Hyperbolic Models for Large Supply Chains

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Introduction

Topic: Overview of conservation law (traffic - like) models for large supply chains.

Joint work with....

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- D. Armbruster (ASU), P. Degond (Toulouse), M. Herty (Aachen)
- K. Kempf , J. Fowler (INTEL Corp.)

Definition of a supply chain

One supplier takes an item, processes it, and hands it over to the next supplier.

Suppliers (Items):

- Machines on a factory floor (product item),
- Agent (client),
- Factory, many items,
- Processors in a computing network (information),

Example: Protocol for a Wafer in a Semiconductor Fab

	diffusion 1	diffusion 2	litho 1	etch clean	etch 1	ion impl	metal dep	litho 2	etch 2	. .	
step	a	b	c	d	e	f	g	h	i		
1				0.25							clean wafer
2	8.00										grow a layer
3			1.00								pattern it
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OUTLINE$_{03}$

- Traffic flow like models
- Clearing functions: Quasi - steady state models - queueing theory.
- First principle models for non - equilibrium regimes (kinetic equations and hyperbolic conservation laws.)
- Stochasticity (transport in random medium).
- Policies (traffic rules).

Traffic flow - like models

- Introduce the stage of the whole process as an artificial 'spatial' variable. Items enter as raw product at $x = 0$ and leave as finished product at $x = X$.
- Define microscopic rules for the evolution of each item.
-
  - many body theory, large time averages
  - fluid dynamic models (conservation laws).
- Analogous to traffic flow models (items $\leftrightarrow$ vehicles).

Similarities between Traffic and Production Modeling $_{05}$

- Complexity and Topology: Complex re - entrant production systems. Networks of roads.
- Many body problem: interaction not given by simple mean fields.
- Control: Policies for production systems. Traffic control mechanisms.
- Random behavior.
- Model Hierarchies: Discrete Event Simulation (DES), Multi - Agent Models (incorporate stochastic behavior) $\Rightarrow$ kinetic equations for densities (mean field theories, large time asymptotics) $\Rightarrow$ fluid dynamics $\Rightarrow$ rate equations (fluid models).
  Simulation $\Rightarrow$ optimization and control.

Quasi - Steady State Models and Clearing functions $_{07}$

- A clearing function relates the expectation of the throughput time in steady state of each item for a given supplier to the expectation of the load, the 'Work in Progress'.
- Derived from steady state queuing theory.
- Yields a formula for the velocity of an item through the stages (Graves '96. Dai - Weiss '99) and a conservation law of the form

$$\boxed{\partial_t \rho + \partial_x [v(x,\rho)\rho] = 0}$$

ρ: item density per stage,

$x \in [0, X]$: stage of the process.

Example: $M/M/1$ queues and simple traffic flow models

Arrivals and processing times governed by Markov processes:

$$\boxed{v(x,\rho) = \frac{c(x)}{1+\rho}, \quad c(x) = \frac{1}{\langle \text{processing times} \rangle}}$$

$c(x)$: service rate or capacity of the processor at stage x.

Simplest traffic flow model (Lighthill - Whitham - Richards)

$$\boxed{v(x,\rho) = v_0(x)\left(1 - \frac{\rho}{\rho_{jam}}\right)}$$

- In supply chain models the density ρ can become arbitrarily large, whereas in traffic the density is limited by the space on the road ρ_{jam}.

phase velocity: $v_{phase} = \frac{\partial}{\partial\rho}[\rho v(x,\rho)]$

$$\boxed{v_{phase} = \frac{c(x)}{(1+\rho^2)} > 0, \quad v_{phase-traffic} = v_0(x)[1 - \frac{2\rho}{\rho_{max}}]}$$

- In supply chain models the propagation of information (shock speeds) is strictly forward $v_{phase} > 0$, whereas in traffic flow models shock speeds can have both signs.
- Problem: Queuing theory models are based on quasi - steady state regime. Modern production systems are almost never in steady state. (short product cycles, just in time production).
- Goal: Derive non - equilibrium models from first principles (first for automata) and then including stochastic effects.

First principle models for automata$_{12}$

- Assume processors work deterministically like automata. A processor located in the infinitesimal stage interval of length Δx needs a time $\tau(x) = \frac{\Delta x}{v_0(x)}$ to process an item.
- It cannot accept more than $c(x)\Delta t$ items per infinitesimal time interval Δt.

Theorem (Armbruster, CR '03): In the limit $\Delta x \to 0$, $\frac{\Delta t}{T} \to 0$. this yields a conservation law for the density ρ of items per stage of the form

