HYP 2008

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Hyperbolic Problems: Theory, Numerics, Applications

Center for Scientific Computation and Mathematical Modeling & Department of Mathematics
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Traffic flow on networks: conservation laws models

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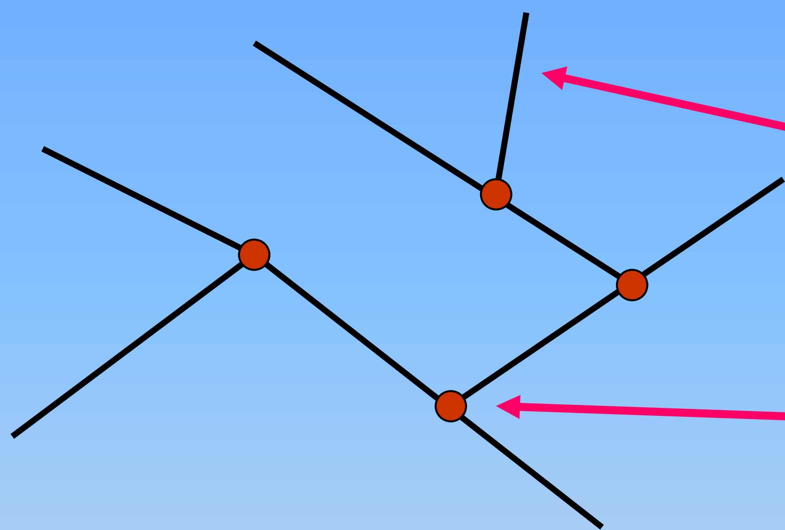
Conservation laws on networks



$u_t + f(u)_x = 0$

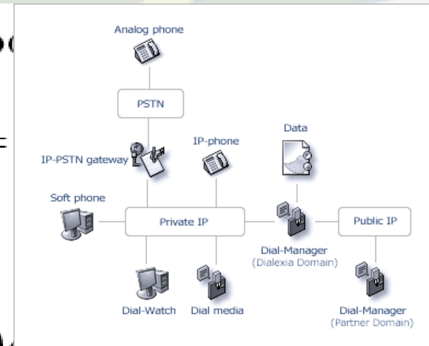
Dynamics at nodes?

1. The only conservation at nodes does not determine the dynamics
2. Additional rules should take into account distribution policies
3. Solutions give rise to boundary value problems on arcs

Car Traffic

Lighthill-Witham-Richards mo

$\rho_t + (\rho\, v(\rho))_x =$ ... sity of cars



Tlc and data networks

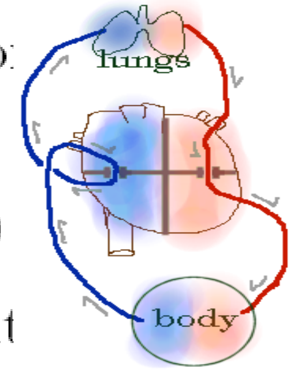
Aw-Rascle model:

Irrigation Channels

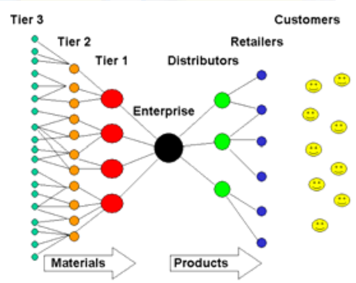
De Saint-Venant equatio

$$\begin{cases} H_t + (H\,v)_x = 0 \\ v_t + [\frac{1}{2} v^2 + g\,H]_x = 0 \end{cases}$$

H water height, v velocit



Blood circulation

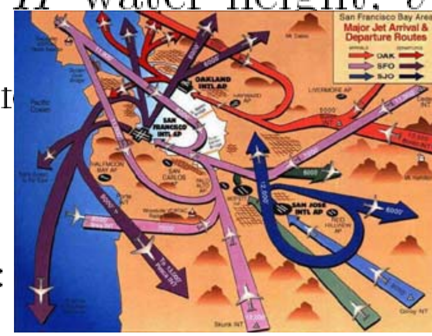


Supply chains

Armbruster-Degond-Ringhof

$\rho_t + \left(\min\{\mu(t,x), v\rho\}\right)_x = 0$ ... ssing rate, v processing velocity

Goettlich-Herty-Klar model: ... e q_j in front of arc j solves



Air traffic management

Isothermal Euler with friction:

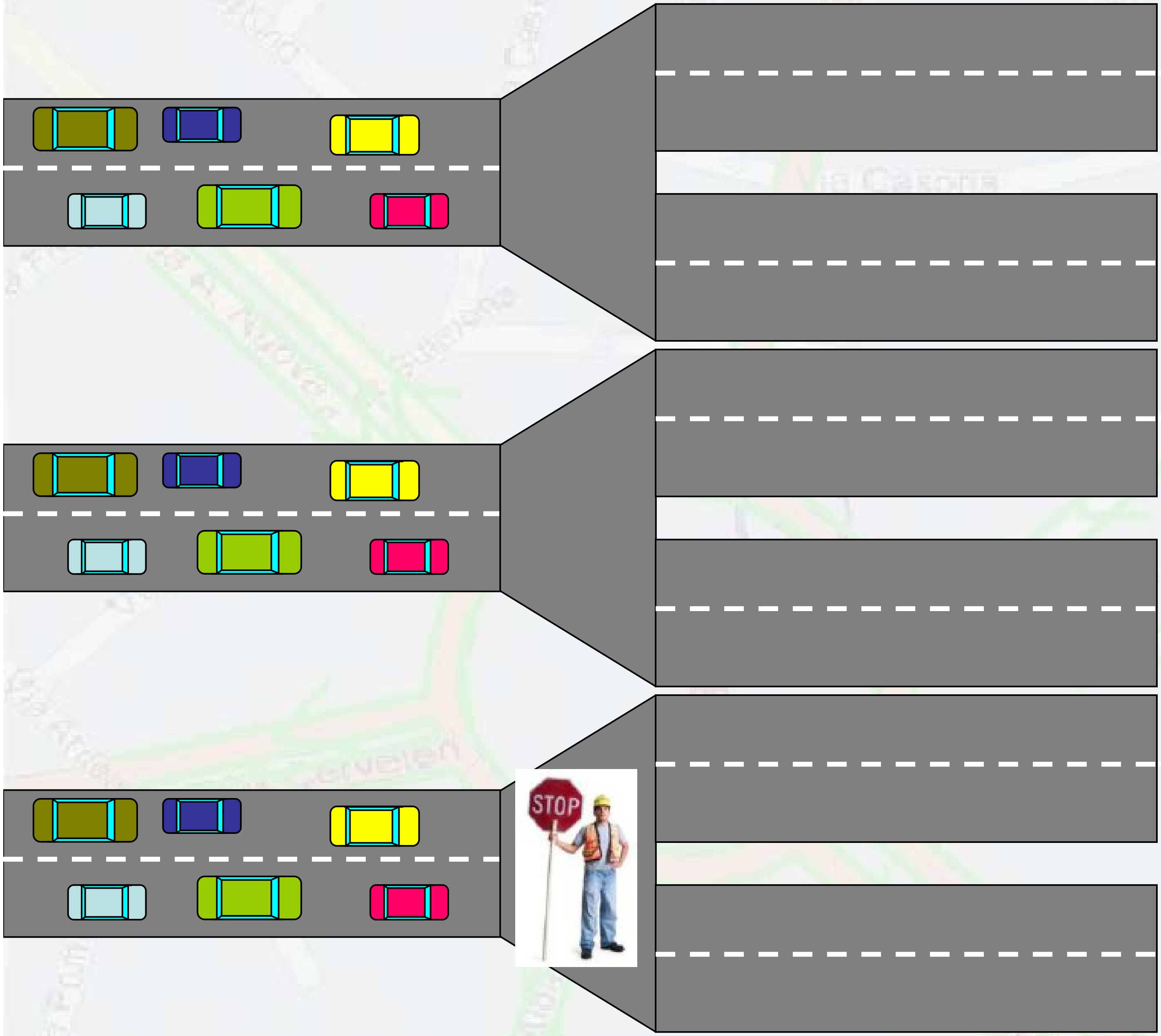
$$\begin{cases} \rho_t + (\rho\, u)_x = 0 \\ (\rho\, u)_t + (\rho\, u^2 + a^2 \rho \end{cases}$$

ρ density, u velocity

Gas pipelines

Social networks

Dynamics at junctions



Dynamics at junctions(2)

Traffic distribution matrix $A = (\alpha_{ji})$, $0<\alpha_{ji}<1$, $\Sigma_j\ \alpha_{ji} = 1$

