Maxwell-Dirac: Null structure and almost optimal local well-posedness

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Outline

- Study Maxwell-Dirac system (MD).
- System of nonlinear wave equations. Describes: electron self-interacting with electromagnetic field.
- We would like to understand the nonlinear structure of MD.
- The structure cannot be seen in each component equation, only in system as whole.
- Structure is expressed in terms of trilinear and quadrilinear integral forms with special cancellation properties expressed in terms of the spatial frequencies.
- 3D case: Use structure to prove multilinear space-time Fourier restriction estimates at scale invariant regularity up to a logarithmic loss.
- As a consequence, we are able to prove local well-posedness almost down to the critical regularity in 3D.

Minkowski space-time $\mathbb{R}^{1+3} = \mathbb{R}_t \times \mathbb{R}^3_x$

- Coordinates/partials:

$$\begin{array}{llll} t = x^0 & \partial_0 = \partial_t & & \text{time} \\ x = (x^1, x^2, x^3) & \partial_j = \partial_{x^j} & \nabla = (\partial_1, \partial_2, \partial_3) & \text{space} \end{array}$$

- Metric (raise/lower indices): $(g^{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
- Summation convention applies for repeated upper/lower indices ($j, k, \dots = 1, 2, 3$; $\mu, \nu, \dots = 0, 1, 2, 3$)
- For example:

$$\square = \partial^\mu \partial_\mu = -\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2 = -\partial_t^2 + \Delta_x$$

Maxwell-Dirac (MD)

- Couple Maxwell's equations and Dirac equation:

$$\begin{cases} \nabla \cdot \mathbf{E} = \rho, & \nabla \cdot \mathbf{B} = 0, \\ \nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0, & \nabla \times \mathbf{B} - \partial_t \mathbf{E} = \mathbf{J}, \end{cases} \qquad \text{(Maxwell)}$$

$$(\boldsymbol{\alpha}^\mu D_\mu + m\boldsymbol{\beta})\,\psi = 0. \qquad \text{(Dirac)}$$

  $m \geq 0$ constant; $\boldsymbol{\alpha}^\mu$ and $\boldsymbol{\beta}$ are 4×4 *Dirac matrices*.

- Unknowns:

$\mathbf{E}, \mathbf{B} \colon \mathbb{R}^{1+3} \to \mathbb{R}^3$	electric and magnetic fields
$\psi \colon \mathbb{R}^{1+3} \to \mathbb{C}^4$	Dirac four-spinor

- Represent EM field by real four-potential A_μ, $\mu = 0, 1, 2, 3$:

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = \nabla A_0 - \partial_t \mathbf{A} \qquad (\mathbf{A} = (A_1, A_2, A_3))\,.$$

Maxwell-Dirac (MD)

- We have

$$\begin{cases} \nabla \cdot \mathbf{E} = \rho, & \nabla \cdot \mathbf{B} = 0, \\ \nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0, & \nabla \times \mathbf{B} - \partial_t \mathbf{E} = \mathbf{J}, \end{cases} \qquad \text{(Maxwell)}$$

$$(\boldsymbol{\alpha}^\mu D_\mu + m\boldsymbol{\beta})\,\psi = 0, \qquad \text{(Dirac)}$$

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = \nabla A_0 - \partial_t \mathbf{A} \qquad \text{(Potentials)}$$

- Complete the coupling:

$J^\mu = \langle \boldsymbol{\alpha}^\mu \psi, \psi \rangle_{\mathbb{C}^4}$ — Dirac four-current

$\rho = J^0 = |\psi|^2$ — charge density

$\mathbf{J} = (J^1, J^2, J^3)$ — three-current density

$D_\mu = D_\mu^{(A)} = \frac{1}{i}\partial_\mu - A_\mu$ — gauge covariant derivative

Gauge invariance

- Result is the following nonlinear system:

$$\Box A_\mu - \partial_\mu(\partial^\nu A_\nu) = -\langle \boldsymbol{\alpha}_\mu \psi, \psi \rangle_{\mathbb{C}^4}, \qquad \text{(Maxwell)}$$
$$(-i\boldsymbol{\alpha}^\mu \partial_\mu + m\boldsymbol{\beta})\,\psi = A_\mu \boldsymbol{\alpha}^\mu \psi, \qquad \text{(Dirac)}$$

- Invariant under the *gauge transformation*

$$\psi \longrightarrow \psi' = e^{i\chi}\psi, \qquad A_\mu \longrightarrow A'_\mu = A_\mu + \partial_\mu \chi, \qquad \text{(GT)}$$

  for any $\chi : \mathbb{R}^{1+3} \to \mathbb{R}$ (the *gauge function*).
- Observables $\mathbf{E}, \mathbf{B}, \rho, \mathbf{J}$ not affected, so solutions related by GT are physically undistinguishable; considered *equivalent*.
- Pick representative whose potential A_μ simplifies the analysis.
- Natural: impose *Lorenz gauge condition*

$$\partial^\mu A_\mu = 0 \quad (\iff \partial_t A_0 = \nabla \cdot \mathbf{A}) \qquad \text{(LG)}$$

Initial data

- System becomes

$$\begin{aligned} \Box A_\mu &= -\langle \boldsymbol{\alpha}_\mu \psi, \psi \rangle_{\mathbb{C}^4}, && \text{(Maxwell)} \\ (-i\boldsymbol{\alpha}^\mu \partial_\mu + m\boldsymbol{\beta})\,\psi &= A_\mu \boldsymbol{\alpha}^\mu \psi, && \text{(Dirac)} \\ \partial^\mu A_\mu &= 0. && \text{(LG)} \end{aligned}$$

