Energy

$$\partial_t u_j + \partial_j \left( u_i u_j \right) + \partial_j \rho = 0$$

$$\rho \partial_t u_j + u_j \left( \theta_i u_i \right) v_j + v_i \left( \theta_i v_i \right) v_j +$$

$$+ u_j \partial_j \rho = 0$$

$$\partial_t \left( \frac{|u|^2}{2} \right) + \nabla \cdot \left( \frac{|u|^2}{2} + \rho \right) = 0$$

$$\partial_t \left( \frac{|u|^2}{2} + \text{div} \left[ \left( \frac{|u|^2}{2} + \rho \right) u \right] \right) = 0$$

$$\frac{d}{dt} \int_{\mathbb{R}^n} \frac{|u|^2}{2} (x,t) \, dx = 0$$

Dissipating means

$$\frac{d}{dt} \int_{\mathbb{R}^n} \frac{|u|^2}{2} (x,t) \, dx \leq 0$$

and \text{SOMEBWHERE}
JOINT WORK WITH
LÁSZLÓ SZEKELYHIDI JR.
(UNIV. BONN)

\[ p \text{-system} \]

isentropic gas dynamics in
Eulerian coordinates n space dimensions.

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho \mathbf{u}) &= 0 \\
\partial_t (\rho \mathbf{u}) + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla [P(e)] &= 0 \\
\rho(0, \cdot) &= \rho^0 \\
\mathbf{u}(0, \cdot) &= \mathbf{u}^0
\end{align*}
\]

\( (\rho^0 > 0 \text{; typical} \) \)

\( \rho(P) = k \rho^\gamma \) \)

Entropy solutions (admissible solutions) := distributional solutions satisfying the energy (entropy) inequality
Internal energy $\varepsilon : \mathbb{R}^+ \to \mathbb{R}$

$p(r) = r^2 \varepsilon'(r)$

Entropy inequality

$\partial_t \left[ (p \varepsilon(p) + p \frac{|v|^2}{2} \right] + \text{div} \left[ (p \varepsilon(p) + p \frac{|v|^2}{2} + p(p)) u \right] \leq 0$

(which indeed should be understood taking into account the initial condition:

$\int_{\mathbb{R}^n \times \mathbb{R}^+} \left[ (p \varepsilon(p) + p \frac{|v|^2}{2} \right] \partial_t \psi$

$+ (p \varepsilon(p) + p \frac{|v|^2}{2} + p(p)) u \cdot \nabla \psi$

$+ \int_{\mathbb{R}^n} \left( p^0 \varepsilon(p^0) + p^0 \frac{|v^0|^2}{2} \right) \psi(\cdot, 0) \geq 0$

$\forall \psi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^n), \psi \geq 0.$
THEOREM (2008)

Let $n \geq 2$. Then, for any given function $p$, there exist bounded initial data $(p^0, u^0)$ with $p^0 \geq c > 0$ for which there are infinitely many bounded admissible solutions $(p, u)$ with $p \geq c > 0$.

Suggested by Elling.

Remark

The solutions constructed satisfy the energy equality. Therefore:

COROLLARY

The same result holds for the full Euler system.
STARTING POINT:

Incompressible Euler

\[ \begin{cases} \partial_t u + \text{div}(u \otimes u) + \nabla p = 0 \\ \text{div} u = 0 \end{cases} \]

THEOREM (Scheffer 1993) $n=2$

There exists a weak solution of (IE) which is compactly supported in space and time and non-trivial.

Shnirelman 1997

Different proof.
Remark
The solutions of Schaffer and Shnirelman do not belong to the energy space.

**THEOREM (O-S 2007)**
There exist nontrivial bounded weak solutions of (1E) that are compactly supported in space and time.

- Short, elementary proof
- Surprising application of a (by now) well-known technique in differential inclusions

→ connections with the C' isometric embedding of Riem. manifolds
THEOREM (Shnirelman 2000)

\( n = 3 \) \( \Rightarrow \) There are weak solutions of (1E) that dissipate energy.

**Important:**

Connection to the Kolmogorov Theory of Turbulence

CONJECTURE (Onsage 1949)

\( n = 3 \) \( \rightarrow \) \( u \in C^0,\alpha \) solution of (1E)

- \( \alpha \geq \frac{1}{3} \) \( \Rightarrow \) Conservation of energy
- \( \alpha < \frac{1}{3} \) \( \Rightarrow \) Solution dissipating energy

Constantin - E - Titi 1994

\( \alpha \geq \frac{1}{3} \) \( \Rightarrow \) Energy Conservation
Smielma's Theorem is a trivial corollary of our construction (in any $n \geq 2$ and with bounded solutions).

Remark: In fact the solutions might display a more surprising behavior, like the instantaneous loss of energy.
Q: Is there a condition that might restore uniqueness of (1E)?