Rule (A) : Out. Fluxes Vector = $A \cdot$ Inc. Fluxes Vector

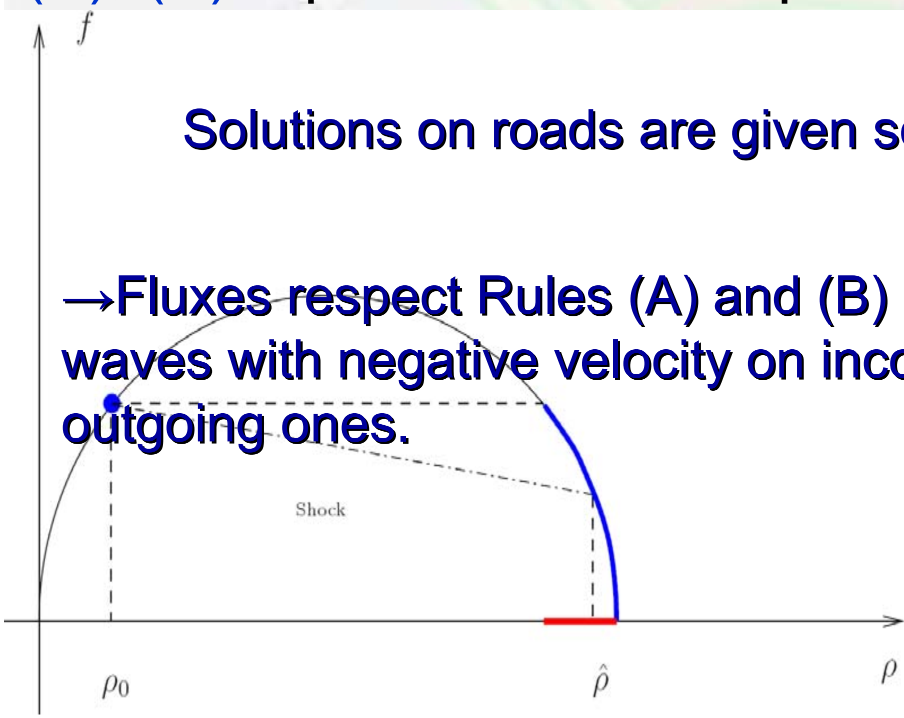
Rule (B) : Max $\|$Inc. Fluxes Vector$\|_1$

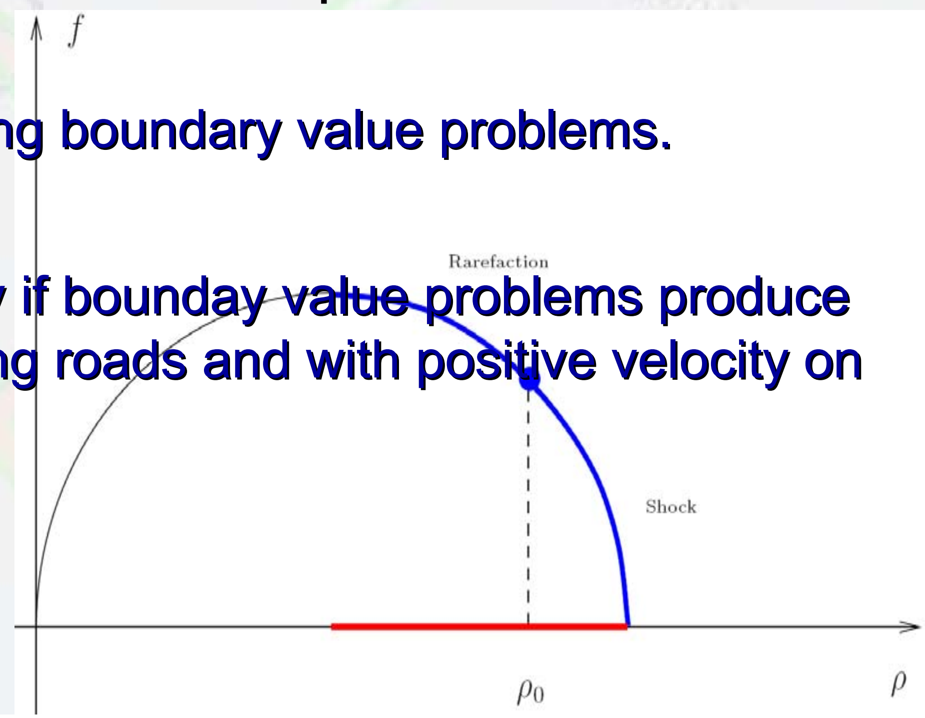
(A) implies conservation at the junction

(A), (B) equivalent to a LP problem and a unique solution to RPs

Solutions on roads are given solving boundary value problems.

→Fluxes respect Rules (A) and (B) only if bounday value problems produce waves with negative velocity on incoming roads and with positive velocity on outgoing ones.