- Initial data:

$$\psi|_{t=0} = \psi_0, \qquad \mathbf{E}|_{t=0} = \mathbf{E}_0, \qquad \mathbf{B}|_{t=0} = \mathbf{B}_0.$$

Maxwell imposes constraints

$$\nabla \cdot \mathbf{E}_0 = |\psi_0|^2, \qquad \nabla \cdot \mathbf{B}_0 = 0.$$

- Scale invariant data regularity (3D):

$$\psi_0 \in L^2(\mathbb{R}^3; \mathbb{C}^4), \qquad (\mathbf{E}_0, \mathbf{B}_0) \in \dot{H}^{-1/2}(\mathbb{R}^3; \mathbb{R}^3).$$

Construction of Lorenz data

- The data for the four-potential,

$$A_\mu\big|_{t=0} = a_\mu, \qquad \partial_t A_\mu\big|_{t=0} = \dot{a}_\mu,$$

  must be constructed from the observables $(\mathbf{E}_0, \mathbf{B}_0)$.

- Set

$$a_0 = \dot{a}_0 = 0.$$

  Then $\mathbf{a} = (a_1, a_2, a_3)$ and $\dot{\mathbf{a}} = (\dot{a}_1, \dot{a}_2, \dot{a}_3)$ determined by

$$\underbrace{\nabla \cdot \mathbf{a} = 0,}_{\text{by LG condition}} \qquad \underbrace{\nabla \times \mathbf{a} = \mathbf{B}_0, \qquad \dot{\mathbf{a}} = -\mathbf{E}_0}_{\text{by Maxwell}}.$$

- LG condition automatically satisfied in the evolution starting from Lorenz data. LG equation can be discarded from the system.
- Next step: Solve wave equation for A_μ and plug into Dirac equation. Result is a single nonlinear Dirac equation.

Reduction to nonlinear Dirac equation

- Recall Duhamel's formula for $\square u = F$, $(u, \partial_t u)|_{t=0} = (u_0, u_1)$:

$$u(t) = \underbrace{\cos(t\,|\nabla|)u_0 + \frac{\sin(t\,|\nabla|)}{|\nabla|}u_1}_{u^{\text{hom.}}} + \underbrace{\int_0^t \frac{\sin((t-s)\,|\nabla|)}{|\nabla|}F(s)\,ds}_{\frac{1}{\square}F}$$

- Thus, $A_\mu = A_\mu^{\text{hom.}} - \frac{1}{\square}\langle \boldsymbol{\alpha}_\mu \psi, \psi \rangle_{\mathbb{C}^4}$.
- Result: MD in LG becomes a single nonlinear Dirac equation

$$(-i\boldsymbol{\alpha}^\mu \partial_\mu + m\boldsymbol{\beta})\,\psi = A_\mu^{\text{hom.}} \boldsymbol{\alpha}^\mu \psi - \mathcal{N}(\psi, \psi, \psi), \qquad \text{(MDL)}$$

where

$$\mathcal{N}(\cdot, \cdot, \cdot) = \left(\frac{1}{\square} \langle \boldsymbol{\alpha}_\mu \cdot, \cdot \rangle_{\mathbb{C}^4} \right) \boldsymbol{\alpha}^\mu \cdot$$

Local well-posedness (almost optimal)

Theorem

Let $s > 0$. Assume given initial data

$$\psi_0 \in H^s(\mathbb{R}^3;\mathbb{C}^4), \qquad \mathbf{E}_0, \mathbf{B}_0 \in H^{s-1/2}(\mathbb{R}^3;\mathbb{R}^3),$$

satisfying the Maxwell constraints. Prepare Lorenz data:

$$a_0 = \dot{a}_0 = 0, \qquad \nabla \cdot \mathbf{a} = 0, \qquad \nabla \times \mathbf{a} = \mathbf{B}_0, \qquad \dot{\mathbf{a}} = -\mathbf{E}_0,$$

and use these to define $A_\mu^{\mathrm{hom.}}$. Then $\exists T > 0$, depending continuously on the data norm, and there exists

$$\psi \in C([-T,T];H^s(\mathbb{R}^3;\mathbb{C}^4))$$

which solves the MDL equation on $(-T,T) \times \mathbb{R}^3$:

$$(-i\boldsymbol{\alpha}^\mu \partial_\mu + m\boldsymbol{\beta})\,\psi = A_\mu^{\mathrm{hom.}} \boldsymbol{\alpha}^\mu \psi - \mathcal{N}(\psi,\psi,\psi), \qquad \psi|_{t=0} = \psi_0.$$

Persistence of regularity for electromagnetic field

Theorem

Assume same hypotheses as in previous theorem, and let ψ be the solution of MDL on $(-T, T) \times \mathbb{R}^3$. Define

$$\rho = |\psi|^2 \qquad \mathbf{J} = \left\{ \langle \boldsymbol{\alpha}^j \psi, \psi \rangle_{\mathbb{C}^4} \right\}_{j=1,2,3},$$

and solve Maxwell's equations

$$\begin{cases} \nabla \cdot \mathbf{E} = \rho, & \nabla \cdot \mathbf{B} = 0, \\ \nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0, & \nabla \times \mathbf{B} - \partial_t \mathbf{E} = \mathbf{J}. \end{cases}$$

