Study on Mach Reflection and Mach Configuration

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Hyp-2008, Maryland
Outline

- Physical phenomena
- Some crucial issues
  - Stability of Mach configuration
  - von Neumann paradox
  - Other irregular configurations
  - Global wave pattern
- A result on global existence of stationary Mach reflection
- Summarize open problems
**Part I: Physical phenomena**

Unsteady Case

(a shock front hits ground)
Two cases of shock reflection

Regular reflection

Mach reflection
polars emanate from their corresponding $I$-polars at the pressure. Figure 4.7 shows a $D_1$MR at a point 'a' (the $I_1$-polars intersect on the right branch of the $I_1$-polar), an $S_1$MR at 'b' (the $I_2$- and $R_2$-polars intersect on the p-axis), and a reflection at point 'c' (the $I_3$- and $R_3$-polars intersect on the left branch of the $I_3$-polar) and finally an RR at point 'd'. Consequently, the multi shock polar suggests that the reflection goes through a dashed line up to point 'e'. When the $I_n$-MR is maintained at a point 'f', the following sequence of events: a $D_1$MR from point 'a' to a momentary $S_1$MR to an $I_n$MR to an $R_1$-polar will be generated in the flow. Therefore, based on the
Pseudo-steady case

Invariance under transformation: \( t \rightarrow \alpha t, x \rightarrow \alpha x, y \rightarrow \alpha y \)

Self-similar solution: \( U(t, x, y) = U(x/t, y/t) \)
Regular reflection
Regular reflection

S. Canic, B. L. Keyfitz, E. H. Kim, : Unsteady Transonic Small Disturbance Eq.
Nonlinear wave equation

Y. X. Zheng: Gradient Pressure Equation
G. Q. Chen & M. Feldman: Potential Equation
T. P. Liu & V. Elling: Potential Equation
Mach reflection
Mach reflection
Steady case
(Mach reflection)

- Wedge
- Incident
- Reflected
- Mach stem
- Ground
Similar wave pattern occurs for the compressible flow in a duct
Part II: Some crucial issues

Problem : What is the right wave configuration near the triple intersection point ?
Three shocks separating three zones of different continuous states are impossible!

R.Courant & K.O.Friedrichs
C.Morawetz
D.Serre
von Neumann suggestion: Mach configuration
The flat configuration can be obtained by using shock polar
$(\theta, p)$ shock polar (E-E type)
$(\theta, \rho)$ shock polar (E-H type)
Confirm the stability of Mach configuration under perturbation

Conclusion

When the supersonic part of a given flat Mach configuration (E-E type) is slightly perturbed, then

1. whole structure of the configuration still holds.

2. all elements of the subsonic part are also slightly perturbed.
E-H type Mach configuration will lead us to study on nonlinear mixed type equation of Lavrentiev type:

An (nonlinear) equation is hyperbolic type in a part of the domain, and is elliptic type in other part of the domain. The coefficients have discontinuity on the line, where the equation changes its type. The line and a part of the boundary are determined together with the solution.
Problem on transition

Regular reflection for small $\beta_i$ (wave angle of the incident)  
Mach reflection for large $\beta_i$

How does a regular reflection transit to a Mach reflection?
Dual-solution Domain

Upper part: Mach reflection
Lower part: Regular reflection
Overlapped: Dual-solution domain
von Neumann Criterion

(Mechanical Equilibrium Criterion)
Detachment Criterion
Transition criterion

- von Neumann criterion
  (Mechanism equilibrium criterion)
- Detachment criterion
- Sonic criterion
- Hysteresis phenomenon
\[ \beta_D > \beta_S > \beta_N \ (\beta_D \sim \beta_S) \]

Henderson & Lozzi (1979, experiment) support (N)
Hornung & Robison (1982, experiment) support (N)
Teshukov (1989, linear stability) RR is stable in “dual”
H. Li & Ben-Dor (1996) RR is stable in most of “dual”
D. Li (2007, stability on linearized system), support (S)
V. Elling (2008, PDE) Find a solution above \( \beta_s \)
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Chpoun (1994) Hysteresis in “dual”
Ben-Dor, Ivanov, Vasilev, Elperin (2002) Hysteresis in “dual”
No Mach reflection

(According to von Neumann model)
Von Neumann Paradox

Discrepancies between von Neumann’s three shock theory and experiments (first reported by White)

Particularly for weak incident shock
Von Neumann Paradox

If (the incident shock) $i$ is sufficiently weak, von Neumann’s model has no physical solution for MR but experiments produce MR-like phenomena. Apparent persistence of RR and MR into regions where von Neumann’s model has no realistic predictions was called “the von Neumann paradox” by Birkhoff (1950)

---- Colella, P. & Henderson, L.F.
Problem: Are there other irregular configurations?