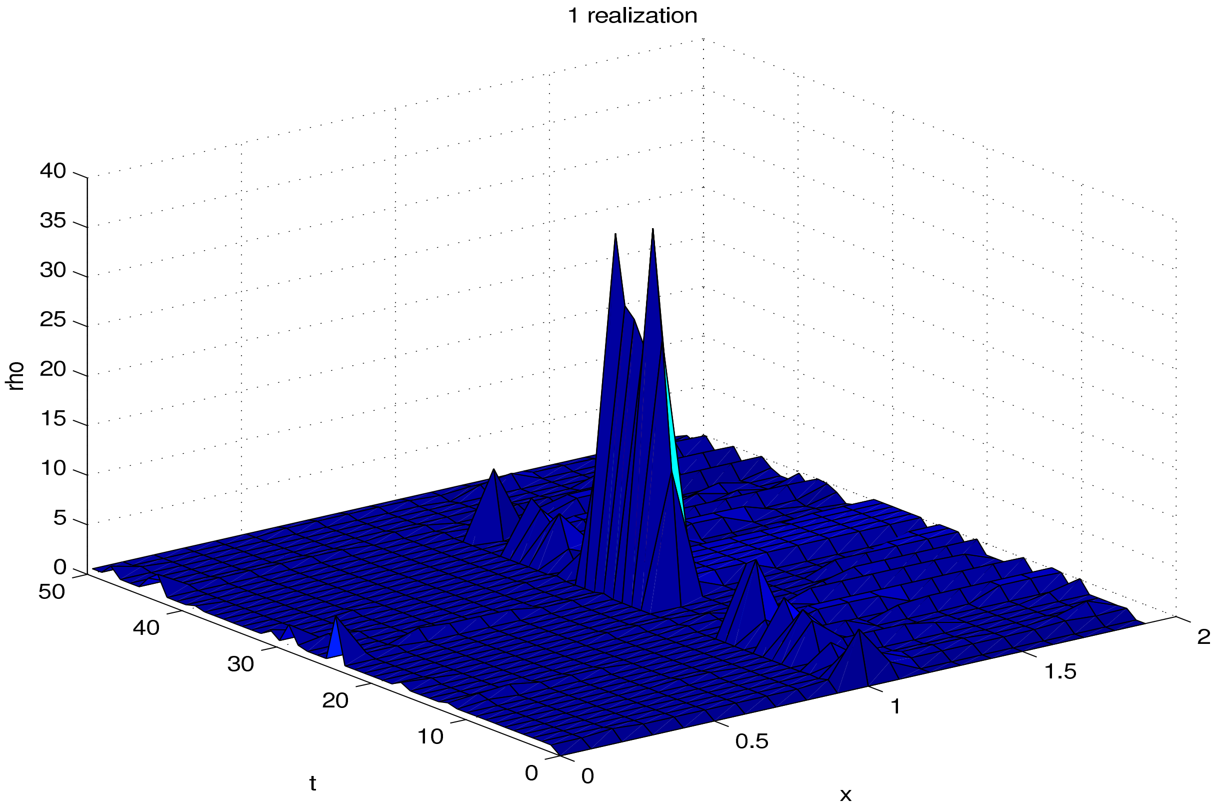
$$\boxed{\partial_t \rho + \partial_x F(x, \rho) = 0, \quad F(x, \rho) = \min\{c(x), v_0(x)\rho\}}$$

Bottlenecks

$$\boxed{\partial_t \rho + \partial_x F(x,\rho) = 0, \quad F(x,\rho) = \min\{c(x), v_0(x)\rho\}}$$

- No maximum principle (similar to pedestrian traffic with obstacles).
- The capacity $c(x)$ is discontinuous if nodes in the chain form a bottleneck.
- Flux F discontinuous $\Rightarrow$ density ρ distributional. (alternative model by Klar, Herty '04).
- Random server shutdowns $\Rightarrow$ bottlenecks shift stochastically.

A bottleneck in a continuous supply chain



Temporary overload of the bottleneck located at $x = 1$.

Stochasticity: Random breakdowns and random media 15

	diffusion 1	diffusion 2	litho 1	etch clean	etch 1	ion impl	metal dep	litho 2	etch 2	. .	
step	a	b	c	d	e	f	g	h	i		
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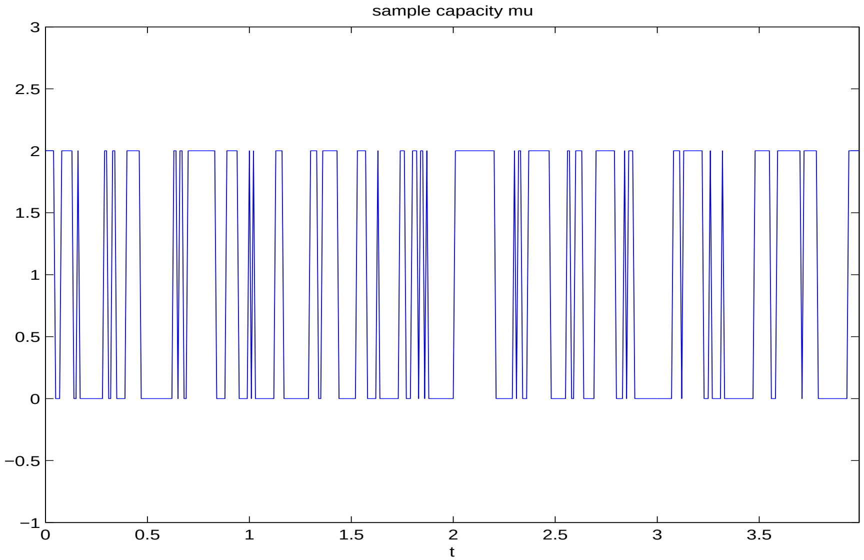
Availability

diffusion 1	diffusion 2	litho 1	etch clean	etch 1	ion impl	metal dep	litho 2	etch 2		
0.00	0.00	0.00	0.00	0.00	0.00	4.50	5.00	8.50		total hours required per lot
0.00	0.00	0.00	0.00	0.00	0.00	900.00	1000.00	1700.00		total hours needed per week
0.80	0.75	0.90	0.70	0.75	0.85	0.85	0.90	0.65		(average availability)
134.40	126.00	151.20	117.60	126.00	142.80	142.80	151.20	109.20		total hours available per machine per week
0.00	0.00	0.00	0.00	0.00	0.00	6.30	6.61	15.57		tools needed as time req / time avail
1.25	1.25	1.00	2.00	1.50	1.25	1.25	1.10	1.50		degree of constrainedness desired
0.00	0.00	0.00	0.00	0.00	0.00	7.88	7.28	23.35		number of tools needed
0	**0**	**0**	**0**	**0**	**0**	**8**	**8**	**24**		number of tools installed

Random capacities

Random breakdowns modeled by a Markov process setting the capacity to zero in random intervals.