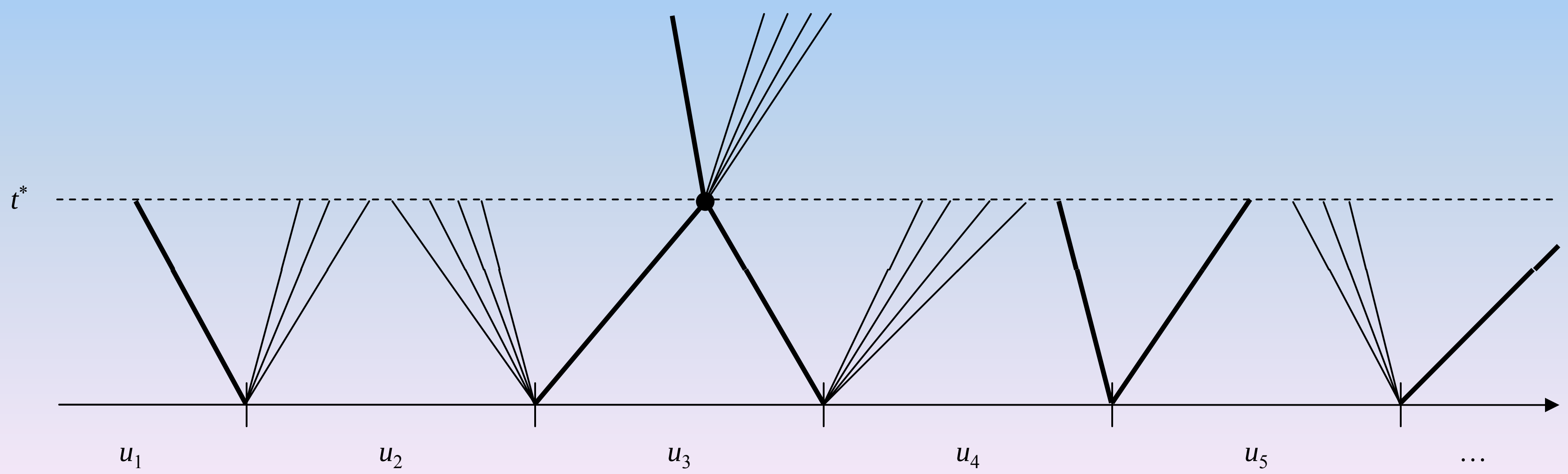
Wave Front Tracking

1. Approximate initial datum by a piecewise constant function

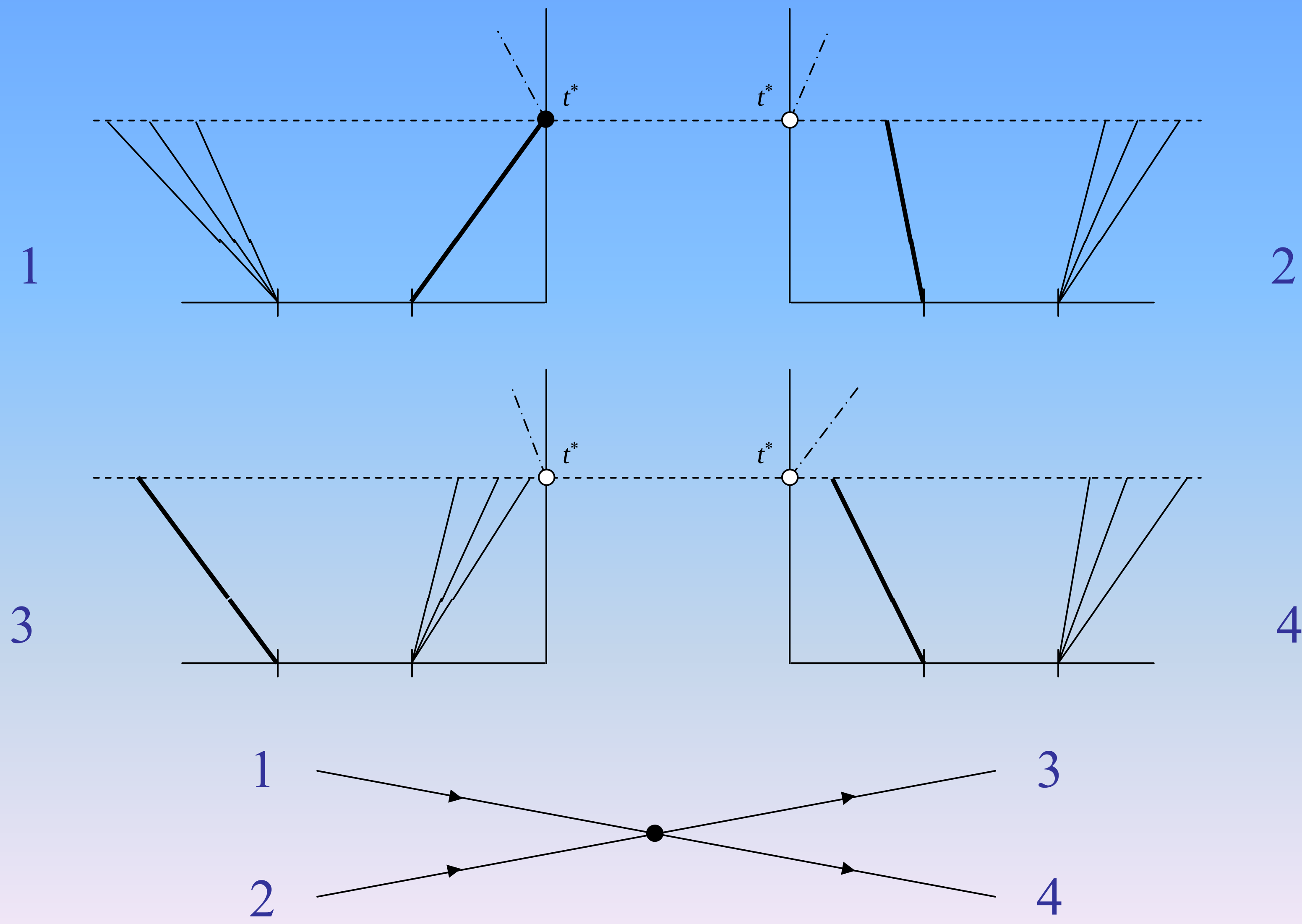
2. Solve RPs, replace rarefactions by rarefaction shocks fans: initially waves evolve independently of one another

3. At time t^* > 0 a first interaction between two of such discontinuities occurs (two shocks collide in this example)

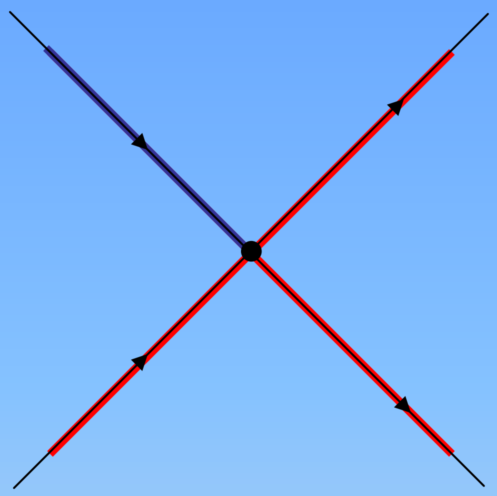
4. Then we solve a new Riemann problem and so on



