Progress on orbiting particles in a Kerr background

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Capra 15

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I. Intro

II. Summary of EMRI results in a Kerr spacetime

   A. Dissipative (“adiabatic”) approximation (only dissipative part of self-force used)

   B. Full self force for scalar particle

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   A. Mode-sum renormalization

   B. $\Delta\,\Omega$, $\Delta\, u^t$ for circular orbit

   C. Self-force (not yet completed)

I. Intro

The departure from geodesic motion to order m/M has two parts:

Dissipative part associated with the loss of energy to gravitational waves,

odd under ingoing $\longleftrightarrow$ outgoing

Conservative part

even under ingoing $\longleftrightarrow$ outgoing

The dissipative part of self force plays the dominant role and is much easier to handle:

The part of the field odd under ingoing ⟷ outgoing is

$\frac{1}{2}(h_{retarded} - h_{advanced})$.

Because $h_{retarded}$ and $h_{advanced}$ have the same source, the odd combination is sourcefree and regular at the particle.

The conservative part of the force, is computed from

$\frac{1}{2}(h_{retarded} + h_{advanced})$

a field singular at the particle. One must renormalize the field.

II. Summary of EMRI results in a Kerr spacetime

A. Dissipative ("adiabatic") approximation: only dissipative part of self-force used

Method and discussion:
Mino '05,
Drasco, Flanagan, Hughes '05,
Pound, Poisson, Nickel '05
Hinderer, Flanagan '08

Point-mass computations with only dissipative part of self force are well in hand:

Kennefick, Ori '06
Drasco, Flanagan, Hughes, Franklin 05, 06
Ganz, Hikida, Nakano, Sago, Tanaka 06, 07
Burko, Khanna 07
Mino 08…

Review: T. Tanaka, Prog. Theor. Phys. Suppl. 163, 120 (2006) [arXiv:gr-qc/0508114].

Sundararajan, Khanna, Hughes, Drasco '08

Orbit constructed as set of short geodesics:

Using black hole perturbation theory compute the evolution of three constants of geodesic motion, *E(t)*, $L_z(t)$, and *Q(t)*.

Choose initial conditions and find the inspiral trajectory [$r(t)$, $\theta(t)$, $\phi(t)$] .

From this trajectory, find EMRI waveform.

http://gmunu.mit.edu/viz/emri_viz/emri_viz.html

Drasco movie: orbit with a/M = 0.9, initial eccentricity = 0.7, inclined at 60^o to equatorial plane

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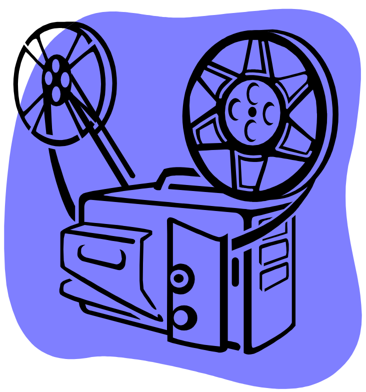


Drasco movie: orbit with a/M = 0.9, initial eccentricity = 0.7, inclined at 60^{o} to equatorial plane

B. Full self-force for scalar particles

Computations in Kerr background that include conservative part of self-force for a particle with scalar charge:

Static Ottewill, Taylor ’12

Circular orbits Warburton, Barack ’10 (frequency-domain)

Dolan, Barack, Wardell ’11 (time-domain)

Eccentric orbits Warburton, Barack ’11 (frequency-domain)

C. Massive particles in circular orbit

Perturbed metric renormalized, quantities $\Delta\Omega$ and Δu^t, invariant under helically symmetric gauge transformations computed.

Shah, JF, Keidl
Dolan

Self-force in progress . . .

