Recent results on 2D Surface Quasi-Geostrophic Equation

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2D Surface Dissipative Quasi-Geostrophic Equation (SQGE)

Appears in the studies of strongly rotating fluids (as a boundary condition on the planar boundary).

$$\theta_t = u \cdot \nabla\theta - (-\Delta)^\alpha \theta, \;\; \theta(x,0) = \theta_0(x),$$

θ scalar, real valued. Here

$$\widehat{u}(\xi) = (-R_2\widehat{\theta}(\xi), R_1\widehat{\theta}(\xi)) = \left(\frac{i\xi_2}{|\xi|}\widehat{\theta}(\xi), \frac{-i\xi_1}{|\xi|}\widehat{\theta}(\xi)\right).$$

In this talk, I consider the periodic setting (T^2).

Introduced by Constantin, Majda and Tabak (1994). Nonlinearity exhibits many properties of the 3D Euler equation, but is simpler. In the conservative case, equation for $\nabla^\perp\theta$ is very similar to 3D Euler equation for vorticity. Finite time blow up?

Studied recently in particular by Caffarelli, Constantin, A. Cordoba, D. Cordoba, Dong, Fefferman, T. Hou, Ju, Li, Majda, Resnick, Rodrigo, Tabak, Stefanov, Vasseur, Wu, Yu.

Perhaps the simplest equation of fluid dynamics for which existence of global regular solution is not known.

The next step towards simplification is 1D dissipative Burgers equation.

$$\theta_t = \theta\theta_x - (-\Delta)^\alpha\theta, \;\; \theta(x,0) = \theta_0(x).$$

Plan of the talk:

1. Background results and history.

2. The critical dissipation case: global regularity by the moduli of continuity method.

3. Other applications: Burgers equation.

4. Growth of high order Sobolev norms for the conservative SQGE.

SQGE: Basic properties and some known results.

The energy $\int_{T^2} \theta^2 \, dx$ is non-increasing.

Resnick, Cordoba&Cordoba showed

$$\int |\theta|^{p-2}\theta(-\Delta)^{\alpha}\theta \, dx \geq 0, 1 < p < \infty.$$

This yields a maximum principle: $\|\theta\|_{L^\infty}$ is non-increasing. It makes the value $\alpha = 1/2$ critical.

If $\alpha > 1/2$, for any $\theta_0 \in H^s$, $s > 2 - 2\alpha$, $s \geq 0$ there exists global smooth (for any $t > 0$) solution. (Resnick, Ju, Wu)

If $\alpha < 1/2$, only local existence is known.
For the Burgers equation, blow up can happen if $\alpha < 1/2$ (Kiselev, Nazarov, Shterenberg; Alibaud, Droniou and Vovelle - whole line case).

The critical case $\alpha = 1/2$.

Has been studied especially actively since this case has physical significance.

Constantin, D. Cordoba, Wu: Global smooth (analytic in x for any $t > 0$) solution exists if $\theta_0 \in H^2$, $\|\theta_0\|_{L^\infty}$ is small. Other results due to A. Cordoba and D. Cordoba, Dong, Ju, Li, Stefanov.

Theorem 1 (KNV) *Assume the initial data θ_0 is periodic and $\theta_0 \in H^1$. Then the critical quasi-geostrophic equation has a unique global solution which is real analytic in x for any $t > 0$.*

Caffarelli-Vasseur: a similar result by a completely different method.

Main idea: a nonlocal maximum principle.

Let us call $\omega(x)$ a modulus of continuity if $\omega(x) : [0,\infty) \mapsto [0,\infty)$ is increasing, continuous, concave, and $\omega(0) = 0$. We say f has ω if $|f(x) - f(y)| \leq \omega(|x-y|)$.

We will construct a family of unbounded moduli of continuity $\omega_A(\xi) = \omega(A\xi)$, $A > 0$, which will be preserved by the evolution: if the initial data θ_0 has ω_A, then so does $\theta(x,t)$ for any $t > 0$.

We will have $\omega'(0) = 1$, thus giving us global control of $\|\nabla\theta(x,t)\|_{L^\infty}$.

It is sufficient to show preservation of ω, ω_A follows by scaling.