Wave Front Tracking on networks

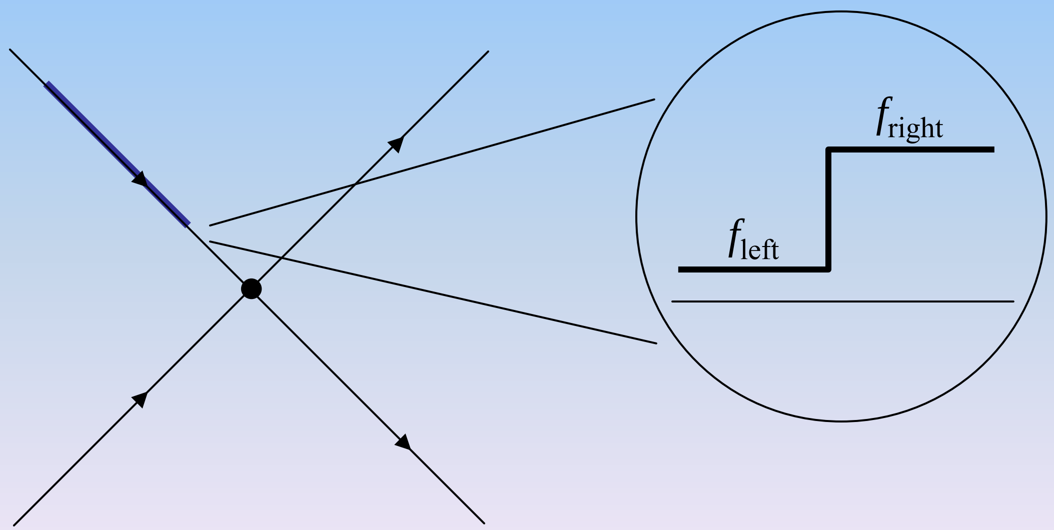


Existence of solutions



(*P*1) $\Delta TV(f) \leq C \min\{TV(f)^{-}, \Delta\Gamma\}$

where Γ is the incoming flux

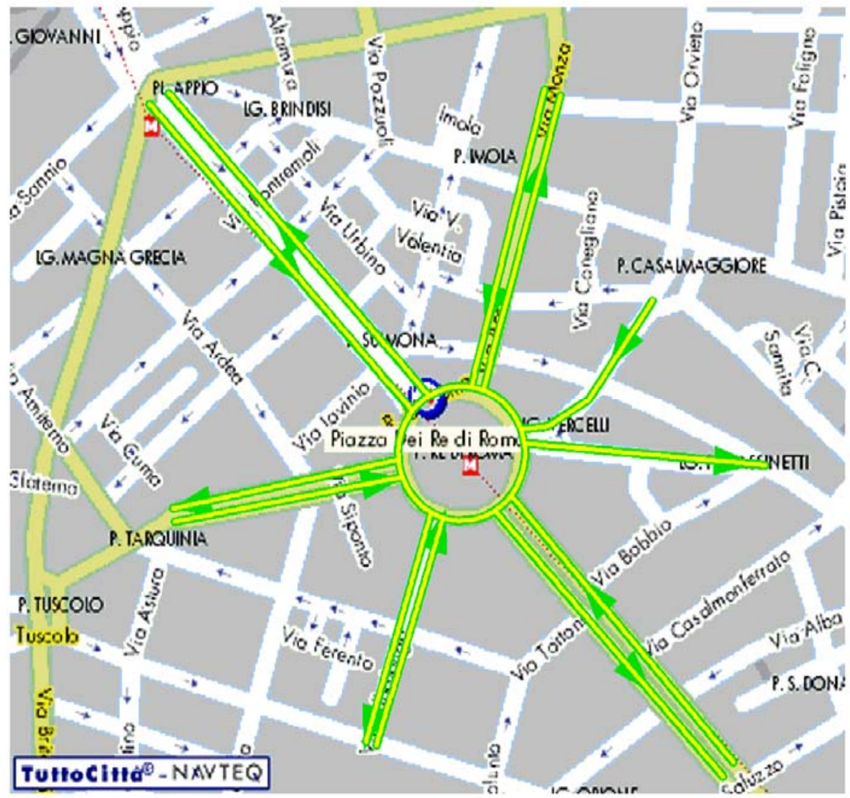


(P2) $\Delta\Gamma \leq 0$

Simulation of Re di Roma square

MOVIE

ZOOM

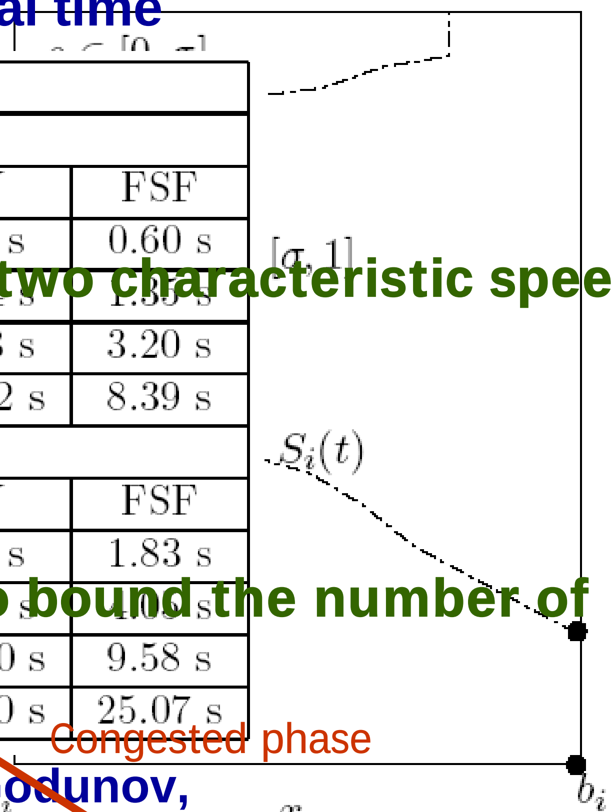


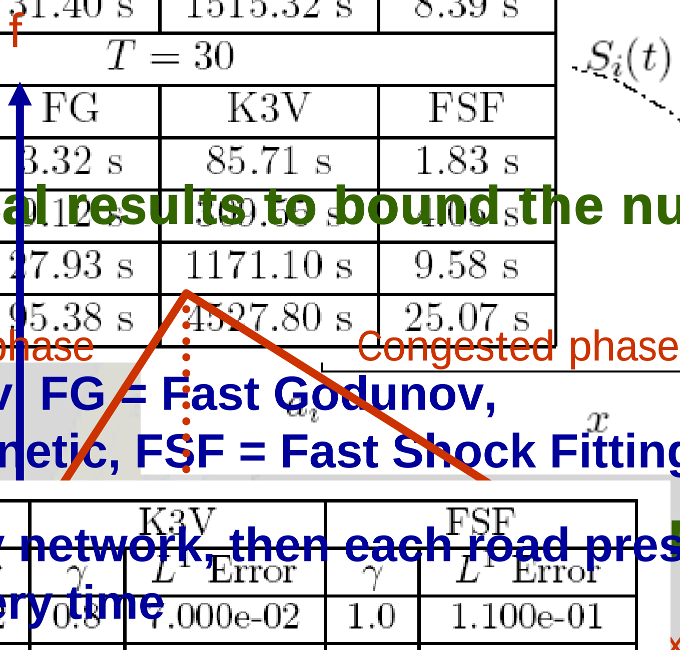
Numerics and FSF scheme

Network with 5000 roads parametrized by [0,1],
h space mesh size, T real time

CPU time				
	$T = 10$			
h	G	FG	K3V	FSF
0.2	1.78 s	1.12 s	29.37 s	0.60 s
0.1	5.68 s	3.05 s	104.74 s	1.35 s
0.05	19.83 s	9.30 s	394.03 s	3.20 s
0.025	73.86 s	31.40 s	1515.32 s	8.39 s
	$T = 30$			
h	G	FG	K3V	FSF
0.2	5.34 s	3.32 s	85.71 s	1.83 s
0.1	[illegible]	[illegible]	[illegible]	[illegible]
0.05	59.95 s	27.93 s	1171.10 s	9.58 s
0.025	223.03 s	95.38 s	4527.80 s	25.07 s

1. **Use simplified flux function with two characteristic speeds**
2. **Make use of theoretical results to bound the number of regime changes**





G = Godunov, FG = Fast Godunov,
K3V = 3-velocities Kinetic, FSF = Fast Shock Fitting

	FG		K3V		FSF	
h	γ	L^1 Error	γ	L^1 Error	γ	L^1 Error
0.2	3.9	5.000e-02	0.8	7.000e-02	1.0	1.100e-01
0.1	1.0	5.000e-03	1.4	4.000e-02	-	5.500e-02
0.05	1.0	2.500e-03	1.6	1.500e-02	-	0.000e+00
0.025	-1.3	6.250e-03	0.2	5.000e-03	-	0.000e+00

3. **Track ... characteristic) and**

Lemma: If we start from empty network, then each road presents at most one regime change for every time

Real data

Problems :

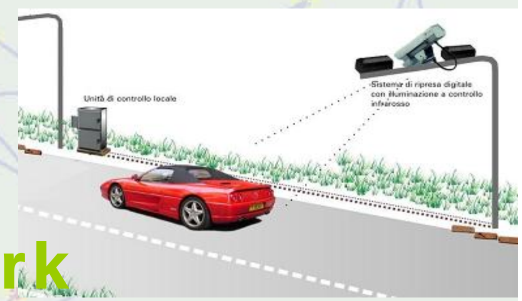
1. Data: measurements and elaboration
2. Dimensionality: big networks

Manual counting

Satellite data

Radars

Videocameras

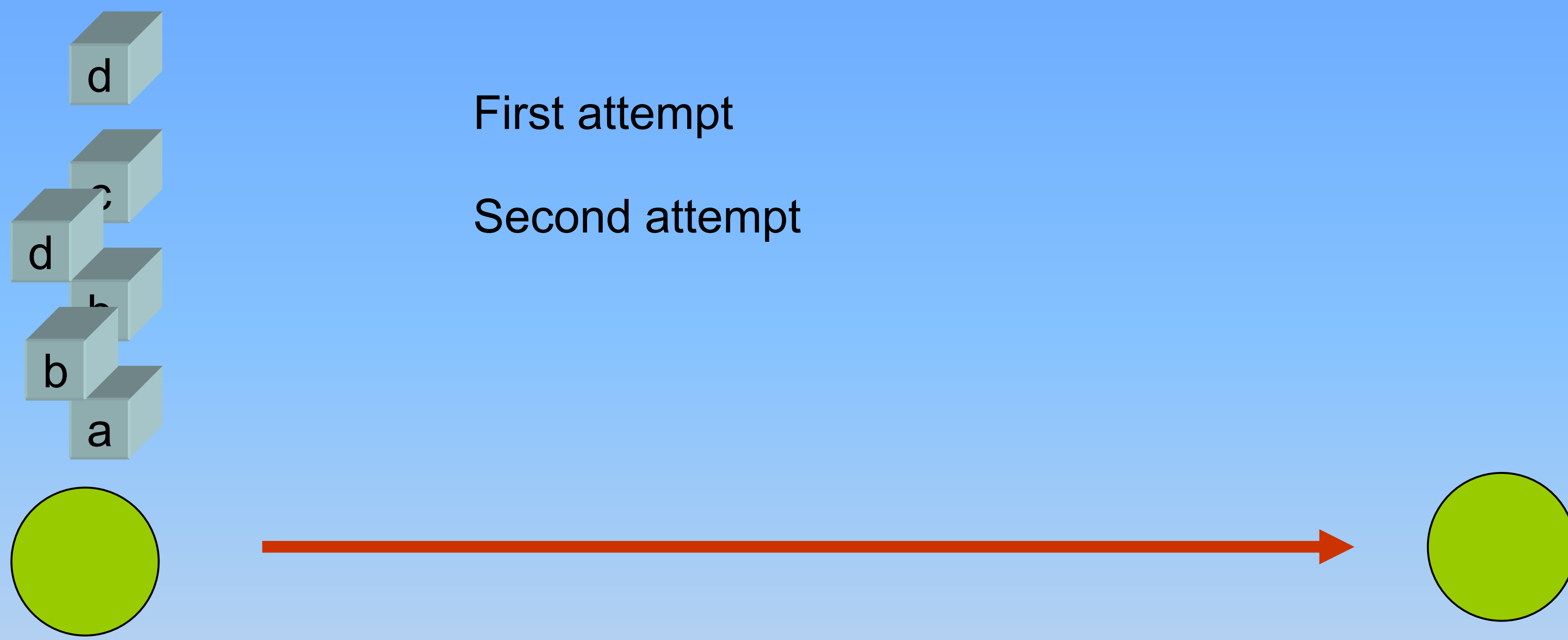


Plates reading

1500 arcs network

NETWORK of SALERNO

Model for data networks



There is a loss probability function $\mathcal{P} : [0, R_{max}] \to [0, 1]$ such that $(1 - \mathcal{P})(R)$ packets are sent and $\mathcal{P}(R)$ are lost.

In the nth attempt $(1 - \mathcal{P}(R))\mathcal{P}(R)^{n-1}$ packets are sent and $\mathcal{P}(R)^n$ are lost.