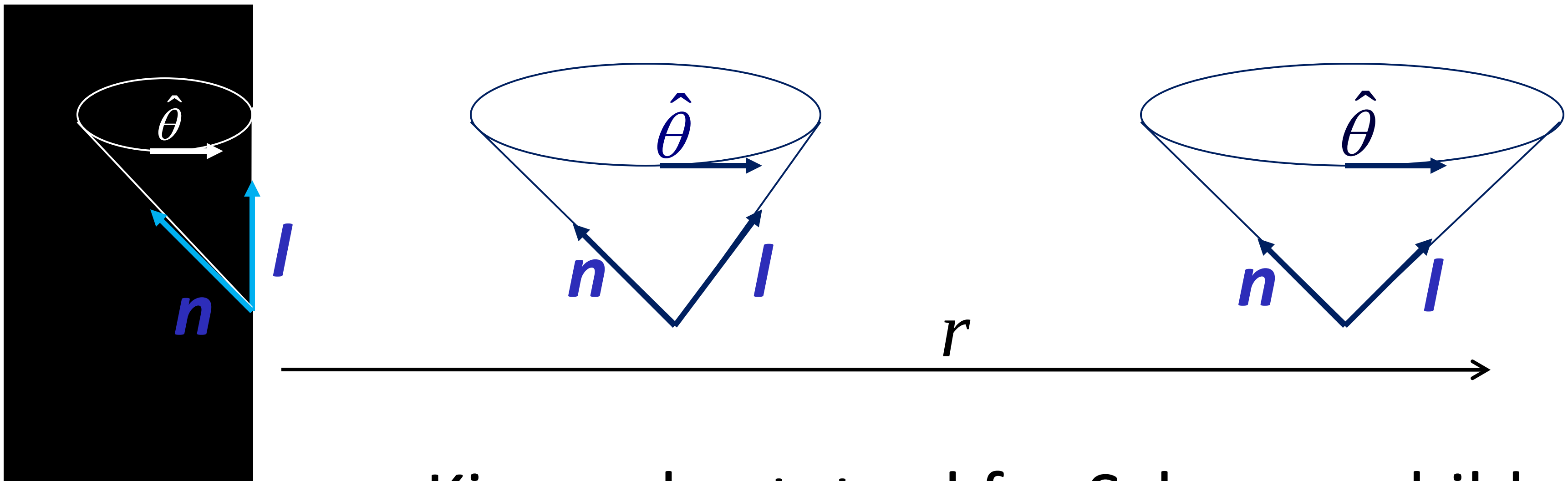
III. Review of method computing self-force for Kerr in a radiation gauge

A single complex Weyl scalar, either ψ_0 or ψ_4 , determines gravitational perturbations of a Kerr geometry (outside perturbative matter sources) up to changes in mass, angular momentum, and change in the center of mass.

ψ_0 and ψ_4 are each a component of the perturbed Weyl tensor along a tetrad associated with the two principal null directions of the spacetime. Each satisfies a separable wave equation, the Teukolsky equation for that component.

Newman-Penrose Formalism

Null tetrad $\quad l^{\alpha}, n^{\alpha}, m^{\alpha}, \overline{m}^{\alpha}$



e.g.,Kinnersley tetrad for Schwarzschild

$$l^{\alpha} = \frac{r^2}{\Delta} t^{\alpha} + r^{\alpha} \qquad n^{\alpha} = \frac{1}{2}\left(t^{\alpha} - \frac{\Delta}{r^2} r^{\alpha}\right)$$

$$m^{\alpha} = \frac{1}{\sqrt{2}}\left(\hat{\theta}^{\alpha} + i\hat{\phi}^{\alpha}\right) \qquad \bar{m}^{\alpha} = \frac{1}{\sqrt{2}}\left(\hat{\theta}^{\alpha} - i\hat{\phi}^{\alpha}\right)$$

$$\Delta = r^2 - 2Mr$$

$$\psi_0 = -\delta C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta \qquad \psi_4 = -\delta C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta$$

Teukolsky equation: $\mathcal{O}_s \psi = S$

$$\mathcal{O}_s = \left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2}{\partial t^2} - 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial}{\partial t} + \frac{4Mar}{\Delta}\frac{\partial^2}{\partial t\partial\phi} - \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial}{\partial r}\right)$$
$$-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial}{\partial\phi}$$
$$+\left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2}{\partial\phi^2} + (s^2\cot^2\theta - s)$$

2nd-order differential operator

Source function $S = \mathcal{T}^{\alpha\beta} T_{\alpha\beta}$, Solution:

ψ_0 is a sum over angular and time harmonics of the form

$$\psi_{0\ell m\omega} = {}_2R_{\ell m\omega}(r)\ \underline{{}_2S_{\ell m\omega}(\theta)\, e^{i(m\phi-\omega t)}}$$

spin-weighted spheroidal harmonic

ψ_0 involves 2 derivatives of the metric perturbation $h_{\alpha\beta}$

To recover the metric from ψ_0 involves 2 net integrations. The method is due to Chrzanowski and Cohen & Kegeles, with a clear and concise derivation by Wald.