$$\boxed{\partial_t \rho + \partial_x [\min\{c(x,t), v_0 \rho\}] = 0}$$



One realization of the capacity $c(x,t)$

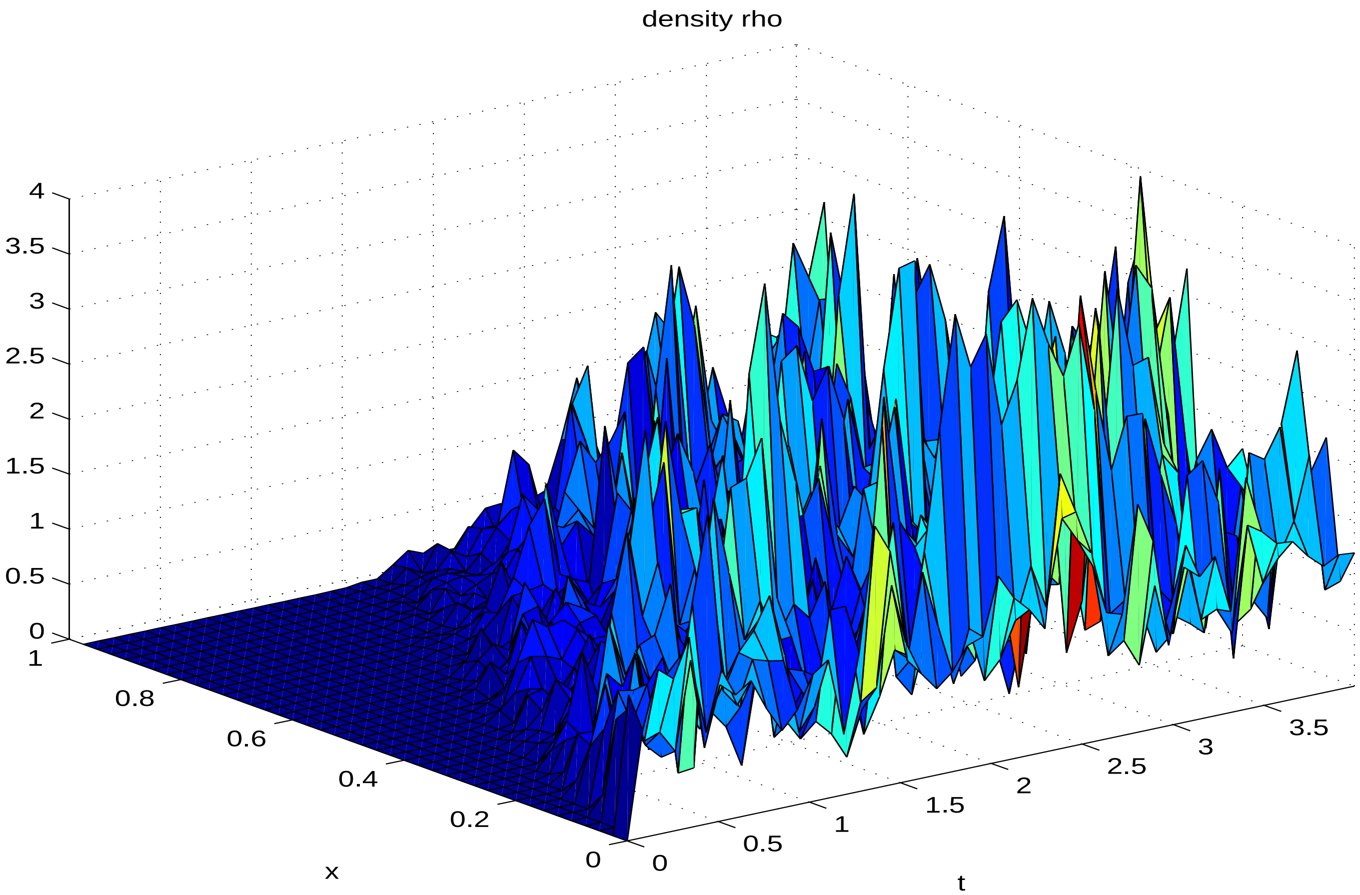
The Markov process:

$c(x,t)$ switches randomly between $c = c^{up}$ and $c = 0$

$$\boxed{c(x, t+\Delta t) = \begin{pmatrix} c(x,t) & prob = 1 - \Delta t \omega(x,c) \\ c^{up}(x) - c(x,t) & prob = \Delta t \omega(x,c) \end{pmatrix}}$$

- Frequency $\omega(x,c)$ given by mean up and down times of the processors.
- Particle moves in random medium given by the capacities.

One realization with flux $F = \min\{c, v\rho\}$ using a stochastic c



Goal: Derive equation for the evolution of the expectation $\langle \rho \rangle$

The many body problem $_{18}$

- Formulate deterministic model in Lagrangian coordinates. $\xi_n(t)$: position of part n at time t.
- A 'follow the leader' model:

$$\boxed{\frac{d}{dt}\xi_n = \min\{c(\xi_n, t)[\xi_n - \xi_{n-1}],\ v_0(\xi_n)\}}$$

- Particles move in a random medium, given by stochastic capacities $c(\xi_n, t)$

Kinetic equation for the many body probability density

$F(t, x_1, .., x_N, y_1, ., y_K)$: probability that

$\xi_1(t) = x_1, .., \xi_N(t) = x_N$ and $c_1(t) = y_1, .., c_K(t) = y_K$.

Satisfies a Boltzmann equation in high dimensional space.

$$\boxed{\partial_t F(t, X, Y) + \nabla_X \cdot [V(X, Y)F] = Q[F]}$$

$$\boxed{Q[F] = \int K(X, Y, Y')F(t, X, Y')\, dY' - \kappa(X, Y)F}$$

$X = (x_1, .., x_N)$: positions

$Y = (y_1, .., y_K),\ Y \in \{0, c^{up}\}^K$: kinetic variable (discrete velocity model).

$Q(F)$: interaction with random background.

- Discrete event simulation corresponds to solving the kinetic many body equation by Monte Carlo.
- Use methodology for many particle systems. Mean field theory, long time averages, Chapman - Enskog.