- Global energy inequality
  ( Leray type
    see Navier-Stokes )

- Strong global energy inequality
  ( i.e. strong $L^2$ continuity )

- Local energy inequality *
  ( i.e. $\exists t ^* \frac{|u|^2}{2} + \text{div} \left( \left( \frac{|u|^2}{2} + \rho \right) u \right) \leq 0$ )

- Equalities and all the possible cases of the requirements.

* proposed in the literature by Danchin-Robert.
THEOREM (D.S. 2008)

None of the previous requirements restores uniqueness for (1E).

Remark

In fact, the local energy inequality is an "entropy condition." It can be derived by a formal vanishing viscosity approximation.

NAVIER STOKES $\rightarrow$ EULER

"Formal, because it is not known whether weak (viscous) solutions of NS do satisfy it."
Remarks

The THEOREM for compressible Euler is an easy corollary of the construction for incompressible Euler.

IMPORTANT

The behaviour of the solutions is quite wild. Classical arguments do not apply.
FIRST PAPER:

Differential inclusion → Euler
Convex integration

Clean approach
Many aspects of Euler fit nicely into this framework

SECOND PAPER:

Bridging together some tools of DI with typical issues in evolutionary PDE.

Some new tools for DI also.
MAIN IDEAS IN THE FIRST PAPER:

- Refamulatation of Euler
  (in Tartar's Spirit)

- Tartar's wave analysis

- Cauer integration
  Baire category arguments

Müller - Šverak in CI

Cellina
Bressan
De Cecco - Marcellini
Kirschhein
Step 1

\[ \begin{align*}
\frac{\partial u}{\partial t} + \text{div } u + \nabla q &= 0 \\
\text{div } u &= 0
\end{align*} \]  \quad (LPDE)

\[ u = u \otimes u - \frac{1}{n} |u|^2 \text{Id} \]  \quad (AC)

\[ q = p + \frac{1}{n} |u|^2 \]  \quad \text{fake constraint}

\[ \begin{align*}
\Omega &= (x, t) \\
U &= \left( \begin{array}{c}
u \times v \\
r \times r \\
r \times 0
\end{array} \right)
\end{align*} \]

\[ \text{div}_T U = 0 \]  \quad (LPDE)

\[ U = \left( \begin{array}{c}
u \otimes v - \frac{1}{n} |v|^2 \text{Id} \\
r \times r \\
r \times 0
\end{array} \right) \]  \quad (AC)
Step 2 Plane wave analysis

Wave case $\Lambda$ :=

State $\Lambda$ (in this case $\in \mathbb{R}^{(n+1)\times(n+1)}$)

s.t. $\exists \xi$ (direction of oscillation)

for which

$U(\mathbf{z}) = A h(\mathbf{z} \cdot \xi)$

is a solution $\forall h : \mathbb{R} \rightarrow \mathbb{R}$

Remark

$\Lambda$ is very large in the case at hand.
Step 3

Building solutions adding waves

TOY EXAMPLE

$|\Delta u| = 1$ \quad \rightarrow \quad \nabla \times u = 0 \quad (LAPDE)

+ \quad |u| = 1 \quad (AC)

\[ u \mid_{\partial \Omega} = 0 \]
$|\nabla u_0| = 1$

But $u_0|_{\partial\Omega} \neq 0$
\[ \hat{u}_0 = (1 - \epsilon) u_0 \psi \]

\[ \nabla \hat{u}_0 = \left( (1 - \epsilon) \psi \nabla \psi \right) \]

Almost \( l \), say in \( L^2 \)

\[ + \frac{u_0}{\epsilon} \nabla \psi (1 - \epsilon) \]

Almost bounded

Small if many oscillations!

Uniformly small \(( < \frac{\epsilon}{3} )\)

\[ |\nabla \hat{u}_0| < 1 \]
Cut off the wave: \( \tilde{u}_0 + w \tilde{\phi} \)

\( \tilde{y} \) supported in the circle

\[ \Delta u = \text{constant} \]

Add a \( \Delta w \) "wave" to get \( \Delta \tilde{u}_0 \) close to 1.1 = 1
Carrying argument:
\( \tilde{u}_0 \to \tilde{u}, \) with

- \( |\nabla \tilde{u}| < 1 \)
- \( \| \nabla \tilde{u} \|_{L^2} \) much smaller

Then \( \| \nabla \tilde{u}_0 \|_{L^2} \)
Step 4: Iteration

If $\nabla \bar{u}: \rightarrow \nabla u$

Strongly

Then $|\nabla u| = 1$

Rem.: The strong convergence is not at all obvious.

Convex integration (Baire's) provides it.

Main idea in C.I.: separation of scales.
Remark

In our formulation (Euler)

. There are many waves

. it is difficult to find suitable potentials

. apparently time does not play a special role (but in fact it does and it is possible to implement
  . entropy inequalities
  . strong continuity in time)
De Bellis, Szekeleyidi

- The Euler equations as a differential inclusion
- Admissibility criteria for weak solutions of the Euler equations.