First integrate 4 times to obtain a potential Ψ, the Hertz potential.

Then take two derivatives of Ψ to find $h_{\alpha\beta}$

The resulting metric is in a *radiation gauge*.

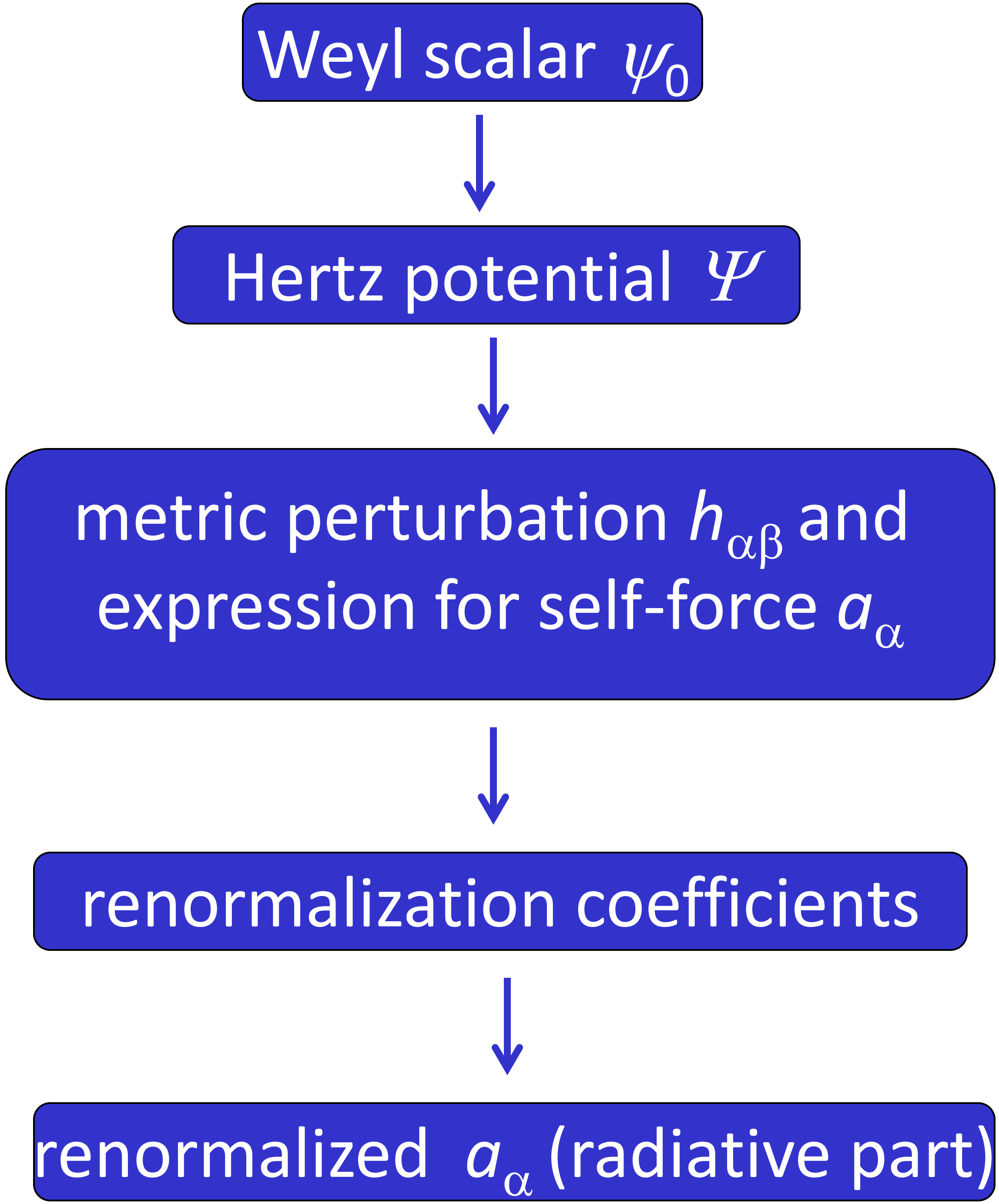
• Outgoing Radiation Gauge (ORG)