Solution retains data regularity:

$$(\mathbf{E}, \mathbf{B}) \in C([-T, T]; H^{s-1/2})$$

Some earlier existence results for MD

- Gross '66: Local existence for smooth data
- Georgiev '91: Global existence for small, smooth data
- Bournaveas '96: Local well-posedness (LWP) for data

$$\psi_0 \in H^s(\mathbb{R}^3; \mathbb{C}^4), \qquad \mathbf{E}_0, \mathbf{B}_0 \in H^{s-1/2}(\mathbb{R}^3; \mathbb{R}^3), \qquad s > \frac{1}{2}$$

- Masmoudi and Nakanishi '03: LWP $s = \frac{1}{2}$ (Coulomb gauge)
- Latter result analogous to Klainerman and Machedon's result for Maxwell-Klein-Gordon (MKG) from 1993 (finite energy well-posedness). But for MD, energy is not positive definite.
- MKG: almost optimal LWP proved by Machedon and Sterbenz '03 (Coulomb gauge)

Nonlinear structure of MD in Lorenz gauge

- Isolate most difficult part: Consider model

$$-i\boldsymbol{\alpha}^\mu \partial_\mu \psi = -\mathcal{N}(\psi,\psi,\psi), \qquad \mathcal{N}(\cdot,\cdot,\cdot) = \left(\frac{1}{\square}\langle \boldsymbol{\alpha}_\mu \cdot, \cdot \rangle_{\mathbb{C}^4}\right) \boldsymbol{\alpha}^\mu \cdot$$

- Diagonalize Dirac operator:

$$-i\boldsymbol{\alpha}^\mu \partial_\mu = -i\partial_t + \underbrace{-i\boldsymbol{\alpha}^j \partial_j}_{=|\nabla|\mathbf{\Pi}_+ - |\nabla|\mathbf{\Pi}_-}$$

where the *Dirac projections* $\mathbf{\Pi}_\pm$ are multipliers

$$\widehat{\mathbf{\Pi}_\pm f}(\xi) = \mathbf{\Pi}(\pm\xi)\widehat{f}(\xi), \qquad \mathbf{\Pi}(\xi) = \frac{1}{2}\left(\mathbf{I}_{4\times 4} + \frac{\xi^j \boldsymbol{\alpha}_j}{|\xi|}\right) \qquad (\xi \in \mathbb{R}^3).$$

- Split $\psi = \psi_+ + \psi_-$ where $\psi_\pm = \mathbf{\Pi}_\pm \psi$.

Nonlinear structure of MD in Lorenz gauge

- Result is system

$$
\begin{aligned}
(-i\partial_t + |\nabla|)\,\psi_+ &= -\mathbf{\Pi}_+\mathcal{N}(\psi,\psi,\psi), \qquad & \psi_+|_{t=0} &= \mathbf{\Pi}_+\psi_0, \\
(-i\partial_t - |\nabla|)\,\psi_- &= -\mathbf{\Pi}_-\mathcal{N}(\psi,\psi,\psi), \qquad & \psi_-|_{t=0} &= \mathbf{\Pi}_-\psi_0,
\end{aligned}
$$

where $\psi = \psi_+ + \psi_-$ and $\mathcal{N}(\cdot,\cdot,\cdot) = \left(\frac{1}{\square}\langle \boldsymbol{\alpha}_\mu \cdot, \cdot\rangle_{\mathbb{C}^4}\right)\boldsymbol{\alpha}^\mu \cdot$.

- Note that $\psi_\pm = \mathbf{\Pi}_\pm \psi$, since $\mathbf{\Pi}_+\mathbf{\Pi}_- = 0$.
- Iterate in space $X_\pm^{s,b}$ with norm

$$
\|u\|_{X_\pm^{s,b}} = \left\| \langle \xi \rangle^s \langle \tau \pm |\xi| \rangle^b \,\widehat{u}(\tau,\xi) \right\|_{L^2_{\tau,\xi}},
$$

where $\widehat{u}(\tau,\xi) =$ F.t. of $u(t,x)$, and $\langle \xi \rangle = (1+|\xi|^2)^{1/2}$.

- For all $\varepsilon > 0$,

$$
X_\pm^{s,\frac{1}{2}+\varepsilon} \hookrightarrow BC(\mathbb{R}; H^s).
$$

Linear estimate in $X_\pm^{s,b}$

- Solution of linear IVP

$$\begin{aligned} (-i\partial_t \pm |\nabla|)\, u &= F \qquad\qquad \text{on } S_T = (-T,T)\times\mathbb{R}^3, \\ u\big|_{t=0} &= u_0, \end{aligned}$$

satisfies

$$\|u\|_{X_\pm^{s,\frac12+\varepsilon}(S_T)} \le C_\varepsilon \|u_0\|_{H^s} + C_\varepsilon T^\varepsilon \|F\|_{X_\pm^{s,-\frac12+2\varepsilon}(S_T)}$$

- Apply to

$$\begin{aligned} (-i\partial_t \pm_4 |\nabla|)\, \psi_{\pm_4} &= -\mathbf{\Pi}_{\pm_4}\mathcal{N}(\psi,\psi,\psi) \\ &= -\sum_{\pm_1,\pm_2,\pm_3} \mathbf{\Pi}_{\pm_4}\mathcal{N}(\psi_{\pm_1},\psi_{\pm_2},\psi_{\pm_3}) \\ &= -\sum_{\pm_1,\pm_2,\pm_3} \mathbf{\Pi}_{\pm_4}\mathcal{N}(\mathbf{\Pi}_{\pm_1}\psi,\mathbf{\Pi}_{\pm_2}\psi,\mathbf{\Pi}_{\pm_3}\psi). \end{aligned}$$