- Von Neumann Reflection (NR)
- Guderley Reflection (RR)
- ?R

4 - Wave Theory

E.I. Vasilev, T. Elperin & G. Ben-Dor

Von Neumann Configuration
(suggested by Collela & Henderson)
Guderley Reflection
Four wave configuration
Global wave pattern
( pseudo-steady case )

E-H type Mach configuration causes more complicated wave pattern
Global wave pattern
(pseudo-steady case)

• Single Mach reflection
• Transition Mach reflection
• Double Mach reflection
• Transitional-double Mach reflection
• Triple Mach reflection

( G. Ben Dor Shock Waves v.15, 2006 )
Double Mach Reflection
Transition Mach Reflection
Classification of steady Mach reflection

• Direct Mach reflection
• Stationary Mach reflection
• Inverted Mach reflection

(R.Courant & K.O.Friedrichs)
Direct Mach reflection
Stationary Mach reflection
Inverted Mach reflection
Inverted Mach configuration is unstable

By subtracting the velocity of the upstream flow from all velocity vectors, these configurations are reduced to reflection configurations, moving into quiet gas, then for inverted Mach conf. the triple point moves towards the wall, so that would be quickly destroyed.
Part III:

Stability of stationary Mach reflection
Perturbed stationary Mach reflection
2-D Stationary Euler System

\[
\begin{align*}
\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} &= 0, \\
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} &= 0, \\
u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} &= 0.
\end{align*}
\]

**Bernoulli Law:**

\[
\frac{1}{2} (u^2 + v^2) + \frac{a^2}{\gamma - 1} = \text{const.,}
\]
Rankine-Hugoniot Conditions

\[
\begin{align*}
[\rho u] &= \psi'[\rho v] \\
[p + \rho u^2] &= \psi'[\rho uv] \\
[\rho uv] &= \psi'[p + \rho v^2]
\end{align*}
\]
• The main task is to consider the solution in $\Omega_{2,3}$

• Two relations from R-H conditions

• The location of triple point, the shock, the contact are to be determined
Free boundary problem (FB)

System in $\Omega_{2,3}$

Two boundary conditions from R-H

- on perturbed shock fronts
  - $p$ and $v/u$ are continuous
  - $p$ is given on $L$

- on perturbed contact discontinuity

- $v=0$ on $B$
Approach to some crucial points

• Free triple point (Monotonicity)
• Lagrange transformation to straighten slip line
• Reduce to a fixed boundary value problem
• Decompose the system to elliptic part and hyperbolic part
• Singular integral equation on the “contact”
**Free triple point**

Take a point \((x_{1t}, y_{1t})\) on the perturbed incident shock as a temporary fixed triple point. If the problem can be solved, we obtain the intersection 
\[
(x_2, y_{2t}) = \tilde{r} \cap L
\]

The monotonicity of \(y_{2t}\) with respect to \(x_{1t}\) helps us to find the right location of the triple point.
Lagrange transformation

Define $y = y(x, h)$ by

\[
\begin{cases}
\frac{dy(x, h)}{dx} = \frac{v}{u}, \\
y(x^*, h) = h
\end{cases}
\]

The conservation of mass implies

\[
\int_{y(x, 0)}^{y(x, \eta)} \rho u dy
\]

in independent of $x$. Then the transform

$T: x = \xi, \ y = y(\xi, \eta)$ can straighten all stream lines, including the slip line.
Fix free boundary

- Form two relations from the R-H conditions on the free boundary, construct a fixed boundary value problem.
- Choose one condition from R-H conditions to update the free boundary.
Decomposition

• The system is elliptic-hyperbolic composite system, which can be decomposed in its principal part.

• The elliptic part can be reduced to a second order equation
Potential theory and singular integral equation on “slip line”

• It is reduced to solve a second order equation with discontinuous coefficients on “slip line”.
• The consistency condition on “slip line” leads to a singular integral equation
• Giraud’s approach to reduce the singular integral equation to a Fredholm equation.
$$(FB)$$

$$\Rightarrow (FB)_t$$  Fix triple point

$$\Rightarrow (\widehat{FB})$$  Lagrange trans.

$$\Rightarrow (NL) + \ldots$$  Fix free boundary

$$\Rightarrow (L)$$  Linearization

$$\Rightarrow (L_1)$$  Decompose system

$$\Rightarrow (L_2)$$  Reduce to 2-order eq.

$$\Rightarrow (SIE)$$  potential theory

singular integral equation
The solvability of the singular integral equation implies the solvability of

--- Boundary value problem of elliptic equation of order 2

--- All intermediate boundary value problems
  (Linear system, Nonlinear system, Free boundary value problems …)

--- Original physical problem
Conclusion:

Global existence of stationary Mach reflection.

Stability with respect to the upstream data and the downstream pressure:

\[ D(\tilde{U}, U) \leq C(D(\tilde{U}_I, U_0^0) + D(\tilde{p} |_L, p_0^0 |_L)) \]
Remark

“The global wave pattern” and “The height of Mach stem” strongly depends on the downstream condition
Open questions for mathematicians

• Confirmation of E-H type Mach configuration
• Choose a correct transition criterion
• Prove or disprove other irregular configuration
• Existence and stability of global wave pattern involving Mach configuration
Thank you