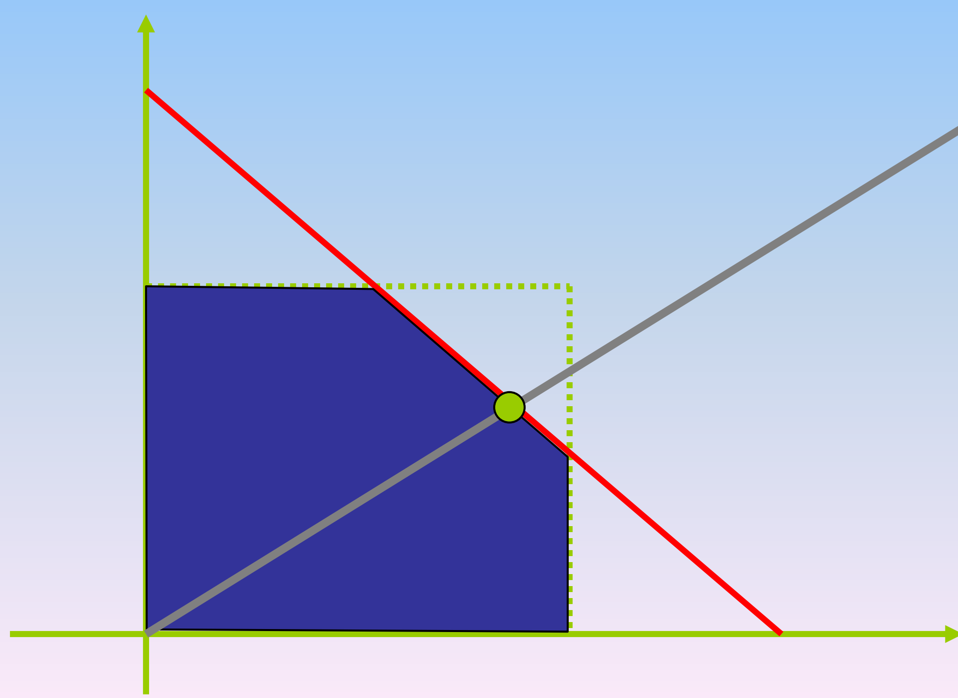
$$t_{av} = \bar{t} \sum_n n(1 - \mathcal{P})\mathcal{P}^{n-1} = \frac{\bar{t}}{(1 - \mathcal{P})} \to v_{av} = \bar{v}(1 - \mathcal{P})$$

Riemann solver for Tlc networks

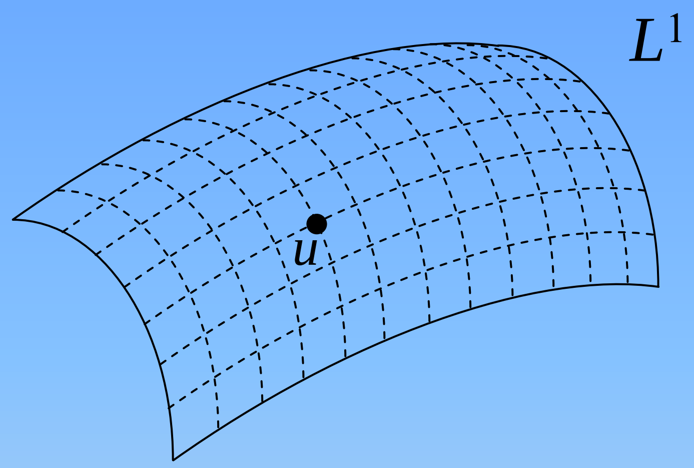
We essentially invert Rules (A) and (B), giving more importance to through flux than traffic distribution.

Define the maximal fluxes as before γ_i^{max} and γ_j^{max}.

The through flux is $\Gamma = \min\{\sum_i \gamma_i^{max} \ , \ \sum_j \gamma_j^{max}\}$.



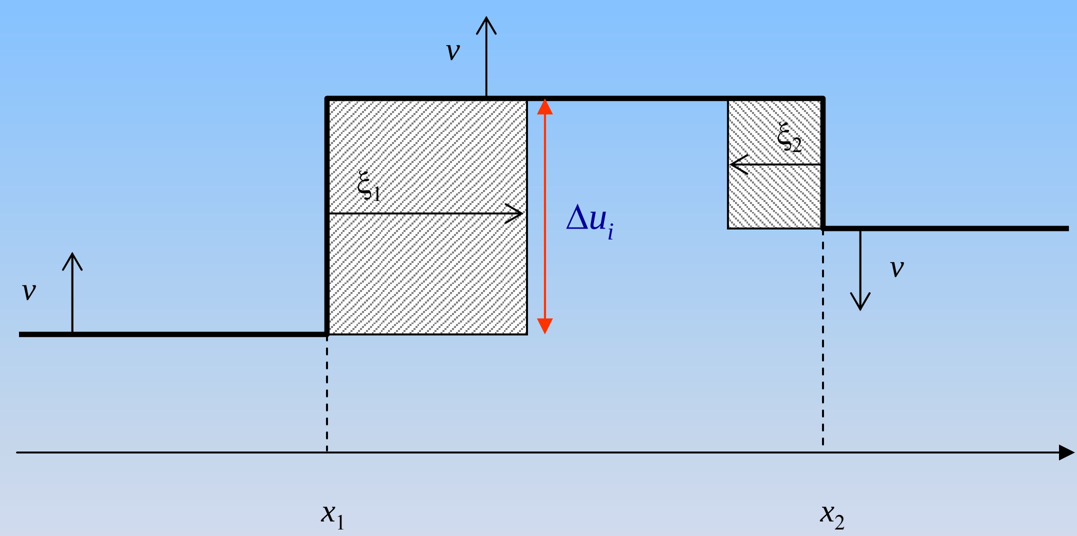
Finsler metric on L^1



Piecewise constant functions

$$u \in \mathrm{PC} \subset L^1$$

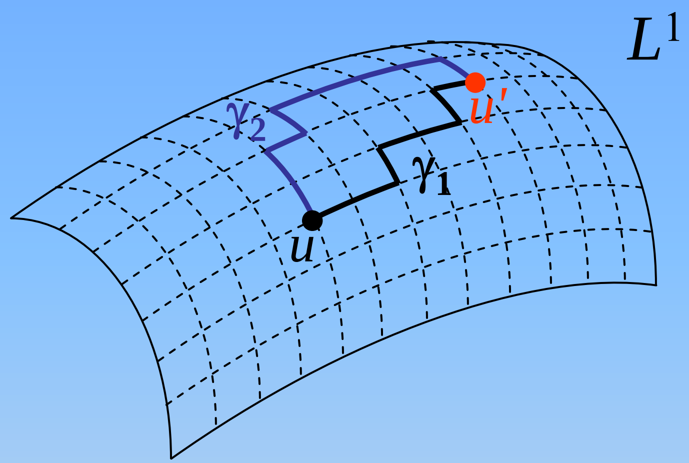
Perturbations:



$$\|(v, \xi)\| = \|v\|_{L^1} + \sum_i |\xi_i| \Delta u_i$$

Finsler metric on L^1 (2)

u, $u' \in \mathrm{PC}$



Family of piecewise smooth curves in PC connecting u and u':

$$\gamma : [0, 1] \to \mathrm{PC}$$

$$\gamma(0) = u,\ \gamma(1) = u'$$

Define the length of each of these curves as

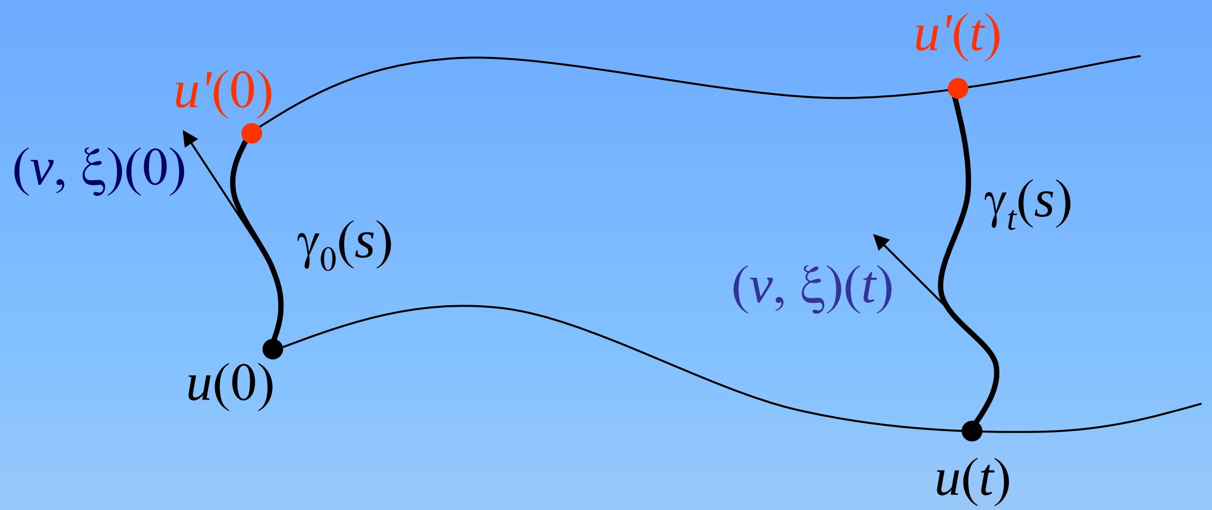
$$L(\gamma) = \int_0^1 \|(v, \xi)(s)\| \, ds$$

and the distance between u and u' (**Finsler metric**) as

$$d(u, u') = \inf_{\gamma : u \to u'} L(\gamma)$$

This metric is (compatible with) the usual L^1 metric, therefore it can be completed on the basis of the latter.