Nonlinear estimate in $X_\pm^{s,b}$

- To close the iteration, thus need following nonlinear estimate:

$$\Big\| \underbrace{\rho(t)}_{\text{cut-off}} \mathbf{\Pi}_{\pm_4}\mathcal{N}(\mathbf{\Pi}_{\pm_1}\psi_1, \mathbf{\Pi}_{\pm_2}\psi_2, \mathbf{\Pi}_{\pm_3}\psi_3)\Big\|_{X_\pm^{s,-\frac{1}{2}+2\varepsilon}} \le C \prod_{j=1,2,3} \|\psi_j\|_{X_{\pm_j}^{s,\frac{1}{2}+\varepsilon}}$$

- In dual form:

$$|I(\psi_1,\psi_2,\psi_3,\psi_4)| \le C \|\psi_1\|_{X_{\pm_1}^{s,\frac{1}{2}+\varepsilon}} \|\psi_2\|_{X_{\pm_2}^{s,\frac{1}{2}+\varepsilon}} \|\psi_3\|_{X_{\pm_3}^{s,\frac{1}{2}+\varepsilon}} \|\psi_4\|_{X_{\pm_4}^{-s,\frac{1}{2}-2\varepsilon}}$$

where

$$\begin{aligned} I(\psi_1,\psi_2,\psi_3,\psi_4) &= \iint \rho \langle \mathcal{N}(\mathbf{\Pi}_{\pm_1}\psi_1, \mathbf{\Pi}_{\pm_2}\psi_2, \mathbf{\Pi}_{\pm_3}\psi_3), \mathbf{\Pi}_{\pm_4}\psi_4\rangle_{\mathbb{C}^4} \,\mathrm{d}t\,\mathrm{d}x \\ &= \iint \rho \frac{1}{\Box} \langle \boldsymbol{\alpha}_\mu \mathbf{\Pi}_{\pm_1}\psi_1, \mathbf{\Pi}_{\pm_2}\psi_2\rangle_{\mathbb{C}^4} \cdot \langle \boldsymbol{\alpha}^\mu \mathbf{\Pi}_{\pm_3}\psi_3, \mathbf{\Pi}_{\pm_4}\psi_4\rangle_{\mathbb{C}^4} \,\mathrm{d}t\,\mathrm{d}x \end{aligned}$$

Pass to Fourier space by Plancherel

- Frequencies $X_j = (\tau_j, \xi_j) \in \mathbb{R}^{1+3}$, $j = 0, 1, 2, 3, 4$:

$$\begin{array}{cccccc} \psi_1 & \psi_2 & \langle \psi_1, \psi_2 \rangle & \psi_3 & \psi_4 & \langle \psi_3, \psi_4 \rangle \\ X_1 & X_2 & X_0 = X_1 - X_2 & X_3 & X_4 & -X_0 = X_3 - X_4 \end{array}$$

- L^2-normalization of spinor-valued $\psi \in X_\pm^{s,b}$:

$$\left|\widehat{\psi}(X)\right| = \frac{F(X)}{\langle \xi \rangle^s \langle \tau \pm |\xi| \rangle^b}, \qquad F \in L^2(\mathbb{R}^{1+3}),\ F \geq 0,$$

$$\widehat{\psi} = z \left|\widehat{\psi}\right|, \qquad z : \mathbb{R}^{1+3} \to \mathbb{C}^4 \text{ meas., } |z| = 1$$

  Apply for index $j = 1, 2, 3, 4$.

- For simplicity replace $\rho_{\frac{1}{\square}}$ by multiplier with Fourier symbol

$$\frac{1}{\langle \xi_0 \rangle \langle \tau_0 \pm_0 |\xi_0| \rangle} \qquad (\pm_0 \text{ arbitrary})$$

Dyadic decomposition

- Assign dyadic sizes to Fourier-weights:

$$\begin{aligned} \langle \xi_j \rangle &\sim N_j, && \text{size of spatial frequency} \\ \langle \tau_j \pm_j |\xi_j| \rangle &\sim L_j, && \text{distance from null cones } (\pm) \end{aligned} \qquad (j = 0, \ldots, 4)$$

where the N's and L's are dyadic numbers ≥ 1.

- Write

$$\begin{aligned} \mathbf{N} &= (N_0, \ldots, N_4), \\ \mathbf{L} &= (L_0, \ldots, L_4), \\ \mathbf{\Sigma} &= (\pm_0, \ldots, \pm_4) \\ \mathbf{X} &= (X_0, \ldots, X_4), \end{aligned}$$

$$\chi_{\mathbf{N},\mathbf{L}}(\mathbf{X}) = \prod_{j=0}^{4} \chi_{\langle \xi_j \rangle \sim N_j} \chi_{\langle \tau_j \pm_j |\xi_j| \rangle \sim L_j}$$

