial_t u + \partial_x p(v) &= 0
\end{align*}
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\( x \in ]0, h[ \)

\( v \) specific volume, \( u \) speed, \( p \) pressure
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\(v\) specific volume, \(u\) speed, \(p\) pressure
Stabilization problem for gas dynamic

- a control acting only on speed $u$ at $x = h$:
  \[ u(t, h) = \alpha(t). \]

- a reflection condition at $x = 0$:
  \[ u(t, 0) = 0. \]

**Pb:** given

\[ v(0, x) = \bar{v}(x), \quad u(0, x) = \bar{u}(x) \quad x \in ]0, h[, \]

Stabilize the system at an equilibrium

\[ (v, u) = (v^*, 0). \]
Stabilization problem for gas dynamic

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  \end{align*}
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Multicomponent chromatography

Separate two chemical species in a fluid by selective absorption on a solid medium

\[ \begin{align*}
\partial_x c_1 + \partial_t \left( \frac{\gamma c_1}{1 + c_1 + c_2} \right) &= 0 \quad x \in ]0, L[ \\
\partial_x c_2 + \partial_t \left( \frac{c_2}{1 + c_1 + c_2} \right) &= 0 
\end{align*} \]

\( c_i \) concentration solute \( S_i \) \( (\gamma \in ]0, 1[) \)
Multicomponent chromatography

Separate two chemical species in a fluid by selective absorption on a solid medium.

\[ x = 0 \quad \text{OUTLET} \]
\[ x = L \quad \text{INLET} \]

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\( c_i \) concentration solute \( S_i \) \quad (\gamma \in ]0, 1])
Multicomponent chromatography

- Temple system with GNL characteristic fields
- Ree, Aris & Amundson (1986, 1989)
- control concentration solute $S_i$ entering the tube at $x = 0$:
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Multicomponent chromatography

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Optimization problem for chromatography

Maximize separation of solutes at time $T$

$$\max_{x, \alpha} \left\{ \int_0^x (c_1(T, \xi) - c_2(T, \xi)) \, d\xi + \int_x^L (c_2(T, \xi) - c_1(T, \xi)) \, d\xi \right\}$$

$$\begin{cases} 
\partial_x c_1 + \partial_t \left( \frac{\gamma c_1}{1+c_1+c_2} \right) = 0, \\
\partial_x c_2 + \partial_t \left( \frac{c_2}{1+c_1+c_2} \right) = 0, \\
c_i(0, x) = \bar{c}_i, \\
c_i(t, 0) = \alpha_i(t). 
\end{cases}$$

$x \in ]0, L[$,
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Two Classes of Problems

1. Controllability & Stabilizability
2. Optimal control problems

(Mostly boundary controls will be considered)
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(Mostly boundary controls will be considered)
Boundary conditions (non characteristic boundary)

\[ b^j(u(t, \psi^j(t))) = g^j(\alpha^j(t)) \quad (j = 0, 1) \]

Given:
- initial datum \( \bar{u} \)
- desired terminal profile \( \Phi \) (e.g. a constant state \( \Phi(x) \equiv u^* \))

Do exist:
boundary controls \( \alpha^j \) at \( x = \psi^j \) so that solution \( u_\alpha(t, x) \) of corresponding IBVP satisfies:
Boundary Controllability & Stabilizability

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Boundary Controllability & Stabilizability

\[ u_\alpha(T, \cdot) = \Phi \]

(finite time exact controllability)

or

\[ \lim_{t \to \infty} u_\alpha(t, \cdot) = \Phi? \]

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Optimization problem

\[ \max \left\{ J(u, z, \alpha) : z \in Z, \alpha \in A \right\} \]

\[ J(u, z, \alpha) = \int_0^T \int_0^{+\infty} L(x, u, z) \, dx \, dt + \int_0^{+\infty} \Phi(x, u(T, x)) \, dx + \]

\[ + \int_0^T \Psi(u(t, 0), \alpha(t)) \, dt \]

- single boundary \( \psi^0 \equiv 0 \)
- \( L, \Phi, \psi \) smooth
- \( A \subset L^\infty(0, T) \) admissible boundary controls at \( x = 0 \)
- \( Z \subset L^1_{loc}(0, +\infty \times \mathbb{R}) \) admissible distributed controls
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Finite time exact controllability to constant states $u^*$