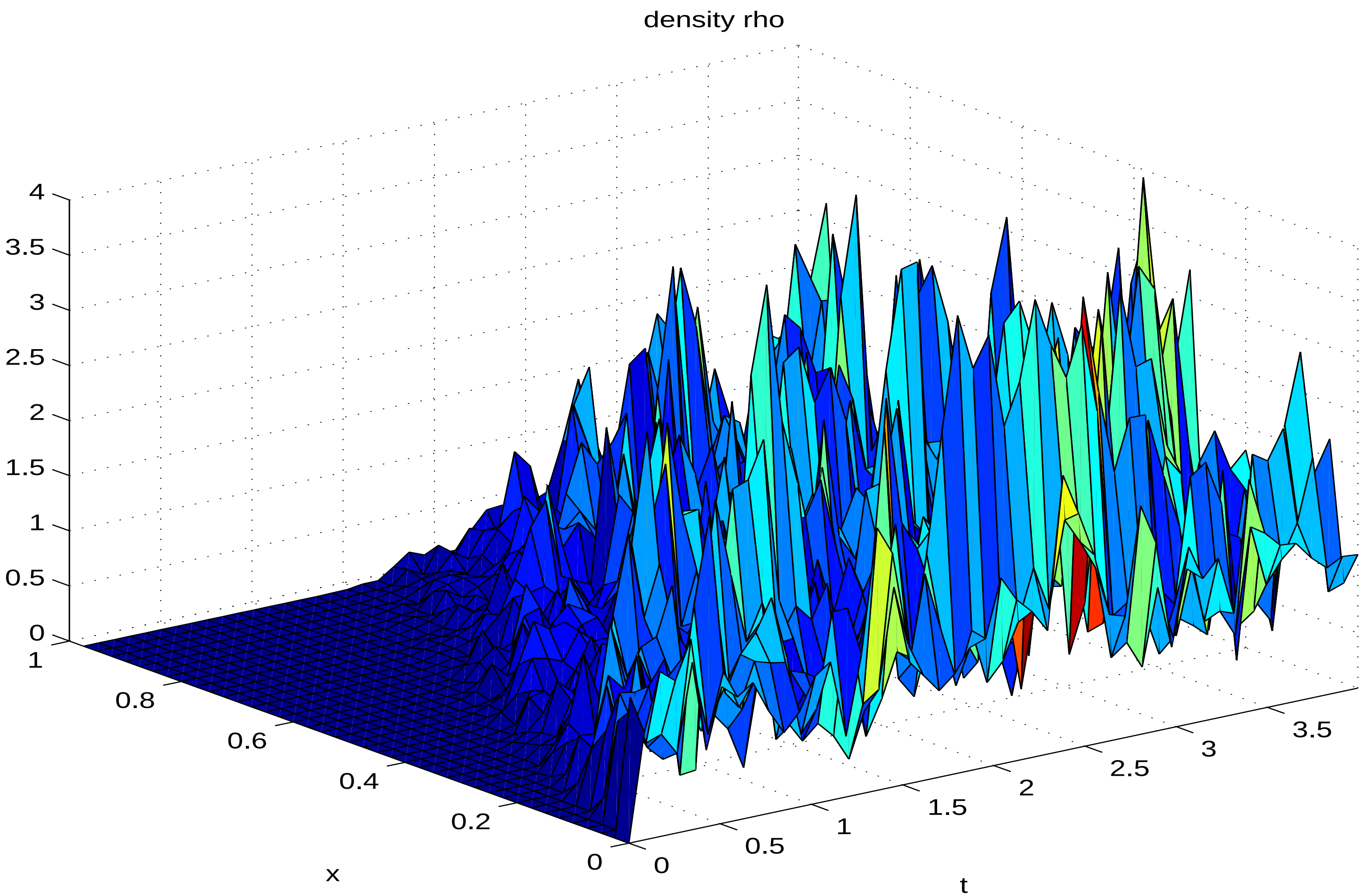
Theorem (Degond, CR '06):

On large time scales (compared to the mean up and down times of the processors) the expectation $\langle\rho\rangle$ of the part - density satisfies

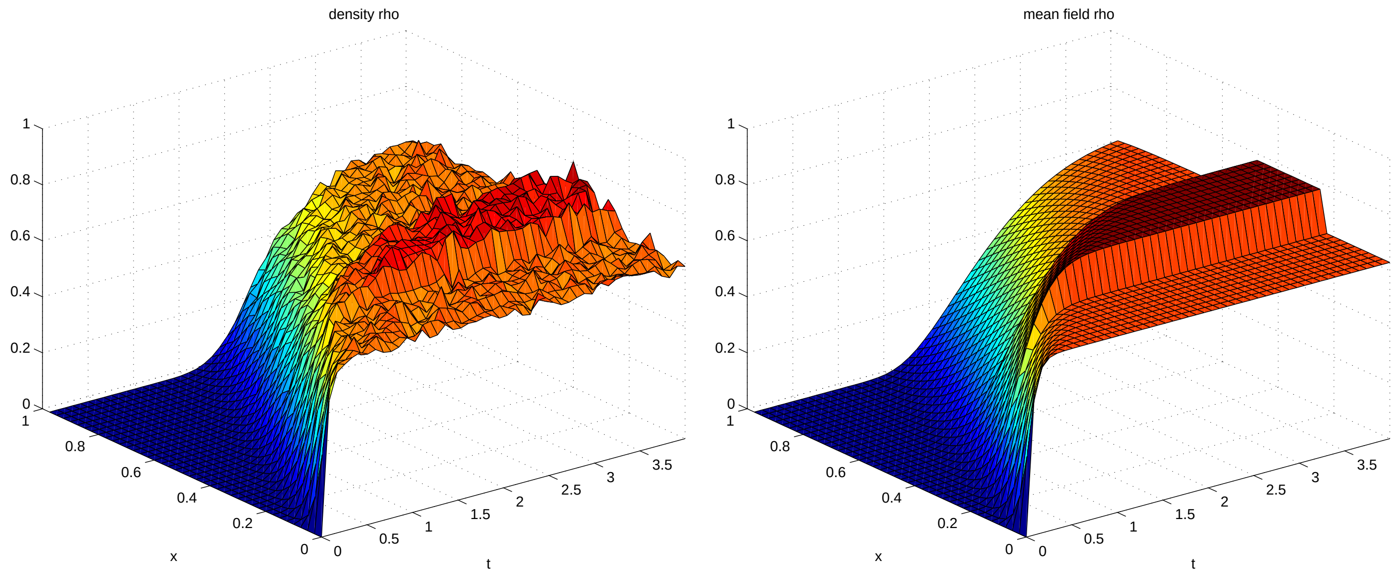
$$\boxed{\partial_t\langle\rho\rangle + \partial_x F = 0, \quad F = \langle a\rangle c^{up}[1 - \exp(-\frac{v_0\langle\rho\rangle}{\langle a\rangle c^{up}}) - \varepsilon\sigma^2(a)\partial_x\langle\rho\rangle]}$$

a: availability $\langle a\rangle = \frac{\langle T_{up}\rangle}{\langle T_{up}\rangle + \langle T_{down}\rangle}$,
ε: ratio of $\langle T_{up/down}\rangle$ to large time scale.