Dyadic decomposition

Thus:

$$|I(\psi_1,\psi_2,\psi_3,\psi_4)| \lesssim \sum_{\mathbf{N},\mathbf{L}} \frac{N_4^s \, J^{\boldsymbol{\Sigma}}_{\mathbf{N},\mathbf{L}}(F_1,F_2,F_3,F_4)}{\underbrace{N_0 L_0}_{\text{from } \rho(t)\frac{1}{\square}} \quad (N_1 N_2 N_3)^s (L_1 L_2 L_3)^{1/2+\varepsilon} L_4^{1/2-2\varepsilon}}$$

where

$$J^{\boldsymbol{\Sigma}}_{\mathbf{N},\mathbf{L}}(F_1,\dots,F_4) = \int \left| q^{\boldsymbol{\Sigma}}(\mathbf{X}) \right| \chi_{\mathbf{N},\mathbf{L}}(\mathbf{X}) F_1(X_1) F_2(X_2) F_3(X_3) F_4(X_4) \, \mathrm{d}\mu(\mathbf{X})$$

and

$$q^{\boldsymbol{\Sigma}}(\mathbf{X}) = \langle \boldsymbol{\alpha}^\mu \mathbf{\Pi}(e_1) z_1(X_1), \mathbf{\Pi}(e_2) z_2(X_2) \rangle \langle \boldsymbol{\alpha}_\mu \mathbf{\Pi}(e_3) z_3(X_3), \mathbf{\Pi}(e_4) z_4(X_4) \rangle,$$

$$e_j = \pm_j \frac{\xi_j}{|\xi_j|} \in \mathbb{S}^2,$$

$$\mathrm{d}\mu(\mathbf{X}) = \delta(X_0 - X_1 + X_2)\delta(X_0 + X_3 - X_4) \, \mathrm{d}X_0 \, \mathrm{d}X_1 \, \mathrm{d}X_2 \, \mathrm{d}X_3 \, \mathrm{d}X_4.$$

Main dyadic estimate

Theorem

Following holds:

$$J^{\mathbf{\Sigma}}_{\mathbf{N},\mathbf{L}} \lesssim N_0 L_0 \left(L_1 L_2 L_3 L_4\right)^{1/2} \log\langle L_0\rangle \prod_{j=1}^{4} \left\| F_j^{\pm_j,N_j,L_j} \right\|,$$

where

$$F_j^{\pm_j,N_j,L_j}(X_j) = \chi_{\langle \xi_j \rangle \sim N_j} \chi_{\langle \tau_j \pm_j |\xi_j| \rangle \sim L_j} F_j(X_j),$$

$$\|\cdot\| = \textit{norm on } L^2(\mathbb{R}^{1+3}).$$

Quadrilinear null structure

- Structure encoded in symbol

$$q(\mathbf{e};\mathbf{z}) = \sum_{\mu=0}^{3} \langle \boldsymbol{\alpha}^{\mu} \boldsymbol{\Pi}(e_1) z_1, \boldsymbol{\Pi}(e_2) z_2 \rangle \langle \boldsymbol{\alpha}_{\mu} \boldsymbol{\Pi}(e_3) z_3, \boldsymbol{\Pi}(e_4) z_4 \rangle,$$

where

$e_j \in \mathbb{S}^2$	represents signed direction of spatial freq. ξ_j
$z_j \in \mathbb{C}^4, \quad \lvert z_j \rvert = 1$	represents the direction of the spinor $\widehat{\psi}_j$

- Denote angles between e_1, e_2, e_3, e_4 on unit sphere by

$$\theta_{jk} = \theta(e_j, e_k)$$

- Six distinct angles:

$\theta_{12}, \ \theta_{34}$	"internal" angles
$\theta_{13}, \ \theta_{14}, \ \theta_{23}, \ \theta_{24}$	"external" angles

Quadrilinear null structure

Set

$$\phi = \text{min. of external angles} = \min\{\theta_{13}, \theta_{14}, \theta_{23}, \theta_{24}\}.$$

Lemma

The symbol

$$q(\mathbf{e};\mathbf{z}) = \sum_{\mu=0}^{3} \langle \boldsymbol{\alpha}^{\mu} \mathbf{\Pi}(e_1) z_1, \mathbf{\Pi}(e_2) z_2 \rangle \langle \boldsymbol{\alpha}_{\mu} \mathbf{\Pi}(e_3) z_3, \mathbf{\Pi}(e_4) z_4 \rangle$$

satisfies

$$|q(\mathbf{e};\mathbf{z})| \lesssim \theta_{12}\theta_{34} + \phi \max(\theta_{12}, \theta_{34}) + \phi^2$$

for all unit vectors $e_1, \dots, e_4 \in \mathbb{R}^3$ *and* $z_1, \dots, z_4 \in \mathbb{C}^4$.

Quadrilinear null structure

Set

$$\phi = \text{min. of external angles} = \min\{\theta_{13}, \theta_{14}, \theta_{23}, \theta_{24}\}.$$

Lemma

The symbol

$$q(\mathbf{e}; \mathbf{z}) = \sum_{\mu=0}^{3} \langle \boldsymbol{\alpha}^\mu \mathbf{\Pi}(e_1) z_1, \mathbf{\Pi}(e_2) z_2 \rangle \langle \boldsymbol{\alpha}_\mu \mathbf{\Pi}(e_3) z_3, \mathbf{\Pi}(e_4) z_4 \rangle$$

satisfies

$$\begin{aligned} |q(\mathbf{e}; \mathbf{z})| &\lesssim \theta_{12}\theta_{34} + \phi \max(\theta_{12}, \theta_{34}) + \phi^2 \\ &\lesssim \theta_{12}\theta_{34} + \theta_{13}\theta_{24} \end{aligned}$$

for all unit vectors $e_1, \ldots, e_4 \in \mathbb{R}^3$ *and* $z_1, \ldots, z_4 \in \mathbb{C}^4$.