1. Quasilinear systems

$$\partial_t u + A(u) \partial_x u = h(u) \quad x \in ]a, b[, $$

with suff. small $C^1$ initial data $\bar{u}$

(Cirinà, 1969; T.Li, B. Rao & co, 2002-2008; M.Gugat & G. Leugering, 2003)
2. Nonlinear scalar convex con laws and GNL Temple systems

\[ \partial_t u + \partial_x (f(u)) = 0 \quad x \in ]a, b[ , \]

with initial data \( \bar{u} \in L^\infty (L^1) \) (discontinuous entropy weak solutions)

3. Isentropic gas dynamic (in Eulerian coord.)

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho \, u) &= 0 \\
\partial_t (\rho \, u) + \partial_x \left( \rho \, u^2 + K \rho^\gamma \right) &= 0
\end{align*}
\]

with T.V.\{bdr controls\} $\lll \| u^* - \bar{u} \|_\infty$

(strong perturbation of the solution)

(O. Glass, 2006)
4. Isentropic gas dynamic for a polytropic gas (in Eulerian coord.)

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0 \\
\partial_t u + \partial_x \left( \frac{u^2}{2} + \frac{K}{\gamma - 1} \rho^{\gamma-1} \right) &= 0
\end{align*}
\]

\[ \exists \text{ initial datum so that corresponding sol. has dense set of discontinuities, whatever bdr controls are prescribed} \]

(A.Bressan & G.M.Coclite, 2002)
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1. Stabilizability with total control on both boundaries

Asymptotic stabilizability around a constant state with exponential rate

(A. Bressan & G. M. Coclite, 2002)
2. Stabilizability with total control on single boundary

\[
\begin{align*}
    b^0(u(t, \psi^0(t))) &= 0, \\
    b^1(u(t, \psi^1(t))) &= g(\alpha(t))
\end{align*}
\]

- Assume $Dg(\alpha)$ has full rank
  \Rightarrow \text{full control on waves entering the domain from } x = \psi^1
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- Assume \(Dg(\alpha)\) has full rank
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2. Stabilizability with total control on single boundary

- Assume $p \geq n - p$ and $Db^0(u)$ with maximum rank

$$rk \left[ Db_0 \cdot r_1(u), \ldots, Db_0 \cdot r_p(u) \right] = n - p$$

use control $\alpha$ acting at $x = \psi^1$ to generate first $p$ components of $u^*$

use reflections at $x = \psi^0$ to generate remaining $n - p$ components of $u^*$
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$\lambda_1, \lambda_{p+1}$

$b^1 = g(\alpha)$
2. Stabilizability with total control on single boundary

- Nonlinear system $\Rightarrow$ waves produced by bndr control interact with each other generating new waves (2nd generation waves)

$\exists \tau, \text{ bdr control } \alpha \text{ s.t.}$

$$
\text{T.V.} u_\alpha(\tau, \cdot) = O(1) \cdot |\bar{u} - u^*|^2
$$

$$
\|u_\alpha(\tau, \cdot) - u^*\|_\infty = O(1) \cdot |\bar{u} - u^*|^2
$$

$\Downarrow$

Asymptotic stabilization to equilibrium $u^*$ \quad (b^0(u^*) = 0)$

(F.A. & A.Marson, 2007)
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- Nonlinear system $\Rightarrow$ waves produced by bndr control interact with each other generating new waves (2nd generation waves)

$\exists \tau, \text{ bdr control } \alpha \text{ s.t.}$

$$\text{T.V.} u_{\alpha}(\tau, \cdot) = O(1) \cdot |\tilde{u} - u^*|^2$$

$$\|u_{\alpha}(\tau, \cdot) - u^*\|_{\infty} = O(1) \cdot |\tilde{u} - u^*|^2$$

$\Downarrow$

Asymptotic stabilization to equilibrium $u^*$ \hspace{1cm} (b^0(u^*) = 0)

(F.A. & A.Marson, 2007)
2. Stabilizability with total control on single boundary

- Nonlinear system $\Rightarrow$ waves produced by bndr control interact with each other generating new waves (2nd generation waves)

$\exists \tau, \ bdr\ control\ \alpha\ s.t.$

$$T.V. u_\alpha(\tau, \cdot) = \mathcal{O}(1) \cdot |\tilde{u} - u^*|^2$$

$$\|u_\alpha(\tau, \cdot) - u^*\|_\infty = \mathcal{O}(1) \cdot |\tilde{u} - u^*|^2$$