Given control over $\|\nabla\theta\|_{L^\infty}$ and local existence of smooth solution, standard techniques allow to show uniform in time estimates for the Sobolev norms of arbitrary order. For example, one can derive a differential inequality

$$\partial_t \|\theta\|_{H^s} \leq C\|\nabla\theta\|_{L^\infty}^{a(s)} \|\theta\|_{H^{s+1/2}}^{3-a(s)} - \|\theta\|_{H^{s+1/2}}^2$$

with $a(s) > 1$ for all s large enough.

Properties of $\omega(\xi)$.

(i) Increasing, concave.

(ii) Near zero, $\omega(\xi) = \xi - \xi^{3/2}$, so $\omega'(0) = 1$ and $\omega''(0) = -\infty$.

(iii) As $\xi \to \infty$, $\omega(\xi) \sim c \log \log \xi$.

Corollary. *The following estimate holds:*

$$\|\nabla\theta(x,t)\|_\infty \le C\|\nabla\theta_0\|_\infty \exp\exp(C\|\theta_0\|_\infty).$$

Reason: we just need to find A so that $\theta_0(x)$ has the modulus of continuity ω_A. Then $|\nabla\theta(x,t)|$ is bounded by A for all times.

Breakdown scenario.

How can solution lose the moduli of continuity ω?

Claim: *If θ_0 has ω and it is lost, there must be a time t_0 and two distinct points x, y where $\theta(x, t_0) - \theta(y, t_0) = \omega(|x - y|)$, θ has ω for $t \leq t_0$ and loses it for $t > t_0$.*

Reason: The only alternative to the Claim is that $|\nabla\theta(x, t_0)| = \omega'(0)$ at some x (instead of two distinct points). But since $\omega''(0) = -\infty$, this would imply that ω has already been violated at time t_0.

Fix x, y, t_0 as in Claim with $|x-y| \equiv \xi$, will show that

$$\partial_t(\theta(x,t) - \theta(y,t))|_{t=t_0} < 0!$$

This contradicts the assumption that solution has modulus of continuity ω up to t_0.

Have to control the flow term and the dissipation term contributions:

$$\frac{\partial}{\partial t}(\theta(x,t) - \theta(y,t)) =$$

$$(u \cdot \nabla\theta)(x) - (u \cdot \nabla\theta)(y) - (-\Delta)^{1/2}\theta(x) + (-\Delta)^{1/2}\theta(y).$$

The flow term.

Lemma. *If the function θ has modulus of continuity ω, then $u = (-R_2\theta, R_1\theta)$ has modulus of continuity*

$$\Omega(\xi) = B\left(\int_0^\xi \frac{\omega(\eta)}{\eta}\, d\eta + \xi \int_\xi^\infty \frac{\omega(\eta)}{\eta^2}\, d\eta\right) \qquad (1)$$

with some universal constant $B > 0$.

Note

$$(u \cdot \nabla\theta)(x) = \frac{d}{dh}\theta(x + hu(x))\Big|_{h=0}.$$

Now

$$\theta(x + hu(x)) - \theta(y + hu(y)) \leq$$

$$\omega(|x-y| + h|u(x) - u(y)|) \leq \omega(\xi + h\Omega(\xi)).$$

Since $\theta(x) - \theta(y) = \omega(\xi)$, we conclude that

$$(u \cdot \nabla\theta)(x) - (u \cdot \nabla\theta)(y) \leq \Omega(\xi)\omega'(\xi).$$

The dissipation term.

Observe that

$$-(-\Delta)^{1/2}\theta(x) = \frac{d}{dh}\mathcal{P}_h * \theta\Big|_{h=0},$$

where $\mathcal{P}_h$ is the 2D planar Poisson kernel, and θ is periodization to R^2.

After a computation, we get that

$$-(-\Delta)^{1/2}\theta(x) + (-\Delta)^{1/2}\theta(y)$$

is bounded from above by

$$\frac{1}{\pi}\int_0^{\frac{\xi}{2}} \frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^2}\,d\eta$$
$$+\frac{1}{\pi}\int_{\frac{\xi}{2}}^{\infty} \frac{\omega(2\eta+\xi)-\omega(2\eta-\xi)-2\omega(\xi)}{\eta^2}\,d\eta\,.$$

Both terms are negative due to concavity.