Quadrilinear space-time estimate

Some key points in the proof of the main dyadic estimate

$$J^{\mathbf{\Sigma}}_{\mathbf{N},\mathbf{L}} \lesssim N_0 L_0 \left(L_1 L_2 L_3 L_4\right)^{1/2} \log\langle L_0 \rangle \prod_{j=1}^{4} \left\| F_j^{\pm_j, N_j, L_j} \right\|.$$

- Apply null estimate for symbol $q(\mathbf{e};\mathbf{z})$.
- To exploit null estimate, make additional angular decompositions of spatial frequencies $\xi_1, \ldots, \xi_4$, based on dyadic sizes of θ_{jk}.
- Eventually apply Cauchy-Schwarz inequality in various ways to reduce to bilinear L^2 space-time estimates (bilinear Fourier restriction estimates for the cone).
- *Klainerman and Machedon* first investigated L^2 bilinear generalizations of the L^4 estimate of *Strichartz* for the 3D wave equation. Also *Klainerman and Foschi*.
- The "standard" estimates not enough for our purposes; apply a number of modifications (*Anisotropic bilinear L^2 estimates related to the 3D wave equation*, S. '08).

Review of some Fourier restriction results

- Stein-Tomas theorem for the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$
- Strichartz' L^4 estimate for the 3D wave equation (cone restriction)
- Klainerman-Machedon type estimates (L^2 bilinear generalizations of Strichartz' estimate)
- Use following notation: If

$$A \subset \mathbb{R}^n,$$

define multiplier $\mathbf{P}_A$ by

$$\widehat{\mathbf{P}_A u} = \chi_A \widehat{u}.$$

Here $n = 3$ or $n = 1 + 3$, depending on context.

Fourier restriction results: Stein-Tomas

- Fourier restriction from $\mathbb{R}^3$ to $\mathbb{S}^2$:

$$f \longmapsto \widehat{f}|_{\mathbb{S}^2}$$

is bounded map

$$L^p(\mathbb{R}^3) \longrightarrow L^2(\mathbb{S}^2, d\sigma) \qquad \text{iff } 1 \le p \le \frac{4}{3}.$$

- Endpoint $p = \frac{4}{3}$ equivalent to, by duality and approximation of $\mathbb{S}^2$ by thickened spheres,

$$\left\| \mathbf{P}_{S(\varepsilon)} f \right\|_{L^4(\mathbb{R}^3)} \le C\sqrt{\varepsilon} \left\| f \right\|_{L^2(\mathbb{R}^3)},$$

where

$$S(\varepsilon) = \varepsilon\text{-thickening of unit sphere } \mathbb{S}^2.$$

Proof of $\left\|\mathbf{P}_{S(\varepsilon)} f\right\|_{L^4(\mathbb{R}^3)} \le C\sqrt{\varepsilon}\left\|f\right\|_{L^2(\mathbb{R}^3)}$

- Naive attempt: Sobolev type estimate

$$\begin{aligned}
\left\|\mathbf{P}_{S(\varepsilon)} f\right\|_{L^4(\mathbb{R}^3)} &\lesssim \left\|\chi_{S(\varepsilon)} \widehat{f}\right\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} && \text{Hausdorff-Young} \\
&\lesssim |S(\varepsilon)|^{\frac{1}{4}} \left\|\widehat{f}\right\|_{L^2(\mathbb{R}^3)} && \text{Hölder} \\
&\simeq \varepsilon^{\frac{1}{4}} \left\|f\right\|_{L^2(\mathbb{R}^3)} && \text{Plancherel}
\end{aligned}$$

- Correct approach: Bilinear
- First step: Equivalent reformulation

$$\left\|\mathbf{P}_{S(\varepsilon)} f \cdot \mathbf{P}_{S(\varepsilon)} g\right\| \le C\varepsilon \left\|f\right\| \left\|g\right\|$$

where $\|\cdot\| =$ norm on L^2.

Proof of $\left\|\mathbf{P}_{S(\varepsilon)} f \cdot \mathbf{P}_{S(\varepsilon)} g\right\| \leq C\varepsilon \|f\| \|g\|$

Apply general fact:

Lemma

Let $A, B \subset \mathbb{R}^n$ be measurable. Then

$$\|\mathbf{P}_A f \cdot \mathbf{P}_B g\| \leq C_{A,B,n} \|f\| \|g\| \qquad (\forall f, g \in \mathcal{S}(\mathbb{R}^n)),$$

where

$$C_{A,B,n} \sim \sup_{\xi \in A+B} |A \cap (\xi - B)|^{\frac{1}{2}}.$$

Proof of $\left\|\mathbf{P}_{S(\varepsilon)} f \cdot \mathbf{P}_{S(\varepsilon)} g\right\| \leq C\varepsilon \|f\| \|g\|$

- Thus, reduce Stein-Tomas to volume estimate

$$|S(\varepsilon) \cap (\xi + S(\varepsilon))| \lesssim \varepsilon^2. \qquad (*)$$

- Fails in concentric case $\xi \to 0$, but since we started with a linear estimate, may use partition of unity and replace $S(\varepsilon)$ by, say,

$$S(\varepsilon) \cap \{\text{first octant}\}.$$

  Then only need $(*)$ for $|\xi| \sim 1$, so OK.

- In general: Let $S_r(\delta) = \delta$-thickening of sphere of radius r in $\mathbb{R}^3$.