$$h_{\alpha\beta} n^{\beta} = 0 \qquad h = 0$$

5 constraints, similar to those for *ingoing* waves in flat space with a transverse-tracefree gauge. The metric perturbation satisfying these conditions is given by

$$h_{\alpha\beta} = L_{\alpha\beta} \Psi$$

where $L_{\alpha\beta}$ is a 2nd-order differential operator involving only ð (angular derivative operator) and ∂_t



Weyl scalar ψ_0
Hertz potential Ψ
metric perturbation $h_{\alpha\beta}$ and expression for self-force a_α
renormalization coefficients
renormalized a_α (radiative part)

Weyl scalar ψ_0

Compute ψ_0^{ret} from the Teukolsky equation as a mode sum over l, m, ω.

Weyl scalar ψ_0

↓

Hertz potential Ψ

For vacuum:

Find the Hertz potential Ψ^{ret} from ψ_0^{ret} or ψ_4^{ret} either algebraically from angular equation or as a 4 radial integrals from the radial equation.

The angular harmonics of ψ_0^{ret} and ψ_4^{ret} are defined for $r>r_0$ or $r<r_0$, with r_0 the radial coordinate of the particle.

Explicitly,

$$\frac{1}{8}\left[\left(\eth - ia\sin\theta\,\partial_t\right)^4 \overline{\Psi} + 12M\partial_t \Psi\right] = \psi_0$$

Integrate 4 times with respect to θ

Algebraic solution for vacuum, valid for circular orbit:

For each frequency and angular harmonic

$$\Psi_{\ell m\omega} = 8\frac{(-1)^m D\,\overline{\psi}_{0\ell -m-\omega} + 12iM\omega\psi_{0\ell m\omega}}{D^2 + 144M^2\omega^2}$$

Equivalent alternative involves radial derivatives along principal null geodesics:

$$\psi_0 = (l^\mu \partial_\mu)^4 \Psi = \partial_r{}^4 \Psi(u, r, \theta, \widetilde{\phi})$$

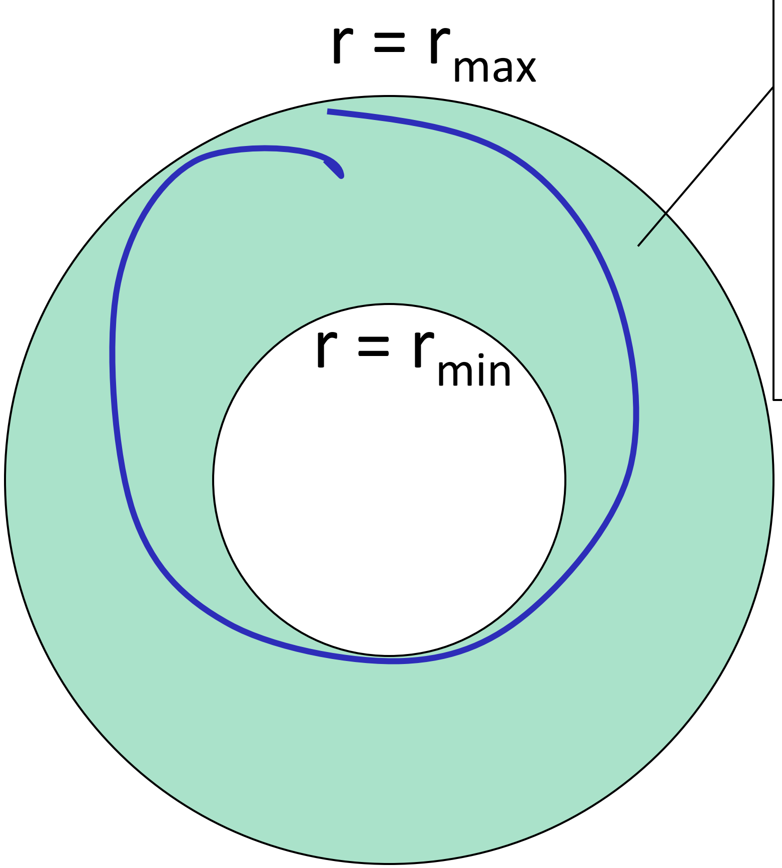
For each angular harmonic of ψ_0, this gives a unique solution satisfying the Teukolsky equation: e.g., for $r > r_0$,