$\downarrow$

Asymptotic stabilization to equilibrium $u^*$  $(b^0(u^*) = 0)$

(F.A. & A.Marson, 2007)
Optimization problem

\[
\max_{z \in \mathcal{Z}, \alpha \in \mathcal{A}} \int_0^T \int_0^{+\infty} L(x, u, z) \, dx \, dt + \int_0^{+\infty} \Phi(x, u(T, x)) \, dx + \\
+ \int_0^T \Psi(u(t, 0), \alpha(t)) \, dt
\]

\[
u = u_{z, \alpha}(t, x) \text{ solution to } (\psi^0 \equiv 0): \\
\begin{cases}
\partial_t u + \partial_x f(u) = h(x, u, z), \\
u(0, x) = \bar{u}(x), \\
b(u(t, 0)) = \alpha(t)
\end{cases}
\]
Optimization problem

$$\max_{z \in \mathcal{Z}, \alpha \in A} \int_0^T \int_0^{+\infty} L(x, u, z) \, dx \, dt + \int_0^{+\infty} \Phi(x, u(T, x)) \, dx + \int_0^T \Psi(u(t, 0), \alpha(t)) \, dt$$

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\end{cases}$$
Goals

1. Establish **existence** of optimal solutions

2. Seek **necessary conditions** for optimality of controls $\hat{z}$, $\hat{\alpha}$

3. Provide algorithm to **construct** (almost) optimal solutions
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Main difficulties

- Lack of regularity of solutions to conservation laws

- Non differentiability of input-to-trajectory map $\left(z, \alpha \right) \mapsto u_{z, \alpha}$ in any natural Banach space
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- Non differentiability of input-to-trajectory map \((z, \alpha) \mapsto u_{z,\alpha}\) in any natural Banach space
Non differentiability

\[
\partial_t u + \partial_x \left( \frac{u^2}{2} \right) = 0, \quad u(0, x) = \bar{u}^\theta(x) \equiv (1 + \theta)x \cdot \chi_{[0,1]}(x)
\]

(1)

Sol. to (1):

\[
u^\theta(t, x) = \frac{(1 + \theta)x}{1 + (1 + \theta)t} \cdot \chi_{[0, \sqrt{1+(1+\theta)t}]}(x)
\]

Notice:

- \(\bar{u}^\theta\) is differentiable in \(L^1\) at \(\theta = 0\)

\[
\lim_{\theta \to 0} \frac{\|\bar{u}^\theta - \bar{u}^0 - \theta \bar{u}^0\|_{L^1}}{\theta} = 0
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Non differentiability

The location of the jump in \( u^\theta(t, \cdot) \) depends on \( \theta \)

\[ \Rightarrow u^\theta(t, \cdot) \text{ is NOT diff. in } L^1 \text{ at } \theta = 0 \text{ for } t > 0 \]
The location of the jump in $u^\theta(t, \cdot)$ depends on $\theta$

$$\Rightarrow u^\theta(t, \cdot) \text{ is NOT diff. in } L^1 \text{ at } \theta = 0 \text{ for } t > 0$$
Non differentiability

\[
\lim_{\theta \to 0} \frac{u^\theta(t, \cdot) - u^0(t, \cdot)}{\theta}
\]
yields a measure \(\mu_t\) with a **nonzero** singular part located at the point of jump \(x(t) = \sqrt{1 + t}\) of \(u^0(t, \cdot)\).

\[
(\mu_t)^s = \Delta u^0(t, x(t)) \cdot \frac{d}{d\theta} \sqrt{1 + (1 + \theta)t} \bigg|_{\theta=0} \cdot \delta x(t)
\]

\[
= \frac{t}{2(1 + t)} \cdot \delta x(t)
\]

\[
(\Delta u^0(t, x(t))) = u^0(t, x(t)^-) - u^0(t, x(t)^+) = \frac{1}{\sqrt{1 + t}}
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   - Linearized evolution equations

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   - Temple systems
   - Evolution of first order variations
   - Pontryagin Maximum Principle
A **generalized tangent vector** generated by a family of solutions \( \{ u^\theta \} \), with \( \frac{u^\theta(t) - u^0(t)}{\theta} \rightarrow \mu_t \), is an element 

\[
(v, \xi) \in L^1(\mathbb{R}) \times \mathbb{R}^\# \text{ jumps in } u
\]