Lemma

$$|S_r(\delta) \cap (\xi + S_R(\Delta))| \lesssim \frac{Rr\delta\Delta}{|\xi|} \qquad (\forall \xi \in \mathbb{R}^3).$$

Fourier restriction results: Strichartz

- Analogous to Stein-Tomas, but for *null cone* in $1+3$ dimensions, i.e., characteristic cone of 3D wave equation:

$$K = K^+ \cup K^-, \qquad K^\pm = \{(\tau,\xi) \in \mathbb{R}^{1+3} \colon \tau = \pm|\xi|\}$$

  Note: Slices $\tau =$ const are 2-spheres.

- Define truncated, thickened cones:

$$K^\pm_{N,L} = \{(\tau,\xi) \in \mathbb{R}^{1+3} \colon \ |\xi| \sim N, \ \tau = \pm|\xi| + O(L)\}$$

- Equivalent formulation of Strichartz' estimate:

$$\left\| \mathbf{P}_{K^\pm_{N,L}} u \right\|_{L^4(\mathbb{R}^{1+3})} \le C\sqrt{NL}\, \|u\|_{L^2(\mathbb{R}^{1+3})}.$$

- Compare Sobolev type estimate:

$$\left\| \mathbf{P}_{K^\pm_{N,L}} u \right\|_{L^4(\mathbb{R}^{1+3})} \le C\left(N^3 L\right)^{\frac{1}{4}} \|u\|_{L^2(\mathbb{R}^{1+3})}.$$

Fourier restriction results: Klainerman-Machedon

- First note obvious bilinear L^2 formulation of Strichartz' estimate:

$$\left\| \mathbf{P}_{K^{\pm_1}_{N_1,L_1}} u_1 \cdot \mathbf{P}_{K^{\pm_2}_{N_2,L_2}} u_2 \right\| \le C \sqrt{N_1 N_2 L_1 L_2} \, \|u_1\| \, \|u_2\| \, .$$

Here $\|\cdot\|$ is norm on $L^2(\mathbb{R}^{1+3})$.

- But *bilinear is better*: Can replace $N_1 N_2$ by square of

$$N^{12}_{\min} = \min(N_1, N_2).$$

- More generally: restrict spatial output frequency ξ_0 to a ball

$$B_{N_0} = \{\xi_0 \in \mathbb{R}^3 \colon \ |\xi_0| \le N_0\}.$$

Then

$$\left\| \mathbf{P}_{B_{N_0}} \left(\mathbf{P}_{K^{\pm_1}_{N_1,L_1}} u_1 \cdot \mathbf{P}_{K^{\pm_2}_{N_2,L_2}} u_2 \right) \right\| \le C \sqrt{N^{012}_{\min} N^{12}_{\min} L_1 L_2} \, \|u_1\| \, \|u_2\| \, .$$

Klainerman-Machedon estimates

- Symmetrized form

$$\left\| \mathbf{P}_{K^{\pm_0}_{N_0,L_0}} \left(\mathbf{P}_{K^{\pm_1}_{N_1,L_1}} u_1 \cdot \mathbf{P}_{K^{\pm_2}_{N_2,L_2}} u_2 \right) \right\| \le C \sqrt{N^{012}_{\min} N^{012}_{\max} L^{012}_{\min} L^{012}_{\mathrm{med}}} \, \|u_1\| \, \|u_2\| \, .$$

- Remark: Spatial frequencies satisfy

$$\begin{array}{ccccc} \xi_0 & = & \xi_1 & + & \xi_2 \\ u_1 u_2 & & u_1 & & u_2 \end{array}$$

Implies that *two largest frequencies always comparable in size.*

- In particular,

$$N^{012}_{\min} N^{012}_{\max} \sim N_0 N^{12}_{\min} .$$

Bilinear null forms

- Standard product of $f = f(x)$, $g = g(x)$ ($x \in \mathbb{R}^3$) has F.t.

$$\widehat{fg}(\xi_0) \simeq \iint \widehat{f}(\xi_1)\widehat{g}(\xi_2)\,\delta(\xi_0 - \xi_1 - \xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2.$$

- Given signs $\pm_1, \pm_2$, define *bilinear null form*

$$\mathfrak{B}_\theta^{\pm_1,\pm_2}(f,g)$$

by inserting angle

$$\theta(\pm_1\xi_1, \pm_2\xi_2)$$

in above convolution formula:

$$\widehat{\mathfrak{B}_\theta^{\pm_1,\pm_2}(f,g)}(\xi_0) \simeq \iint \theta(\pm_1\xi_1, \pm_2\xi_2)\widehat{f}(\xi_1)\widehat{g}(\xi_2)\,\delta(\xi_0 - \xi_1 - \xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2.$$

Bilinear null forms

- Replacing standard product by a null form improves the bilinear space-time estimates of Klainerman-Machedon type.
- Why? Consider space-time bilinear interaction

$$\begin{array}{ccccc} X_0 & = & X_1 & + & X_2 \\ K^{\pm_0}_{N_0,L_0} & & K^{\pm_1}_{N_1,L_1} & & K^{\pm_1}_{N_2,L_2} \\ u_1 u_2 & & u_1 & & u_2 \end{array}$$

I.e., frequencies $X_j = (\tau_j, \xi_j) \in \mathbb{R}^{1+3}$ restricted by

$\lvert\xi_j\rvert \sim N_j$	N's "elliptic" weights
$\tau_j = \pm_j \lvert\xi_j\rvert + O(L_j)$	L's "hyperbolic" weights

- Null interaction: All hyperbolic weights vanish, i.e.,

$$L_0 = L_1 = L_1 = 0$$

(or all small).