$$\psi_0 = (l^\mu \partial_\mu)^4 \Psi = \partial_r{}^4 \Psi(u, r, \theta, \widetilde{\phi})$$

$$\Psi = \int_r^\infty dr_1 \int_{r_1}^\infty dr_2 \int_{r_2}^\infty dr_3 \int_{r_3}^\infty dr_4 \psi_0(u, r_4, \theta, \widetilde{\phi}).$$

(Kerr coordinates: $\tilde{\phi} = \phi + \int_r^\infty \frac{dr}{\Delta}$)

When the orbit is not circular, one cannot use the algebraic method to find Ψ near the particle. Inside the spherical shell between $r_{\min}$ and $r_{\max}$, $\psi_{0lm\omega}$ has a nonzero source and the vacuum algebraic relation fails:



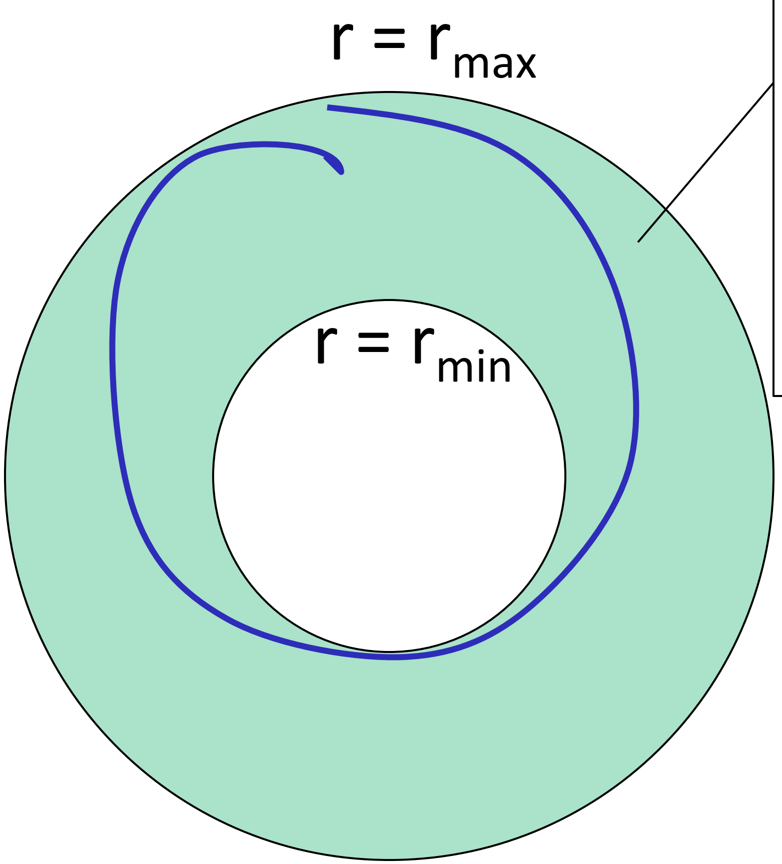
$\psi_0 = \sum \psi_{0lm}$ sourcefree

ψ_{0lm} has a source

$\Psi \neq \sum \Psi_{lm\omega}$ algebraic

Radial integration commutes with decomposition into spherical harmonics: Can use $\Psi_{lm\omega}$ near the particle if computed by radial integration:

$$\Psi_{lm\omega} = \int_r^\infty dr_1 \int_{r_1}^\infty dr_2 \int_{r_2}^\infty dr_3 \int_{r_3}^\infty dr_4 \psi_{0lm\omega}$$



$$\psi = \sum \psi_{0lm\omega} \Rightarrow$$

$$\Psi = \sum \Psi_{lm\omega}$$

Weyl scalar ψ_0

↓

Hertz potential Ψ

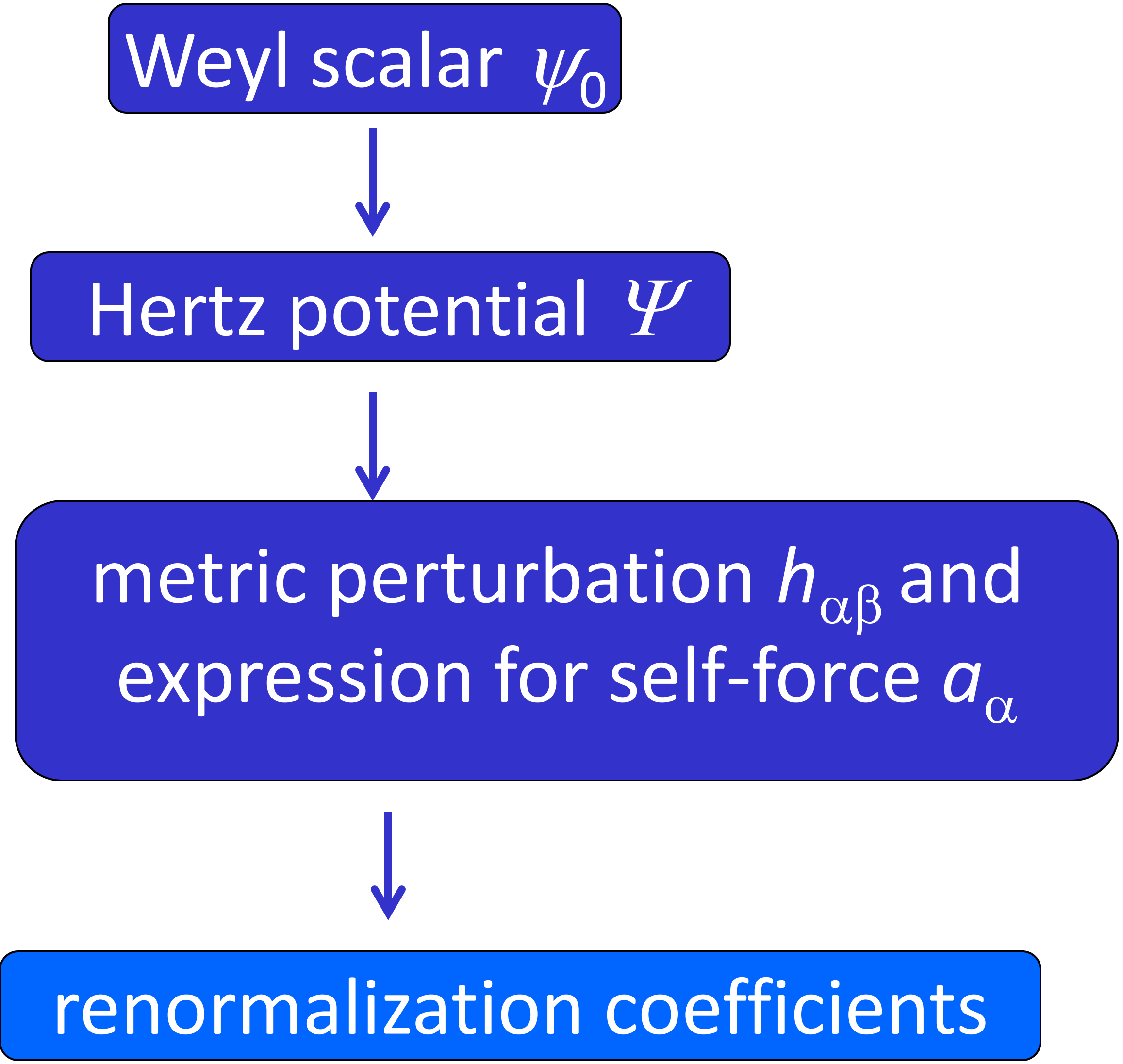
↓

metric perturbation $h_{\alpha\beta}$ and expression for self-force a_α

Find, in a radiation gauge, the components of $h_{\alpha\beta}^{\rm ret}$ and its derivatives that occur in the expression for a^α by taking derivatives of $\Psi^{\rm ret}$.