- \( v \) (vertical displacement) takes into account of the absolutely continuous part of \( \mu_t \)
- \( \xi \) (horizontal displacement) takes into account of the singular part of \( \mu_t \)

(no Cantor part in \( \mu_t \))

(A.Bressan & A.Marson, 1995)
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Vertical displacement

\[ u \approx \theta v \]

\[ u^0(t, \cdot) \approx \theta v \]

\[ u^\theta(t, \cdot) \]

\[ v(t, x) = \lim_{\theta \to 0} \frac{u^\theta(t, x) - u^0(t, x)}{\theta} \]
Vertical displacement

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Horizontal displacement

\[ u_\alpha(t) = \lim_{\theta \to 0} \frac{x_\alpha^\theta(t) - x_\alpha^0(t)}{\theta} \]

rates of horizontal displacement of locations \( x_1^\theta(t) < \cdots > x_N^\theta(t) \)
of jumps in \( u^\theta(t, \cdot) \)
Horizontal displacement

\[ \xi_{\alpha}(t) = \lim_{\theta \to 0} \frac{x_{\alpha}^{\theta}(t) - x_{\alpha}^{0}(t)}{\theta} \]

rates of horizontal displacement of locations \( x_{1}^{\theta}(t) < \cdots > x_{N}^{\theta}(t) \)
of jumps in \( u^{\theta}(t, \cdot) \)
**Admissible variations**

\[
u^\theta(t) \approx u^0(t) + \theta v(t) + \sum_{\xi_\alpha < 0} \Delta u^0(t, x_\alpha(t)) \cdot \chi [x^0(t) + \theta \xi_\alpha(t), x^0(t)]
\]

\[
+ \sum_{\xi_\alpha > 0} \Delta u^0(t, x_\alpha(t)) \cdot \chi [x^0(t), x^0(t) + \theta \xi_\alpha(t)]
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Fabio Ancona
Evolution of generalized tangent vectors

If

- $u^\theta(\bar{t}, \cdot)$ generates a generalized tangent vector
- discontinuities of $u^0$ interact at most two at the time
- $u^\theta$ is piecewise Lipschitz with uniform in $\theta$ Lipschitz constant outside the discontinuities

Then

- $u^\theta(t, \cdot)$ generates a generalized tangent vector $(v(t, \cdot), \xi(t))$ for $t > \bar{t}$

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  \( (\nu(t, \cdot), \xi(t)) \) for \( t > \bar{t} \)

(A.Bressan & A.Marson, 1995)
Evolution of generalized tangent vectors

Moreover

- \( v(t, x) \) is a broad solution of
  \[
  \partial_t v + Df(u)\partial_x v + [D^2f(u) \cdot v]\partial_x u = D_u h(x, u, z) \cdot v
  \]

- \( \xi_\alpha(t) \) satisfies an ODE along the \( \alpha \)-th discontinuity
  \( x = x_\alpha(t) \)

- explicit restarting conditions at the interaction of two discontinuities

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Necessary conditions for optimality obtained by means of generalized cotangent vectors \((\nu^*, \xi^*)\) satisfying

\[
\int v^*(t, x) \cdot v(t, x) \, dx + \sum_j \xi^*_j(t)\xi_j(t) = \text{const}
\]

backward transported along trajectories of

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(A. Bressan, A. Marson, 1995; A. Bressan, W. Shen, 2007)
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Goal

Extend variational calculus on generalized tangent and cotangent vectors to **first order variations** $u^\theta$ that do not satisfy

- structural stability assumption on wave structure of reference solution $u^0$
- uniform Lipschitzianity assumption on continuous part of reference solution $u^0$
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Extend variational calculus on generalized tangent and cotangent vectors to first order variations $u^\theta$ that do not satisfy

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Extend variational calculus on generalized tangent and cotangent vectors to first order variations $u^\theta$ that do not satisfy

- **structural stability** assumption on wave structure of reference solution $u^0$
- **uniform Lipschitzianity** assumption on continuous part of reference solution $u^0$
Shock interactions

- the discontinuities of $u^0$ interact at most two at time $t$

Stability of outgoing wave structure $\Rightarrow$ existence of outgoing tangent vectors
Shock interactions

- the discontinuities of \( u^0 \) interact at most two at time

\[ (\tau^0, \eta^0) \]

\[ (\tau^\theta, \eta^\theta) \]

Stability of outgoing wave structure \( \Rightarrow \) existence of outgoing tangent vectors
If more than two discontinuities interact at the time...