To conclude, have to check the inequality

$$B\left[\int_0^{\xi}\frac{\omega(\eta)}{\eta}\,d\eta+\xi\int_{\xi}^{\infty}\frac{\omega(\eta)}{\eta^2}\,d\eta\right]\omega'(\xi)+$$
$$\frac{1}{\pi}\int_0^{\frac{\xi}{2}}\frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^2}\,d\eta$$
$$+\frac{1}{\pi}\int_{\frac{\xi}{2}}^{\infty}\frac{\omega(2\eta+\xi)-\omega(2\eta-\xi)-2\omega(\xi)}{\eta^2}\,d\eta<0$$

Explicit form of ω : Set

$$\omega(\xi)=\xi-\xi^{3/2},\ \text{ when }\ 0\le\xi\le\delta$$

and

$$\omega'(\xi)=\frac{\gamma}{\xi(4+\log(\xi/\delta))}\ \text{ when } \xi>\delta.$$

Here $0<\gamma<\delta$ are small, can be chosen to ensure the inequality is true. Moreover the contribution of the dissipative term is $\le -2\Omega(\xi)\omega'(\xi)$.

Consider, for example, $\xi \geq \delta$ case. Then

$$\int_0^\xi \frac{\omega(\eta)}{\eta} \, d\eta \leq \delta + \omega(\xi) \log \frac{\xi}{\delta} \leq \omega(\xi) \left(2 + \log \frac{\xi}{\delta} \right)$$

if δ is small enough. Also

$$\int_\xi^\infty \frac{\omega(\eta)}{\eta^2} \, d\eta = \frac{\omega(\xi)}{\xi} + \gamma \int_\xi^\infty \frac{d\eta}{\eta^2 (4 + \log(\eta/\delta))} \leq \frac{2\omega(\xi)}{\xi}$$

if γ, δ are small enough. Thus the positive term does not exceed

$$B\omega(\xi) \left(4 + \log \frac{\xi}{\delta} \right) \omega'(\xi) = B\gamma \frac{\omega(\xi)}{\xi}.$$

Now if $\xi \geq \delta$ and γ is small, then $\omega(2\xi) \leq \frac{3}{2}\omega(\xi)$. Due to concavity,

$$\frac{1}{\pi}\int_{\xi/2}^{\infty} \frac{\omega(2\eta+\xi) - \omega(2\eta-\xi) - 2\omega(\xi)}{\eta^2}\, d\eta \leq$$

$$-\frac{1}{2\pi}\int_{\xi/2}^{\infty} \frac{\omega(\xi)}{\eta^2}\, d\eta = -\frac{\omega(\xi)}{\pi\xi}.$$

We have

$$-\frac{\omega(\xi)}{\pi\xi} \;\; vs. \;\; B\gamma\frac{\omega(\xi)}{\xi},$$

just take γ small enough!

Analyticity.

Smoothness of solution follows from control of $\|\nabla\theta\|_\infty$ and standard estimates on Sobolev norms. These estimates yield uniform in time control of the Sobolev norms of the solution.
Set $\xi_k(t) = \widehat{\theta}(k,t)\exp(\frac{1}{2}|k|t)$. Define

$$Y(t) = \sum_k |k|^4 |\xi_k(t)|^2.$$

One can show that

$$\frac{dY}{dt} \leq C_1 Y^{3/2} + (C_2 Y^{1/2} t - 1) \sum_k |k|^5 |\xi_k|^2.$$

Thus we have control of Y on a small time interval $[0,\tau]$, which implies analyticity. Uniform in time control of $\|\theta\|_{H^2}$ allows to continue with a fixed time step.

Other applications:

Theorem 2 (KNS). *Let* $\alpha \geq 1/2$, $\theta_0 \in L^p$, $1 < p < \infty$. *Then there exists a global solution real analytic in* x *for* $t > 0$ *such that*

$$\|\theta(x,t) - \theta_0(x)\|_p \to 0$$

as $t \to 0$.

Uniqueness is not known!

A related model: 2D Surface Critical Dissipative Dispersive QGE equation:

$$\theta_t = u \cdot \nabla\theta - (-\Delta)^{1/2}\theta + Fu_2,$$

Theorem 3 (KN). *For* $\alpha \leq 1/2$ *and* $\theta_0 \in H^1$, *the dispersive SQGE has unique global solution which is smooth for* $t > 0$.

Rough solutions.