e.g.:

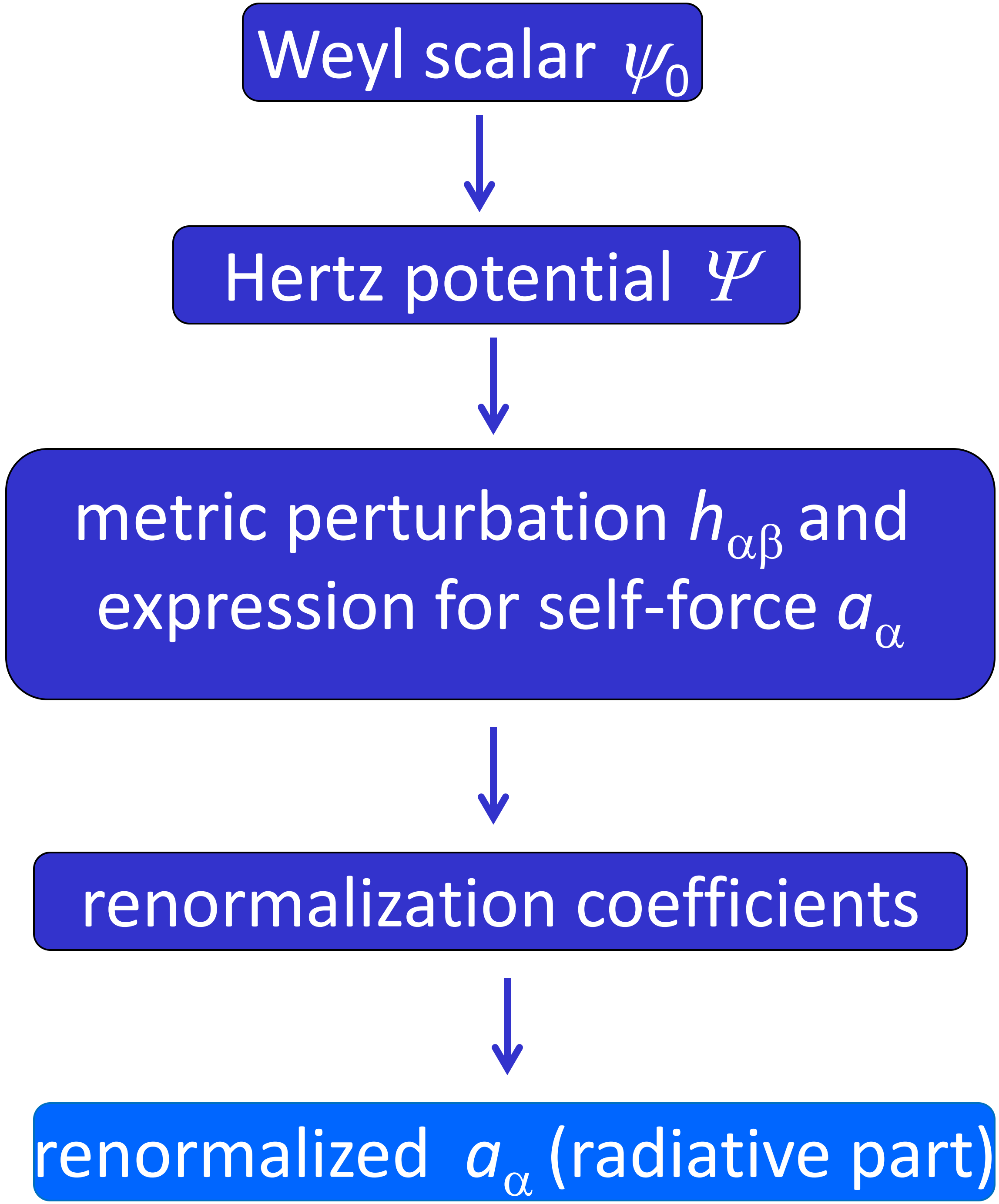
$$h_{\alpha\beta} m^\alpha m^\beta \propto (n\cdot\partial + \Gamma)(n\cdot\partial + \Gamma)\Psi$$