Lipschitz continuous dependence



Lemma: $\|(v,\xi)(t)\| \leq \|(v,\xi)(0)\|$

In view of this lemma one has:

$$d(u(t), u'(t)) = \inf_{\eta : u(t) \to u'(t)} L(\eta) \leq \inf_{\gamma_t : u(t) \to u'(t)} L(\gamma_t)$$

$$\overset{\text{Lemma}}{\leq} \inf_{\gamma_0 : u(0) \to u'(0)} L(\gamma_0) = d(u(0), u'(0))$$

Thank you for your attention!

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An applied mathematics Journal

- http://www.iac.rm.cnr.it/~piccoli/
- http://www.aimsciences.org/journals/NHM/index.htm
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Dynamics at junctions(2)

Conservation through the node:

$$\sum_{i\in\{incoming\ roads\}} f_i = \sum_{j\in\{outgoing\ roads\}} f_j \tag{1}$$

Fix a matrix $A = (\alpha_{ji})$, $0 \leq \alpha_{ji} \leq 1$ and $\sum_j \alpha_{ji} = 1$.

Rule (A) Incoming fluxes $\gamma_1, \ldots, \gamma_n$, outgoing fluxes $\gamma_{n+1}, \ldots, \gamma_{n+m}$:

$$(\gamma_{n+1}, \ldots, \gamma_{n+m}) = A \cdot (\gamma_1, \cdots, \gamma_n) \tag{2}$$

Rule (B) Find the point $(\bar{\gamma}_1, \ldots, \bar{\gamma}_n)$ which maximizes the function

$$E(\gamma_1, \ldots, \gamma_n) = \gamma_1 + \cdots + \gamma_n, \tag{3}$$

and define $(\bar{\gamma}_{n+1}, \ldots, \bar{\gamma}_{n+m}) := A \cdot (\bar{\gamma}_1, \ldots, \bar{\gamma}_n)$.

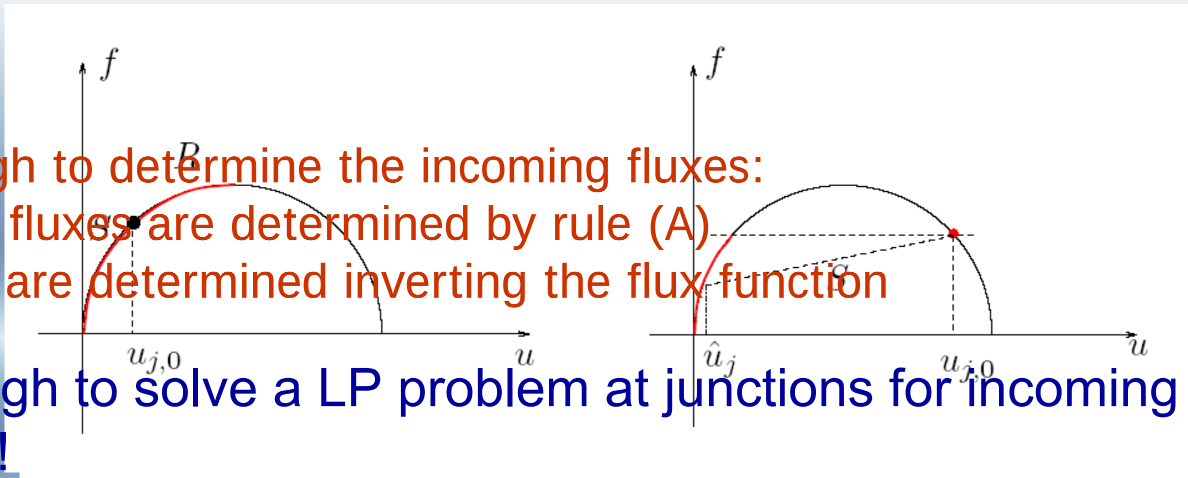
Rule (A) (i.e. (2)) implies the conservation of cars (1).

Rule (A) is not sufficient to determine a unique solution.

Given Rule (A), Rule (B) is essentially equivalent to entropy criteria.

Rules (A) and Rule (B) determine fluxes via solution of a LP problem.

LP problem at junctions



It is enough to determine the incoming fluxes:

-Outgoing fluxes are determined by rule (A)

-Densities are determined inverting the flux function

It is enough to solve a LP problem at junctions for incoming fluxes!

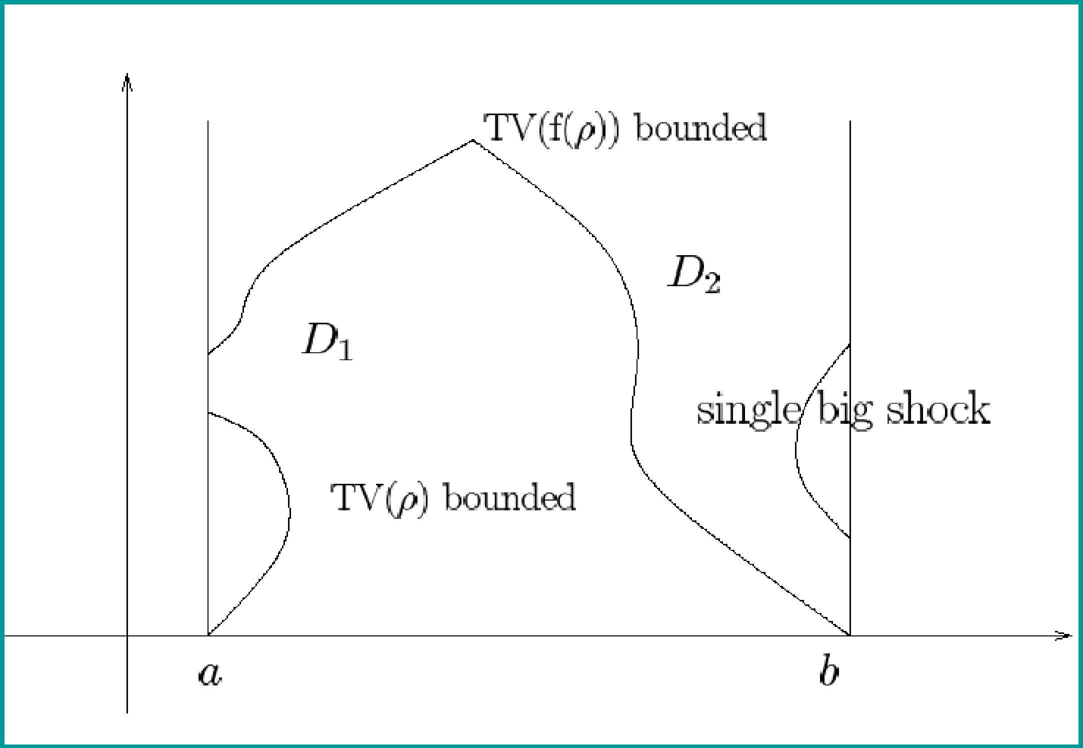
$$\max_{incoming\ fluxes\ \gamma_i} \sum_i \gamma_i$$

$$\gamma_j^{max}(u_{j,0}) = \begin{cases} f(\sigma) & \text{if } u_{j,0} \in [0,\sigma], \\ f(u_{j,0}) & \text{if } u_{j,0} \in]\sigma, 1], \end{cases}$$

$$0 \le \gamma_j = \sum_i \alpha_{ji}\gamma_i \le \gamma_j^{max}$$

Solutions via Wave Front Tracking

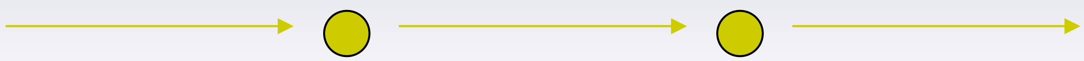
Technique: rules on the Riemann solver to get bounds on the flux variation of the solution



Continuous dynamics estimates by discrete counting of shocks

Packets flow on telecommunication networks

Telecommunication networks as Internet: no conservation of packets at small time scales.