1. Approximate θ_0 in L^p by smooth $\theta_{0,k}$, construct $\theta_k(x,t)$.

2. Get a-priori estimates:

a. Set $M_k(t) = \|\theta_k(x,t)\|_\infty$. Then uniformly in k

$$M_k'(t) \le c_1 M_k^{1+p/2} + c_2 M_k \;\Rightarrow\; t^{2/p} M_k(t) \le C$$

b. Using arguments similar to the proof of conservation of ω for regular data, one can show that $\theta_k(x,t)$ have modulus of continuity $\omega_{A(t)}$, with certain $A(t) > 0$, $A(t) \to \infty$ as $t \to 0$.

c. Passing to the limit gives a regular for $t > 0$ solution.

d. Uniqueness is not known.

Dispersive SQGE.

$$\theta_t = u \cdot \nabla\theta - (-\Delta)^\alpha \theta + F u_2.$$

1. L^2 norm is conserved.

2. L^∞ norm is no longer conserved, but is still controlled:

$$\|\theta(x,t)\|_\infty \le C \left(\|\theta_0\|_\infty + \|\theta_0\|_2^2 (F \log F)^2 \right).$$

3. The equation does not have the scaling properties of the critical SQGE. We have to consider ω_A for all $A > 0$. In the modulus of continuity estimate there is an extra term: if

$$\theta(x,t) - \theta(y,t) = \omega_A(|x-y|), \;\; \xi \equiv |x-y|,$$

then

$$\partial_t(\theta(x) - \theta(y)) \le -\omega_A'(\xi)\Omega_A(\xi) - F\Omega_A(\xi).$$

Since $\|\theta(x,t)\|_\infty \le C$ is controlled, we just need to take A large enough so that $\omega_A'(\xi) > F$ for ξ where $\omega_A \le 2C$.

Thus moduli of continuity with large A are conserved.

The Dissipative Burgers equation:

$$\theta_t = \theta\theta_x - (-\Delta)^\alpha \theta, \;\; \theta(x,0) = \theta_0(x).$$

Theorem 4 (KNS). *Assume that* $\alpha \geq 1/2$, $\theta_0 \in H^{\frac{3}{2}-2\alpha}$. *Then Burgers equation has unique global solution which is real-analytic for* $t > 0$.
If $\alpha < 1/2$, *then there exist smooth initial data which lead to blow up in finite time (in* $H^{\frac{3}{2}-2\alpha}$ *norm).*

The first part is proved similar to the SQGE case (but is easier).

The second part is proved by a time-splitting argument.

What about possible singularity formation?

Constantin, Majda, Tabak (1994) - sharp growth of the gradient of solution (saddle point collapse scenario).

Cordoba (2000) - double exponential in time upper bound on the solution growth in this scenario.

Cordoba, Fefferman (2002) - ruled out some other scenarios.

Theorem 5 (KN) *Fix any sufficiently large s. Given any $A > 0$, there exists the initial data θ_0 such that $\|\theta_0\|_{H^s} < 1$ but*

$$\limsup_{t\to\infty} \|\theta(\cdot,t)\|_{H^s} \geq A.$$

Idea of proof:

Ideally, would like to find a Lyapunov functional capable of measuring H^s norm growth. Look at Fourier coefficients:

$$\partial_t \widehat{\theta}(k) = \frac{1}{2} \sum_{l+m=k} \langle l^\perp, m \rangle \left(\frac{1}{|l|} - \frac{1}{|m|} \right) \widehat{\theta}(l) \widehat{\theta}(m).$$

First try: linear?

$$F_1(\theta) = \sum_k a_k \widehat{\theta}(k).$$

Does not work - time derivative is a quadratic form, coefficients in front of the squares vanish.

Trilinear form? First step - ensure that the expression for time derivative has positive coefficients in front of the squares.

$$F_3(\theta) = \sum_{i+j+l=0} (|i|-|j|)(|j|-|l|)(|l|-|i|)\langle i^\perp, j\rangle \widehat{\theta}(i)\widehat{\theta}(j)\widehat{\theta}(l)$$

works for this. This is just the symmetrized form of

$$\sum_{i+j+l=0} |i|^2|j|\langle i^\perp, j\rangle \widehat{\theta}(i)\widehat{\theta}(j)\widehat{\theta}(l),$$

or

$$\int \langle \nabla^\perp(-\Delta\theta), \nabla(-\Delta)^{1/2}\theta\rangle\theta\, dx.$$

Time derivative of that is positive for "most" points in the phase space! (17000 random R^4 tries are positive, but the steepest descent method shows in general it is not).