Null interaction

- Null interaction is "worst interaction".
- Why? Consider model problem (iteration)

$$\Box v = u_1 u_2 \qquad \text{(zero initial data)},$$

where u_1, u_2 given. Study regularity of v.

- After dyadic decomposition, roughly

$$\Box \approx N_0 L_0.$$

Worse regularity for v when L_0 small.

- Previous iterates u_1, u_2: worse regularity when L_1, L_2 small.
- Absolute worst: All L'small (compared to N's).

Null interaction: Improvement with null form

- Extreme case

$$L_0 = L_1 = L_1 = 0$$

  Then X_0, X_1, X_2 all lie on null cone.

- But

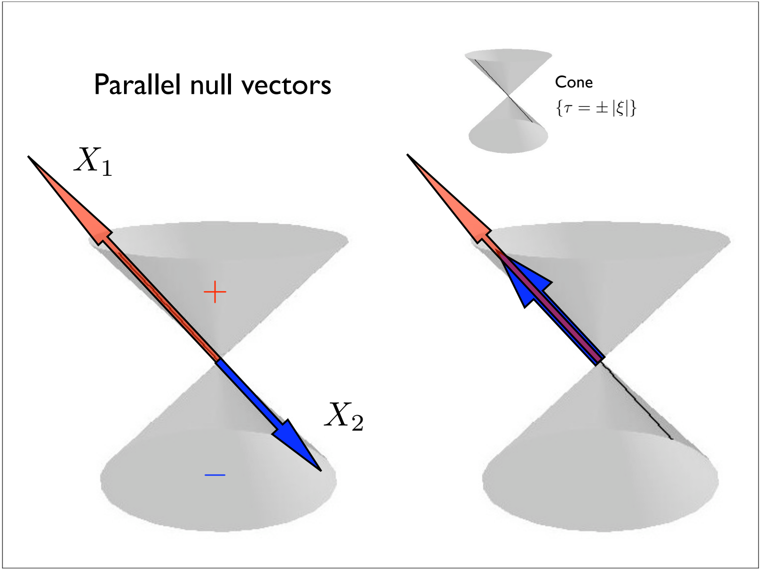
$$X_0 = X_1 + X_2,$$

  so only way X_0 can end up on cone is if X_1, X_2 collinear. Even more:

$$\theta(\pm_1\xi_1, \pm_2\xi_2) \qquad \text{must vanish}.$$

- Hence null form better than standard product.

Null interaction



Null form estimate

- In general:

$$\theta(\pm_1\xi_1, \pm_2\xi_2) \lesssim \sqrt{\frac{L_{\max}^{012}}{N_{\min}^{12}}}$$

- Recall bilinear estimate:

$$\left\| \mathbf{P}_{K_{N_0,L_0}^{\pm_0}} \left(\mathbf{P}_{K_{N_1,L_1}^{\pm_1}} u_1 \cdot \mathbf{P}_{K_{N_2,L_2}^{\pm_2}} u_2 \right) \right\| \le C \sqrt{N_0 N_{\min}^{12} L_{\min}^{012} L_{\mathrm{med}}^{012}} \left\| u_1 \right\| \left\| u_2 \right\| .$$

- Combine to give *null form estimate*

$$\left\| \mathbf{P}_{K_{N_0,L_0}^{\pm_0}} \mathfrak{B}_{\theta}^{\pm_1,\pm_2} \left(\mathbf{P}_{K_{N_1,L_1}^{\pm_1}} u_1, \mathbf{P}_{K_{N_2,L_2}^{\pm_2}} u_2 \right) \right\| \le C \sqrt{N_0 L_0 L_1 L_2} \left\| u_1 \right\| \left\| u_2 \right\| .$$

Application to MD: The easy case

- Recall: $|q(\mathbf{e};\mathbf{z})| \lesssim \underbrace{\theta_{12}\theta_{34}}_{\text{easy part}} + \underbrace{\phi \max(\theta_{12},\theta_{34}) + \phi^2}_{\text{hard part}}$
- Consider easy part. Then by Cauchy-Schwarz inequality, estimate

$$J^{\mathbf{\Sigma}}_{\mathbf{N},\mathbf{L}}(F_1,\dots,F_4) = \int \left| q^{\mathbf{\Sigma}}(\mathbf{X}) \right| \chi_{\mathbf{N},\mathbf{L}}(\mathbf{X}) F_1(X_1) F_2(X_2) F_3(X_3) F_4(X_4)\,\mathrm{d}\mu$$

$$\lesssim \left\| \int \chi_{K^{\pm_0}_{N_0,L_0}}(X_0)\theta_{12} F_1^{\pm_1,N_1,L_1}(X_1) F_2^{\pm_2,N_2,L_2}(X_2)\delta_{X_0-X_1+X_2}\,\mathrm{d}X_1\,\mathrm{d}X_2 \right\|_{L^2_{X_0}}$$

$$\times \left\| \int \chi_{K^{\pm_0}_{N_0,L_0}}(X_0)\theta_{34} F_3^{\pm_3,N_3,L_3}(X_3) F_4^{\pm_4,N_4,L_4}(X_4)\delta_{X_0+X_3-X_4}\,\mathrm{d}X_3\,\mathrm{d}X_4 \right\|_{L^2_{X_0}}$$

$$\lesssim \sqrt{N_0L_0L_1L_2}\sqrt{N_0L_0L_3L_4}\prod_{j=1}^{4}\left\| F_j^{\pm_j,N_j,L_j} \right\|$$

as desired.