...instability of outgoing wave structure
Existence of outgoing tangent vectors?
If more than two discontinuities interact at the time...

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Existence of outgoing tangent vectors?
Shock interactions

If more than two discontinuities interact at the time...

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Existence of outgoing tangent vectors?
\( u^\theta \) is piecewise Lipschitz with uniform in \( \theta \) Lipschitz constant outside the discontinuities

\[ \Rightarrow \text{no gradient catastrophe in } u^0 \]

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A first step ... towards the goal

Provide necessary conditions for optimality of piecewise Lipschitz solutions with finite number of discontinuities, that may contain compression waves

- Extend variational calculus on generalized tangent and cotangent vectors for a particular class of hyperbolic systems (Temple systems)
- Derive a Pontryagin type maximum principle for optimal solutions of such systems

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What a Temple system is

Exists a system of coordinates \( w = (w_1, \ldots, w_n) \) consisting of Riemann invariants so that

\[
\partial_t w_i + \lambda_i(w)\partial_x w_i = \tilde{h}(x, w, z), \quad i = 1, \ldots, n
\]

and the level sets

\[
\{ u : w_i(u) = \text{const} \}, \quad i = 1, \ldots, n
\]

are hyperplanes \( \Rightarrow \) Hugoniot curves \( \equiv \) integral curves of characteristic fields and are straight lines.

Models: chromatography, traffic flow
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**Models:** chromatography, traffic flow
Stability of wave structure at interactions...

...even in the presence of three or more interacting discontinuities (No wave of new families emerges at the interaction)

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A PDE for first order variations

Key point: consider a perturbation \( u^\theta \) that generates a generalized tangent vector \((v, \xi)\) on the domain \([0, T] \times \mathbb{R}\). Then the limit Radon measure

\[
\frac{u^\theta(t) - u^0(t)}{\theta} \to \mu_t = \mu^{\text{ac}} + \mu^s
\]

\[(\mu^s = \sum_{\alpha} \Delta_{\alpha} u^0 \xi_{\alpha} \delta_{x_{\alpha}})\]

is a (measure valued) solution of

\[
\mu_t + (Df(u^0)\mu^{\text{ac}}) + \sum_{\alpha} \left( \Delta_{\alpha} u^0 \xi_{\alpha} \lambda_{k_{\alpha}}(u^0_{\alpha}^-, u^0_{\alpha}^+) \delta_{x_{\alpha}} \right)_x = 0
\]

\[(\lambda_{k_{\alpha}}(u^0_{\alpha}^-, u^0_{\alpha}^+) \text{ is shock speed of jump } \Delta_{\alpha} u^0)\]
if a new shock of $u^0$ is generated at $\bar{t}$, apply divergence theorem for measure valued solutions to obtain $\mu(\bar{t}, \cdot)$, relying on $\mu(t, \cdot)$ for $t < \bar{t}$

in time intervals where no new shock is generated evolution of $\mu$ is determined by the linearized equation for generalized tangent vectors and the corresponding ODE along discontinuities of $u^0$
if a new shock of $u^0$ is generated at $\bar{t}$, apply divergence theorem for measure valued solutions to obtain $\mu(\bar{t}, \cdot)$, relying on $\mu(t, \cdot)$ for $t < \bar{t}$

in time intervals where no new shock is generated, evolution of $\mu$ is determined by the linearized equation for generalized tangent vectors and the corresponding ODE along discontinuities of $u^0$
Outline

1. Introduction
   - General setting
   - Physical Motivations
   - Main problems

2. Controllability & Stabilizability
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   - Asymptotic stabilizability

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   - Generalized tangent vectors
   - Linearized evolution equations

4. Pontryagin Maximum Principle for Temple systems
   - Temple systems
   - Evolution of first order variations
   - Pontryagin Maximum Principle
The Maximum Principle