Assume there exists a loss probability function and packets are re-sent if lost.

$$p : [0, \rho_{max}] \mapsto [0, 1]$$

Then at 1st step: (1-p) packets sent, p lost
at 2nd step: p(1-p) packets sent, p^2 lost
.... at kth step: p^(k-1) (1-p) sent, p^k lost ...

Finally the average transmission time and velocity are:

$$\Delta t_{av} = \sum_{n=1}^{+\infty} n \Delta t_0 (1-p) p^{n-1} = \frac{\Delta t_0}{1-p}$$

$$v = \frac{\delta}{\Delta t_{av}} = \frac{\delta}{\Delta t_0}(1-p) = \bar{v}(1-p).$$

Traffic lights and Viale del Muro Torto

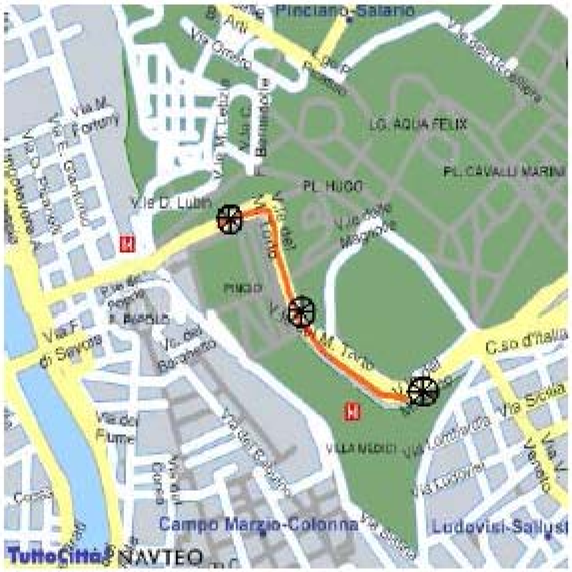
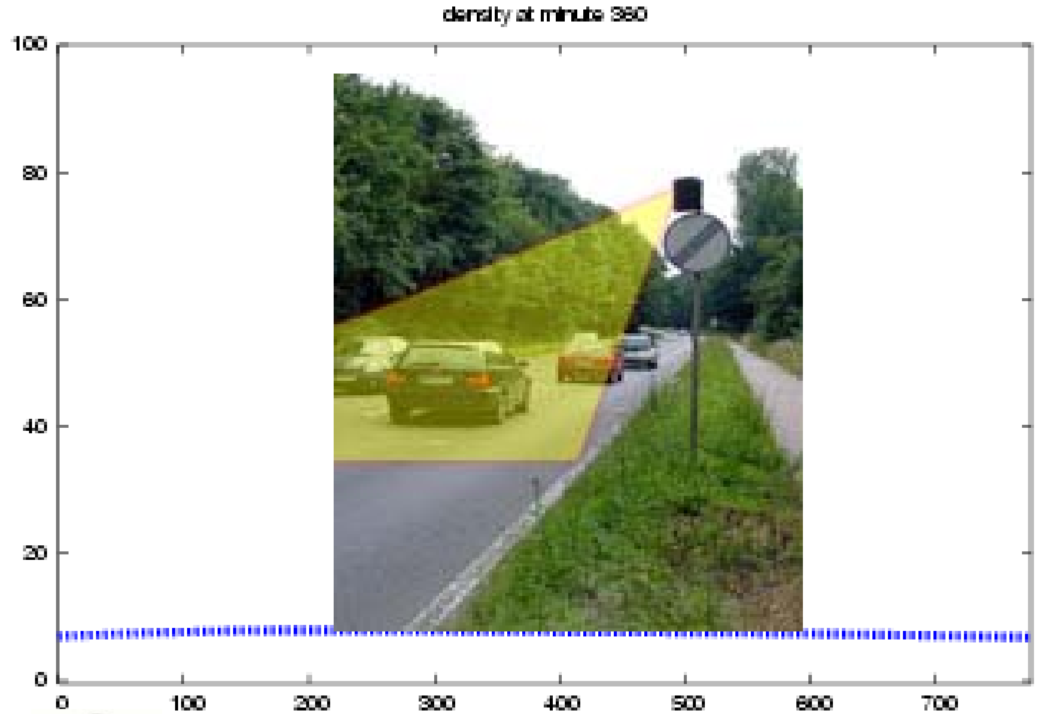
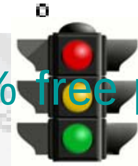


FIGURE 4. Viale del Muro Torto.



Data reconstruction error: 9% free phase, 19% congested phase



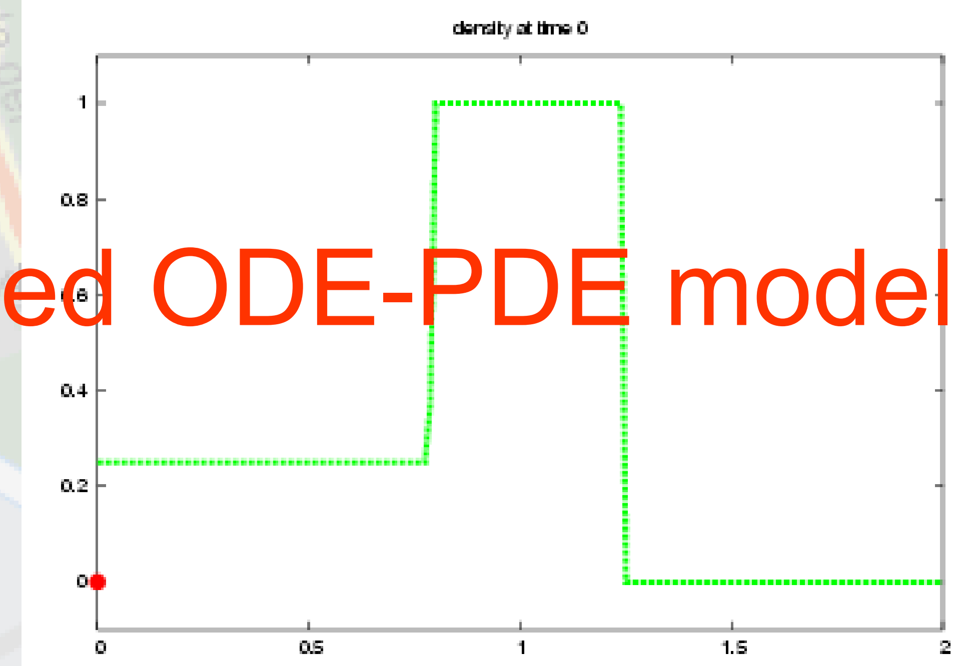
Continuous flow reconstructed from spot (discrete) data

Car trajectory on network

- **Determine the trajectory of a car on a loaded network**

$$\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0, \\ \rho(0, x) = \rho_0(x), \end{cases}$$