One realization



The steady state case



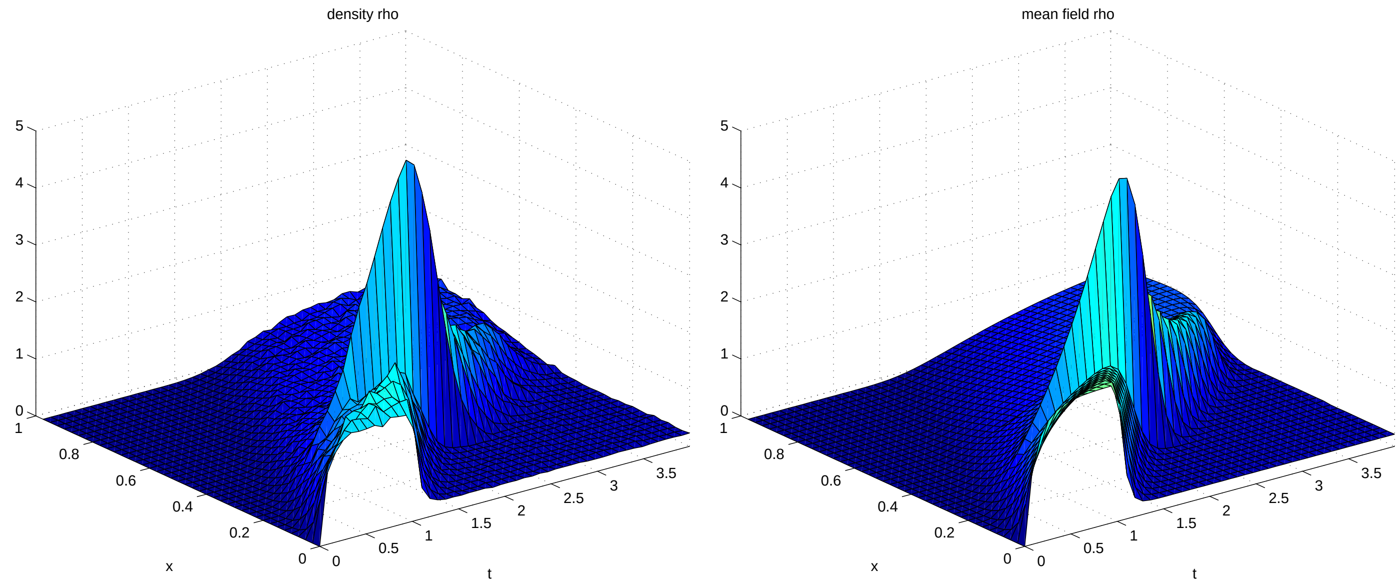
60 stages, bottleneck processors for $0.4 < x < 0.6$

Constant influx;

$F(x = 0) = 0.5\times$ bottleneck capacity

Left: DES (100 realizations), Right: mean field equations

Verification of the transient case



Influx $F(x=0)$ temporarily at $2.0\times$ the bottleneck capacity

Left: 500 realizations, Right: mean field equations

Re - entrant Networks and Scheduling Policies $_{23}$

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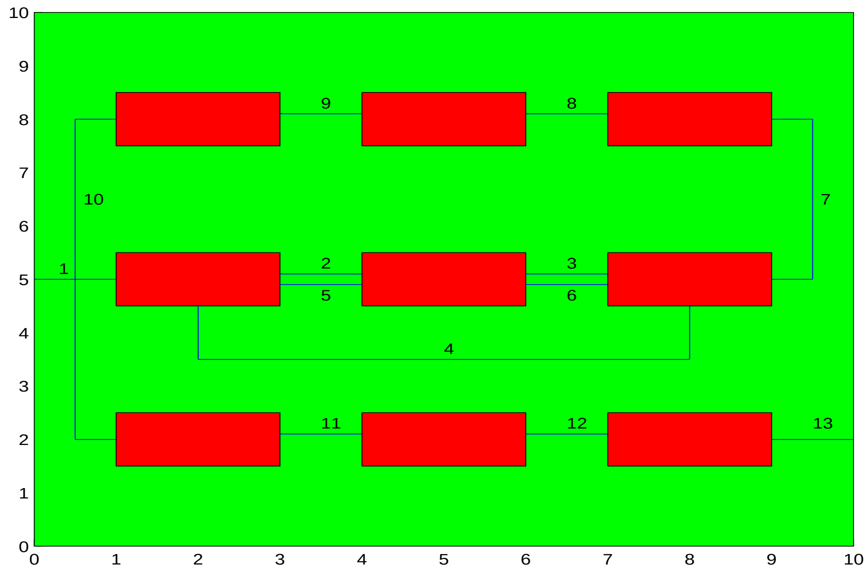
Re - entrant manufacturing lines: One and the same tool is used at different stages of the manufacturing process.



▶ Conservation laws on graphs. Implies that the velocity is computed non - locally. (Different stages of the process correspond to the same physical node.)

▶ Requires the use of a policy governing in what sequence to serve different lines ('the right of way': FIFO, PULL, PUSH).

Priority scheduling $_{25}$

- Equip each part with an attribute vector $y \in \mathbb{R}^K$.
- Define the priority of the part by $p(y) : \mathbb{R}^K \longrightarrow \mathbb{R}$
- The velocity of the part is determined by all the parts using the same tool at the same time with a higher priority.
- Leads to a (nonlocal) kinetic model for stages and attributes (high dimensional).
- Recover systems of conservation laws by using multi - phase approximations for level sets in attribute space.