Assume

- \((\hat{z}, \hat{w}) = (\text{optimal control–optimal trajectory})\) be a solution to the optimal control problem
- \(\hat{w}\) with a finite number of discontinuities
- cotangent vector \((v^*(t, x), \xi^*(t))\) be a backward solution of

\[
\partial_t v^* + \partial_x v^* \cdot \Lambda(\hat{w}) + v^* D\Lambda(\hat{w}) \cdot \partial_x (\hat{w}) = -v^* D_w \tilde{h}(x, \hat{w}, \hat{z}) - D_w L(x, \hat{w}, \hat{z}), \quad \Lambda(\hat{w}) = \text{diag}(\lambda_i(\hat{w}))
\]

\[
v^*(T, x) = D_w \Phi(x, \hat{w}(T, x))
\]

\[
\xi^*_\alpha(T) = \Delta \Phi(x_\alpha, \hat{w}(T, x_\alpha)) + \text{backward ODEs along the jumps for } \xi^*_\alpha
\]

Fabio Ancona
Control Problems for Hyperbolic Equations
The Maximum Principle

Assume

- $(\hat{z}, \hat{w}) = (\text{optimal control}–\text{optimal trajectory})$ be a solution to the optimal control problem
- $\hat{w}$ with a finite number of discontinuities
- cotangent vector $(v^*(t, x), \xi^*(t))$ be a backward solution of

$$
\partial_t v^* + \partial_x v^* \cdot \Lambda(\hat{w}) + v^* D\Lambda(\hat{w}) \cdot \partial_x (\hat{w}) = -v^* Dw \tilde{h}(x, \hat{w}, \hat{z}) - Dw L(x, \hat{w}, \hat{z}), \quad \Lambda(\hat{w}) = \text{diag}(\lambda_i(\hat{w}))
$$

$v^*(T, x) = Dw \Phi(x, \hat{w}(T, x))$

$\xi^*_\alpha(T) = \Delta \Phi(x_\alpha, \hat{w}(T, x_\alpha))$

+ backward ODEs along the jumps for $\xi^*_\alpha$
The Maximum Principle

Assume

- \((\widehat{Z}, \widehat{w}) = \) (optimal control–optimal trajectory) be a solution to the optimal control problem
- \(\widehat{w}\) with a finite number of discontinuities
- cotangent vector \((v^*(t, x), \xi^*(t))\) be a backward solution of

\[
\partial_t v^* + \partial_x v^* \cdot \Lambda(\widehat{w}) + v^* \bar{D} \Lambda(\widehat{w}) \cdot \partial_x(\widehat{w}) =
\]
\[
= -v^* D_w \bar{h}(x, \widehat{w}, \widehat{Z}) - D_w L(x, \widehat{w}, \widehat{Z}), \quad \Lambda(\widehat{w}) = \text{diag}(\lambda_i(\widehat{w}))
\]
\[
v^*(T, x) = D_w \Phi(x, \widehat{w}(T, x))
\]
\[
\xi^*_\alpha(T) = \Delta \Phi(x_\alpha, \widehat{w}(T, x_\alpha))
\]

+ backward ODEs along the jumps for \(\xi^*_\alpha\)
The Maximum Principle

Assume

- \((\hat{Z}, \hat{w}) = (\text{optimal control–optimal trajectory})\) be a solution to the optimal control problem
- \(\hat{w}\) with a finite number of discontinuities
- cotangent vector \((\nu^*(t, x), \xi^*(t))\) be a backward solution of

\[
\partial_t \nu^* + \partial_x \nu^* \cdot \Lambda(\hat{w}) + \nu^* \tilde{D} \Lambda(\hat{w}) \cdot \partial_x(\hat{w}) = -\nu^* D_w \tilde{h}(x, \hat{w}, \hat{z}) - D_w L(x, \hat{w}, \hat{z}), \quad \Lambda(\hat{w}) = \text{diag}(\lambda_i(\hat{w}))
\]

\[
\nu^*(T, x) = D_w \phi(x, \hat{w}(T, x))
\]

\[
\xi^*_\alpha(T) = \Delta \phi(x_\alpha, \hat{w}(T, x_\alpha))
\]

+ backward ODEs along the jumps for \(\xi^*_\alpha\)
The Maximum Principle

Assume

1. \((\hat{z}, \hat{w}) = (\text{optimal control–optimal trajectory})\) be a solution to the optimal control problem
2. \(\hat{w}\) with a finite number of discontinuities
3. cotangent vector \((v^*(t, x), \xi^*(t))\) be a backward solution of

\[
\partial_t v^* + \partial_x v^* \cdot \Lambda(\hat{w}) + v^* \tilde{D} \Lambda(\hat{w}) \cdot \partial_x (\hat{w}) = \\
= -v^* D_w \tilde{h}(x, \hat{w}, \hat{z}) - D_w L(x, \hat{w}, \hat{z}),
\]