$$\begin{cases} \dot{x} = v(\rho(t, x)), \\ x(\bar{t}) = \bar{x}, \end{cases}$$



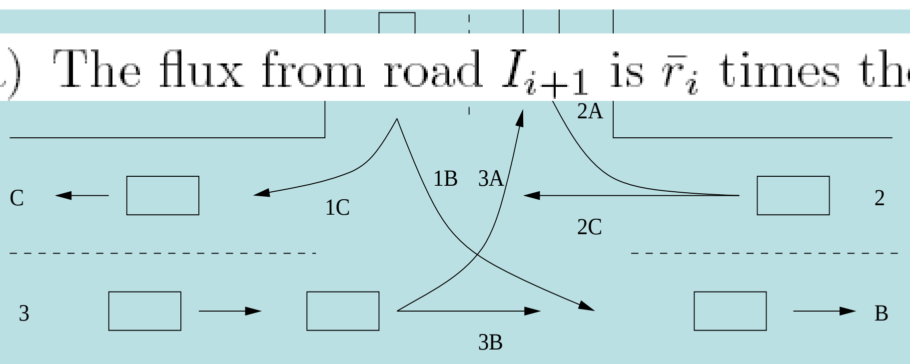
Mixed ODE-PDE model

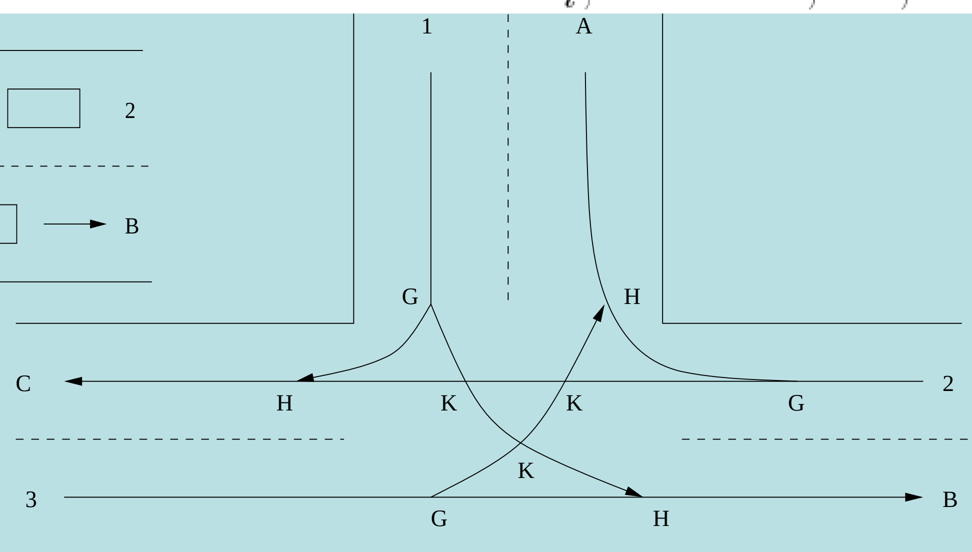
Theory in papers by Colombo and Marson

A model fot T-junctions

1) The flux from road I_i is the same of the corresponding exiting road I_{n+i}.
2) The total flux through J does not exceed its maximum capacity Γ_J.
3) The total flux through J is maximal respecting rules 1) and 2).

FPR) The flux from road I_{i+1} is $\bar{r}_i$ times the flux from road I_i, for $i = 1, \ldots, n-1$.





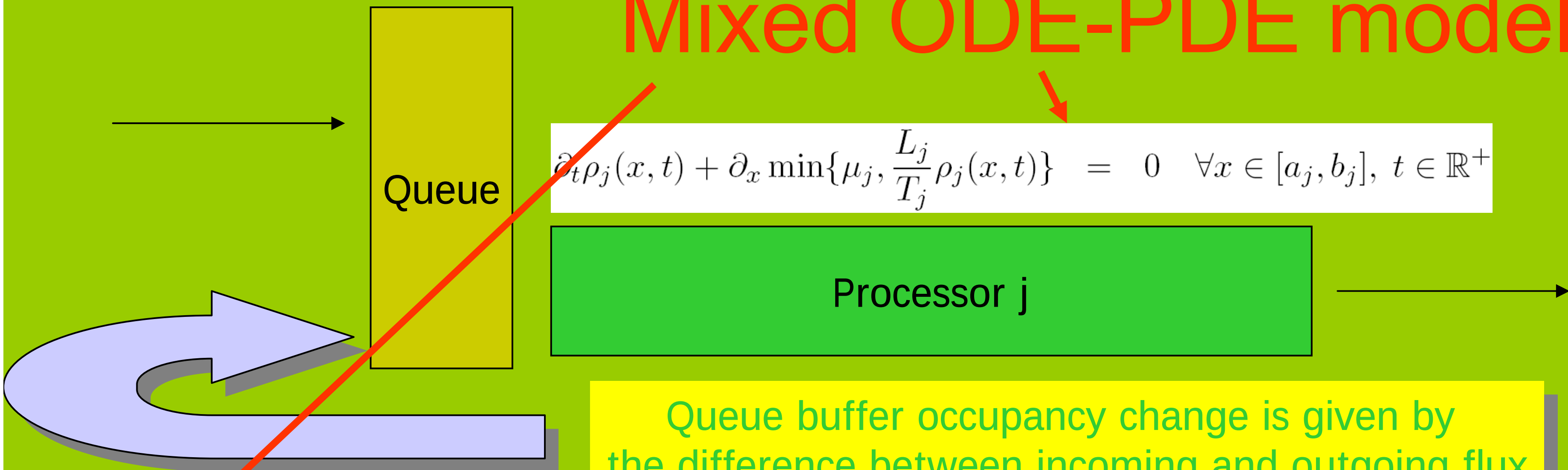
Red lights and jams

MODEL	P1	P2	P3
Lighthill-Whitham-Richards model	yes	yes	yes
Multipopulation model	yes	yes	yes
Aw-Rascle-Zhang model	yes	no	no
Colombo phase transition model	yes	yes	yes
Goatin phase transition model	yes	no	no
Siebel-Mauser BVT model	yes	no	asymptotically
Greenberg-Klar-Rascle multilane model	yes	no	asymptotically
Helbing third order model	yes	no	–

Processor with queue model (Goettlich-Herty-Klar)

Mixed ODE-PDE model

$$\partial_t \rho_j(x,t) + \partial_x \min\{\mu_j, \frac{L_j}{T_j}\rho_j(x,t)\} = 0 \quad \forall x \in [a_j, b_j],\ t \in \mathbb{R}^+$$



Queue buffer occupancy change is given by the difference between incoming and outgoing flux

$$\partial_t q_j(t) = f_{j-1}(\rho_{j-1}(b_{j-1},t)) - f_j(\rho_j(a_j,t))$$

$$f_j(\rho_j(a_j,t)) = \begin{cases} \min\{f_{j-1}(\rho_{j-1}(b_{j-1},t)), \mu_j\} & q_j(t) = 0 \\ \mu_j & q_j(t) > 0 \end{cases}$$

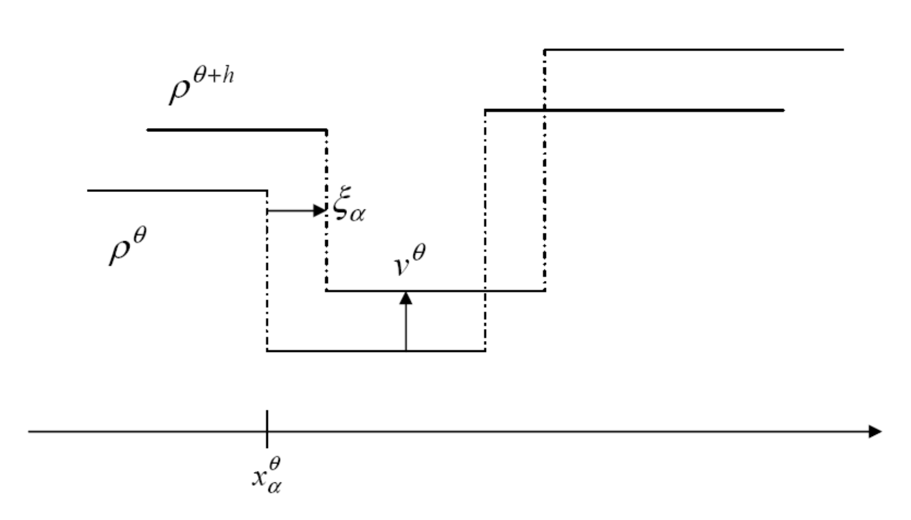
BV estimates for Goettlich-Herty-Klar supply chain model

$$\sum_{j=1}^{N} T.V.(\rho_j^\delta(\cdot,t)) + \sum_{j=2}^{N} |\partial_t q_j^\delta(t)| \le \sum_{j=1}^{N} T.V.(\rho_{j,0}^\delta(\cdot)) + \sum_{j=2}^{N} |\partial_t q_j^\delta(0)|$$

$$\text{and } \rho_j^\delta(x,t) \le \max_j \mu_j \ \forall j, x.$$

$$\sum_{j=2}^{N} T.V.(\partial_t q_j^\delta, [0, K\eta]) \le K \sum_{j=2}^{N} \left(2\ T.V.(\rho_{j-1,0}^\delta(\cdot)) + |\partial_t q_j^\delta(0)|\right)$$

Lipschitz continuous dependence (tlc and GHK supply chain model)



$$\|(v,\xi)\| \doteq \|v\|_{L^1} + \sum_{\beta=1}^{M} |\Delta\rho_\beta|\,|\xi_\beta|\,,$$

$$d(u,u') \doteq \inf\left\{\|\gamma\|_{L^1}\,,\ \gamma \in \Omega(u,u')\right\}.$$

Lemma (tlc) $\big\|(v,\xi)^+\big\| \le \big\|(v,\xi)^-\big\|$

$$\|(\xi_{\beta_i^j},\eta_j)\| = \sum_{j,i} |\xi_{\beta_i^j}||\Delta\rho_{\beta_i^j}| + \sum_j |\eta_j|.$$

η_j is the shift of the queue buffer occupancy q_j

Lemma 2.7 *The norm of tangent vectors are decreasing along wave front tracking approximations.*