Kinetic model (Degond, Herty, CR '07)

(Vlasov - type)

$$\boxed{\partial_t f(x,y,t) + \partial_x[v(\phi(x,p(y)))f] + \nabla_y[Ef] = 0}$$

f: kinetic density of parts at stage x with attribute y.

$p(y)$: priority of parts with attribute y.

$\phi(x,q)$: cumulative density of parts with priority higher than q.

$$\boxed{\phi(x,q) = \int H(p(y)-q) f(x,y,t)\, dy}$$

or (for re-entrant systems):

$$\boxed{\phi(x,q) = \int H(p(y)-q) K(x,x') f(x',y,t)\, dx' dy}$$

Choices:

- Attributes y;
- $p(y)$ determines the policy;
- the velocity $v(\phi)$ (the flux model).

Example: $y \in \mathbb{R}^3$

y_1: cycle time (time the part has spent in the system).

y_2: time to due date.

y_3: type of the part (integer valued).

$$\partial_t f + \partial_x [vf] + \nabla_y \cdot [Ef] = 0 \Rightarrow E = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Policies:

- FIFO: $p(y) = y_1$
- Due date scheduling: $p(y) = -y_2$.
- Combined policy (c.f. for perishable goods)

$$\boxed{p(y) = y_1 H(y_1 - d(y_3)) - y_2 H(d(y_3) - y_1)}$$

The phase velocity:

The microscopic velocity $v(\phi)$ has to be chosen as the phase velocity $\frac{\partial F(\phi)}{\partial \phi}$ of a macroscopic conservation law.

Theorem $_{29}$: The total density of parts (with all attributes) $\rho(x,t) = \int f(x,y,t)\, dy$ satisfies the conservation law

$$\boxed{\partial_t \rho + \partial_x F(\rho) = 0, \quad F(\rho) = \int_{-\infty}^{\rho} v(\phi)\, d\phi}$$

Decide on an over all flux model $F(\rho)$. Set $v(\phi) = \partial_\phi F(\phi)$.

Example: $\boxed{F(\rho) = \min\{c, v_0\rho\} \;\Rightarrow v(\phi) = v_0 H(c - v_0\phi)}$

Multi - phase approximations

- Leads to high dimensional kinetic equation. Reduce to conservation laws via a multi - phase ansatz.
- Approximate $f(x, y, t)$ by a combination of $\delta-$ measures in y.

$$\boxed{f(x, y, t) = \sum_n \rho_n(x, t)\delta(y - Y_n(x, t))}$$

- Derive conservation laws for the number densities $\rho_n(x, t)$ with attributes $y = Y_n(x, t)$.

Standard approach (Jin, Li 03): Moment closures

Level Sets $_{31}$

Almost all information about the microscopic transport picture is contained in the evolution of the level sets of parts with equal priority

$$\Lambda(x,q,t) = \int \delta(p(y)-q) f(x,y,t)\, dy = -\partial_q \phi(x,q,t)$$

Level set equation:

$$\boxed{\partial_t \Lambda(x,q,t) + \partial_x [v(\phi(x,q))\Lambda] + \partial_q A[f] = 0, \quad \Lambda = -\partial_q \phi}$$

$$A[f](x,q,t) = \int \delta(q-p) f [\partial_t p + v(\phi(x,p))\partial_x p + E \nabla_y p] dy$$

The Riemann Problem

The multi - phase approximation implies for the level sets $\Lambda(x,q,t)$ and the cumulative densities $\phi(x,q,t)$

$$\boxed{\Lambda(x,q,t) = \sum_n \rho_n(x,t)\delta(P_n - q)}$$

$$\boxed{\phi(x,q,t) = \sum_n \rho_n(x,t)H(P_n - q),}$$

with $P_n(x,t) = p(Y_n(x,t)) \in \mathbb{R}^1$.

The cumulative density $\phi(x,q,t)$ is piecewise constant in $q \Rightarrow$ solve a Riemann problem for ϕ and compute the motion of $p(Y_n)$ from the Rankine - Hugoniot condition for the shock speeds.

$$\boxed{\frac{d}{dt}P_n + v_n\partial_x P_n = A_n(Y), \quad v_n = \lim_{\varepsilon\to 0}\frac{F(\phi(P_n+\varepsilon))-F(\phi(P_n-\varepsilon))}{\phi(P_n+\varepsilon)-\phi(P_n-\varepsilon)}}$$