\(\Lambda(\hat{w}) = \text{diag}(\lambda_i(\hat{w}))\)

\(v^*(T, x) = D_w \Phi(x, \hat{w}(T, x))\)

\(\xi^*_\alpha(T) = \Delta \Phi(x_\alpha, \hat{w}(T, x_\alpha))\)

+ backward ODEs along the jumps for \(\xi^*_\alpha\)
Assume

- \((\hat{Z}, \hat{w}) = \text{(optimal control–optimal trajectory)}\) be a solution to the optimal control problem
- \(\hat{w}\) with a finite number of discontinuities
- cotangent vector \((v^*(t, x), \xi^*(t))\) be a backward solution of

\[
\begin{align*}
\frac{\partial_t v^* + \partial_x v^* \cdot \Lambda(\hat{w}) + v^* \hat{D} \Lambda(\hat{w}) \cdot \partial_x (\hat{w}) =}
&= -v^* D_{w} \tilde{h}(x, \hat{w}, \hat{z}) - D_{w} L(x, \hat{w}, \hat{z}), \quad \Lambda(\hat{w}) = \text{diag}(\lambda_i(\hat{w}))
\end{align*}
\]

\(v^*(T, x) = D_{w} \Phi(x, \hat{w}(T, x))\)

\(\xi_{\alpha}^*(T) = \Delta \Phi(x_{\alpha}, \hat{w}(T, x_{\alpha})) + \text{backward ODEs along the jumps for } \xi_{\alpha}^*\)
The Maximum Principle

Assume

- \((\hat{Z}, \hat{w}) = (\text{optimal control–optimal trajectory})\) be a solution to the optimal control problem
- \(\hat{w}\) with a finite number of discontinuities
- cotangent vector \((v^*(t, x), \xi^*(t))\) be a backward solution of

\[
\begin{align*}
\partial_t v^* + \partial_x v^* \cdot \Lambda(\hat{w}) + v^* \tilde{D} \Lambda(\hat{w}) \cdot \partial_x(\hat{w}) &= -v^* D_w \tilde{h}(x, \hat{w}, \hat{z}) - D_w L(x, \hat{w}, \hat{z}), \\
n^*(T, x) &= D_w \Phi(x, \hat{w}(T, x)) \\
\xi^*_\alpha(T) &= \Delta \Phi(x_\alpha, \hat{w}(T, x_\alpha)) + \text{backward ODEs along the jumps for } \xi^*_\alpha
\end{align*}
\]
The Maximum Principle

Then

at every point of continuity of $\hat{w}(t, x)$ and $v^*(t, x)$ there holds

$$v^*(t, x) \cdot h(x, \hat{w}, \hat{z}) + L(x, \hat{w}, \hat{z}) =$$

$$= \max_{z \in Z} \left\{ v^*(t, x) \cdot h(x, \hat{w}, z) + L(x, \hat{w}, z) \right\}$$
Future directions

- Consider feedback controls \( z = z(u) \) which yield regular solutions of balance law

\[
\partial_t u + \partial_x f(u) = h(u, z)
\]

- Study the optimization problem within a class of (more regular) approximate solutions, e.g.

\[
\begin{aligned}
\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) &= h(x, u^\varepsilon, z) + \varepsilon \partial_x^2 u^\varepsilon \\
u^\varepsilon(0, x) &= \bar{u}(x), \\
u^\varepsilon(t, 0) &= g(\alpha(t))
\end{aligned}
\]
\( \varepsilon \to 0^+ \)
Future directions

- Consider feedback controls $z = z(u)$ which yield regular solutions of balance law

$$\partial_t u + \partial_x f(u) = h(u, z)$$

- Study the optimization problem within a class of (more regular) approximate solutions, e.g.

$$\begin{cases} 
\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = h(x, u^\varepsilon, z) + \varepsilon \partial_x^2 u^\varepsilon \\
u^\varepsilon(0, x) = \bar{u}(x), \\
u^\varepsilon(t, 0) = g(\alpha(t)) 
\end{cases} \quad \varepsilon \to 0^+$$
Future directions

- Consider feedback controls $z = z(u)$ which yield regular solutions of balance law

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u^\varepsilon(t, 0) = g(\alpha(t))
\end{cases} \quad \varepsilon \to 0^+$$
Thank you for your attention!!