Idea that works:

Find a stable flow that creates small scale structures on the imposed perturbation.

Scenario we work with: perturbed shear flow $\theta_s(x) = \cos x$, $u_s(x) = (0, -\sin x)$.

Define $e = (1,0)$ and $g = (0,2)$. The initial data is given by

$$\widehat{\theta}_0(e) = \widehat{\theta}_0(-e) = 1;$$

$$\widehat{\theta}_0(g) = \widehat{\theta}_0(-g) = \widehat{\theta}_0(e+g) = \widehat{\theta}_0(-e-g) = \tau,$$

τ is small. $\widehat{\theta}(k) = 0$ for all other k. Two conservation laws:

$$\sum_k |\widehat{\theta}(k,t)|^2 = const, \;\; \sum_k |k|^{-1} |\widehat{\theta}(k,t)|^2 = const.$$

Using these laws one can show

$$\sum_{k \neq \pm e} |\widehat{\theta}(k,t)|^2 \leq C\tau^2, \;\; \forall\, t > 0.$$

Can do next best thing to a Lyapunov functional. Define

$$J(\theta) = \sum_{k \in Z^2_+} \left(k_1 + \frac{1}{2}\right) \widehat{\theta}(k) \widehat{\theta}(k+e).$$

Observe $J(\theta_0) \sim \tau^2$.

Take time derivative, separate the terms that have $\widehat{\theta}(\pm e)$ and the terms that don't.

Can show

$$\frac{d}{dt} J(\theta) \geq c \sum_{k \in Z^2_+} |k|^{-3} |\widehat{\theta}(k,t)|^2 - C\tau^2 \sum_{k \in Z^2_+} |k|^2 |\widehat{\theta}(k,t)|.$$

Now we have several possibilities.

$$A. \quad \sum_{k \in Z_+^2} |k|^2 |\widehat{\theta}(k,t)| \geq \tau^{1/2}.$$

Then

$$\sum_{k \in Z_+^2} |k|^2 |\widehat{\theta}(k)| \leq$$

$$\left(\sum_{k \in Z_+^2} |\widehat{\theta}(k)|^2 \right)^{1/3} \left(\sum_{k \in Z_+^2} |k|^{21} |\widehat{\theta}(k)|^2 \right)^{1/6} \left(\sum_{k \in Z_+^2} |k|^{-3} \right)^{1/2} .$$

So

$$\sum_{k \in Z_+^2} |k|^{21} |\widehat{\theta}(k)|^2 \geq C\tau^{-1}$$

B. (A) never occurs but at some point

$$\sum_{k \in Z_+^2} |k|^{-3} |\widehat{\theta}(k,t)|^2$$

becomes comparable to $\tau^{5/2}$. Until this moment, $J(\theta)$ is increasing, and so

$$\sum_{k \in Z_+^2} |k| |\widehat{\theta}(k,t)|^2 \geq J(\theta(t)) \geq J(\theta_0) \geq c\tau^2.$$

But

$$\sum_{k \in Z_+^2} |k| |\widehat{\theta}(k,t)|^2 \leq \left(\sum_{k \in Z_+^2} |k|^{-3} |\theta(k)|^2 \right)^{5/6} \left(\sum_{k \in Z_+^2} |k|^{21} |\widehat{\theta}(k)|^2 \right)^{1/6}.$$

Thus $\|\theta(t)\|_{H^{11}} \geq C\tau^{-1/2}$.

C. Neither (A) nor (B) occur. Then

$$\frac{d}{dt}J(\theta) \geq c \sum_{k \in Z^2_+} |k|^{-3} |\widehat{\theta}(k,t)|^2 - C\tau^2 \sum_{k \in Z^2_+} |k|^2 |\widehat{\theta}(k,t)| \geq c'\tau^{5/2}.$$

Thus $J(\theta)$ grows linearly in time.

Summary:

Nonlocal maximum principle is a new tool natural for PDE with nonlocal terms. Allows to show existence of global regular solutions for the critical SQGE, Burgers and related equations.

Conservative SQGE can generate growth of high order Sobolev norms of the solution at least in the "weak turbulence" sense.

Big open question:

Regularity or blow up for the conservative SQGE?

Many smaller but still interesting questions (better estimates for the rate of growth of Sobolev norms, uniqueness of rough solutions, ...)