The densities ρ_n are evolved according to

$$\boxed{\partial_t \rho_n + \partial_x (v_n \rho_n) = 0}$$

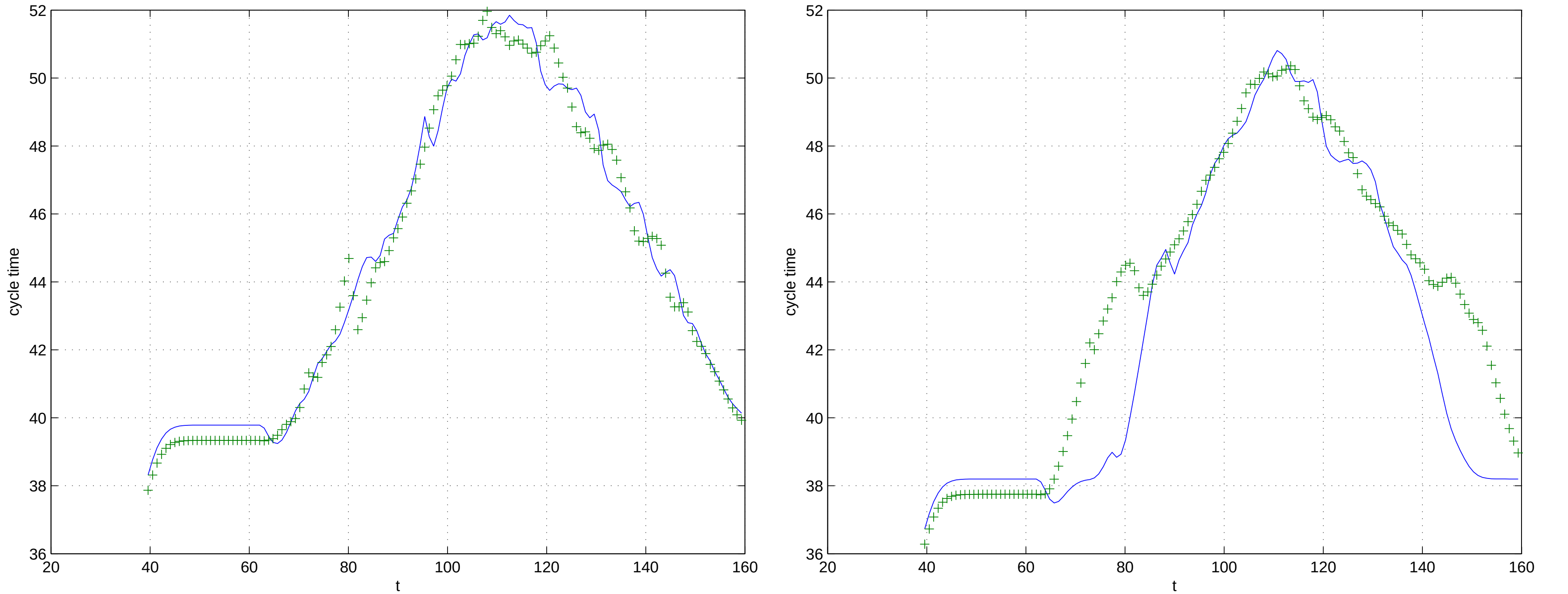
For $y \in \mathbb{R}^1,\ Y_n = P_n$ this is an exact (weak) solution of the kinetic transport equation.

For more than one dimensional attributes the actual attributes Y_n are evolved according to

$$\boxed{\partial_t Y_n + v_n \partial_x Y_n - E = 0}$$

within the level set - subject to the constraints $p(Y_n) = P_n$ (enforced by a projection method).

Policy effect on cycle time

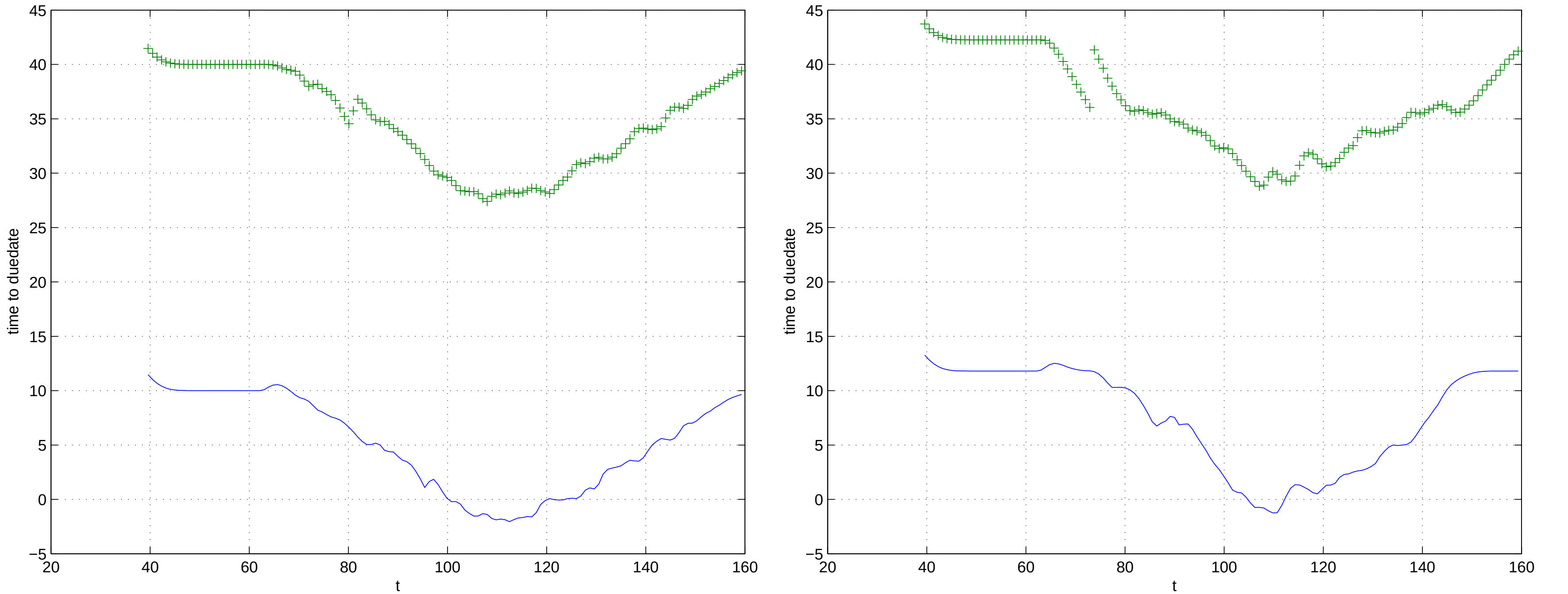


2 products with 2 different delivery due dates.

$+$:slow lots, $-$ hot lots.

Left: FIFO, Right: PERISH

Time to due date at exit



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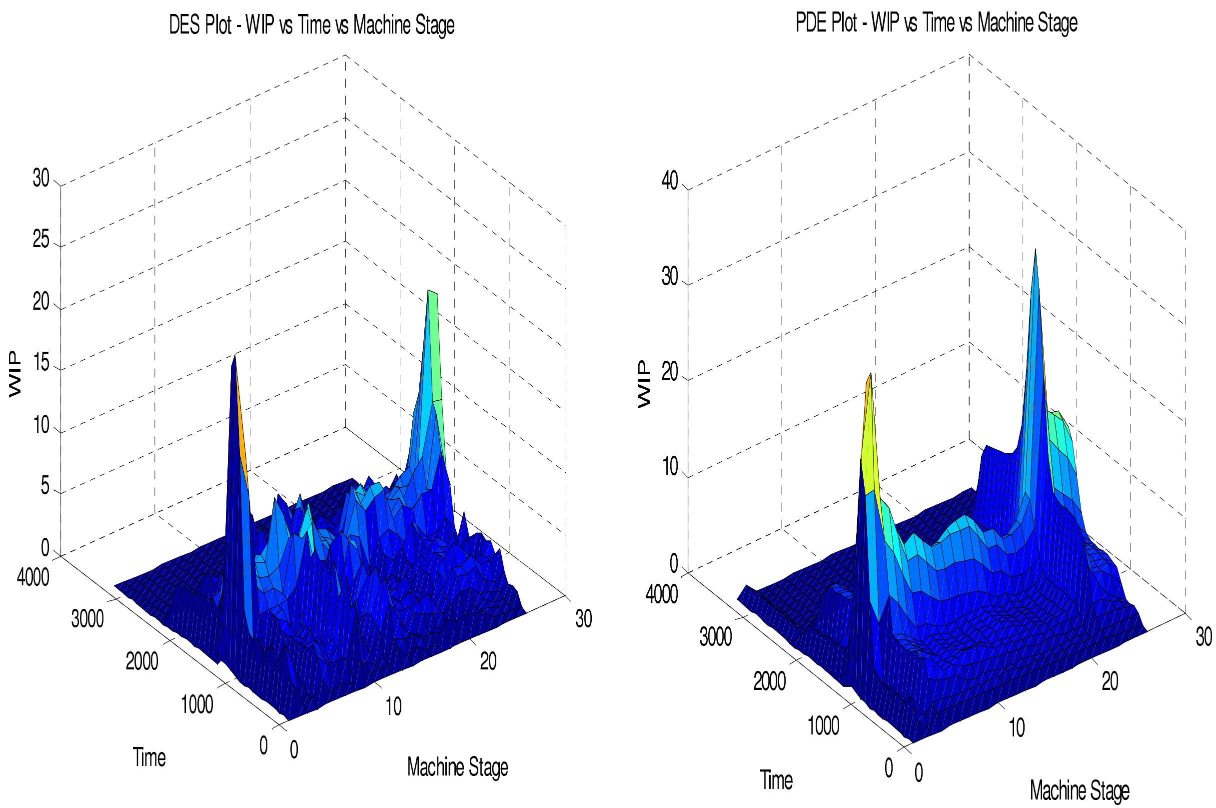
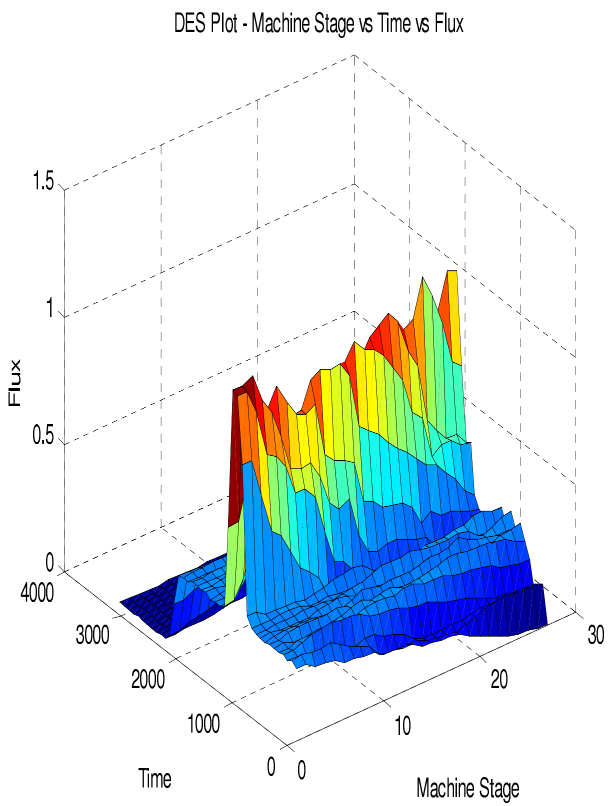
Toy factory - Comparison (WIP) to discrete event simulations

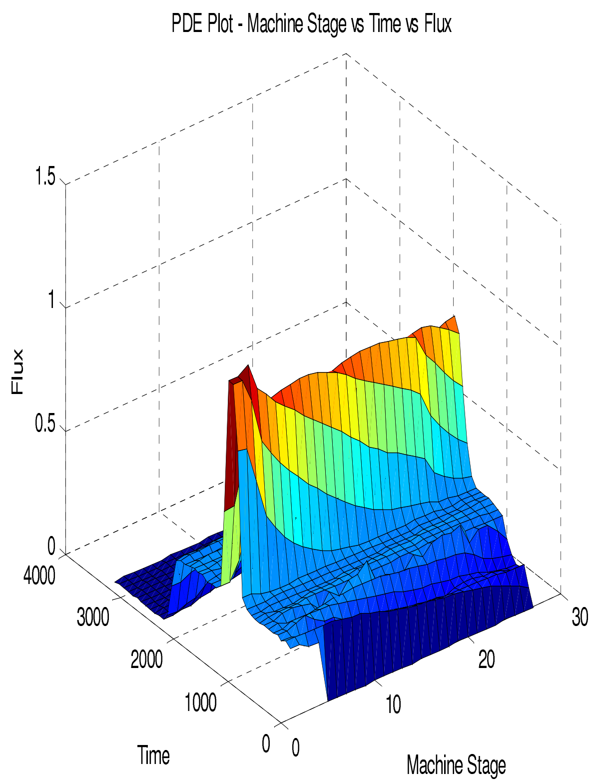


26 processing steps, 200 machines, FIFO

Left: DES: (60,000 lots, 100 realizations), Right: Conservation Law

Toy factory - Comparison (FLUX) to discrete event simulations





26 processing steps, 200 machines, FIFO

Left: DES: (60,000 lots, 100 realizations), Right: Conservation Law

Conclusions $_{36}$

- Value of PDE models: Provide online decision making tools in complex processes in non - equilibrium regimes.
- Less versatile than DES.
- Conservation laws on graphs (nonlocal constitutive relations).
- Future work:
  - Non - Markovian behavior
  - Optimization.