Weyl scalar ψ_0
Hertz potential Ψ
metric perturbation $h_{\alpha\beta}$ and expression for self-force a_α
renormalization coefficients

renormalization coefficients

Compute $a_\ell^{\mathrm{ret}\,\alpha}$ from the perturbed geodesic equation as a mode sum truncated at $\ell_{\max}$. Compute the renormalization vectors A^a and B^a (and C^a ?), numerically matching a power series in to the values of $a_\ell^{\mathrm{ret}\,\alpha}$. (Shah et al)


Weyl scalar ψ_0
Hertz potential Ψ
metric perturbation $h_{\alpha\beta}$ and expression for self-force a_α
renormalization coefficients
renormalized a_α (radiative part)

renormalized a_α (radiative part)

Subtract singular part of expression mode-by-mode

$$a_\ell^{\mathrm{ren}\,\alpha} = a_\ell^{\mathrm{ret}\,\alpha} - \left(A^\alpha L + B^\alpha + \frac{C^\alpha}{L}\right)$$

$$a^{\mathrm{ren}\,\alpha} = \lim_{\ell_{\max}\to\infty} \sum_{\ell=0}^{\ell_{\max}} a_\ell^{\mathrm{ren}\,\alpha}$$

Shah uses $l_{max} = 75$

The missing pieces

ψ_0 and ψ_4 do not determine the full perturbation:
Spin-weight 0 and 1 pieces undetermined.

There are algebraically special perturbations of Kerr,
perturbations for which ψ_0 and ψ_4 vanish:
changing mass δm

changing angular momentum δJ
(and singular perturbations –
to C-metric and to Kerr-NUT).

And gauge transformations

$h^{\mathrm{ret}}_{\alpha\beta}[\psi_0]$ via CCK procedure

$h^{\mathrm{ret}}_{\alpha\beta}[\delta m]$ from the conserved current associated with the background Killing vector t^α.

$h^{\mathrm{ret}}_{\alpha\beta}[\delta J]$ from the conserved current associated with the background Killing vector ϕ^α, for the part of $\delta\boldsymbol{J}$ along background $\boldsymbol{J}$.

(L. Price)

$$h_{\alpha\beta}[\delta m], \quad h_{\alpha\beta}[\delta J]$$

$$j_{(t)}{}^{\alpha} = \delta(2T^{\alpha}{}_{\beta} - \delta^{\alpha}_{\beta} T)t^{\beta} \qquad j_{(\phi)}{}^{\alpha} = -\delta T^{\alpha}{}_{\beta}\phi^{\beta}$$

Background $T^{\alpha}{}_{\beta} = 0 \quad \Rightarrow$

$$\nabla_{\alpha} j_{(t)}{}^{\alpha} = 0, \qquad \nabla_{\alpha} j_{(\phi)}{}^{\alpha} = 0$$

$$\delta m = \int j_{(t)}{}^{\alpha} dS_{\alpha} \qquad \delta J = -\int j_{(\phi)}{}^{\alpha} dS_{\alpha}$$

$$= m(2u^{\alpha}\nabla_{\alpha} t - \frac{1}{u_{\alpha} t^{\alpha}}) \qquad = -m u_{\alpha}\phi^{\alpha}$$

This is enough to compute the self-force-induced change in two related quantities, a change invariant under gauge transformations generated by helically symmetric gauge vectors:

$\Delta U = \Delta u^t$ at fixed Ω

$\Delta\Omega$ (at fixed u^t)

Each computable in terms of $h^{ren}{}_{\alpha\beta}\, u^\alpha\, u^\beta$

$\Delta\Omega$ for circular orbits in a Kerr background

a < 0 counter-rotating

a > 0 corotating

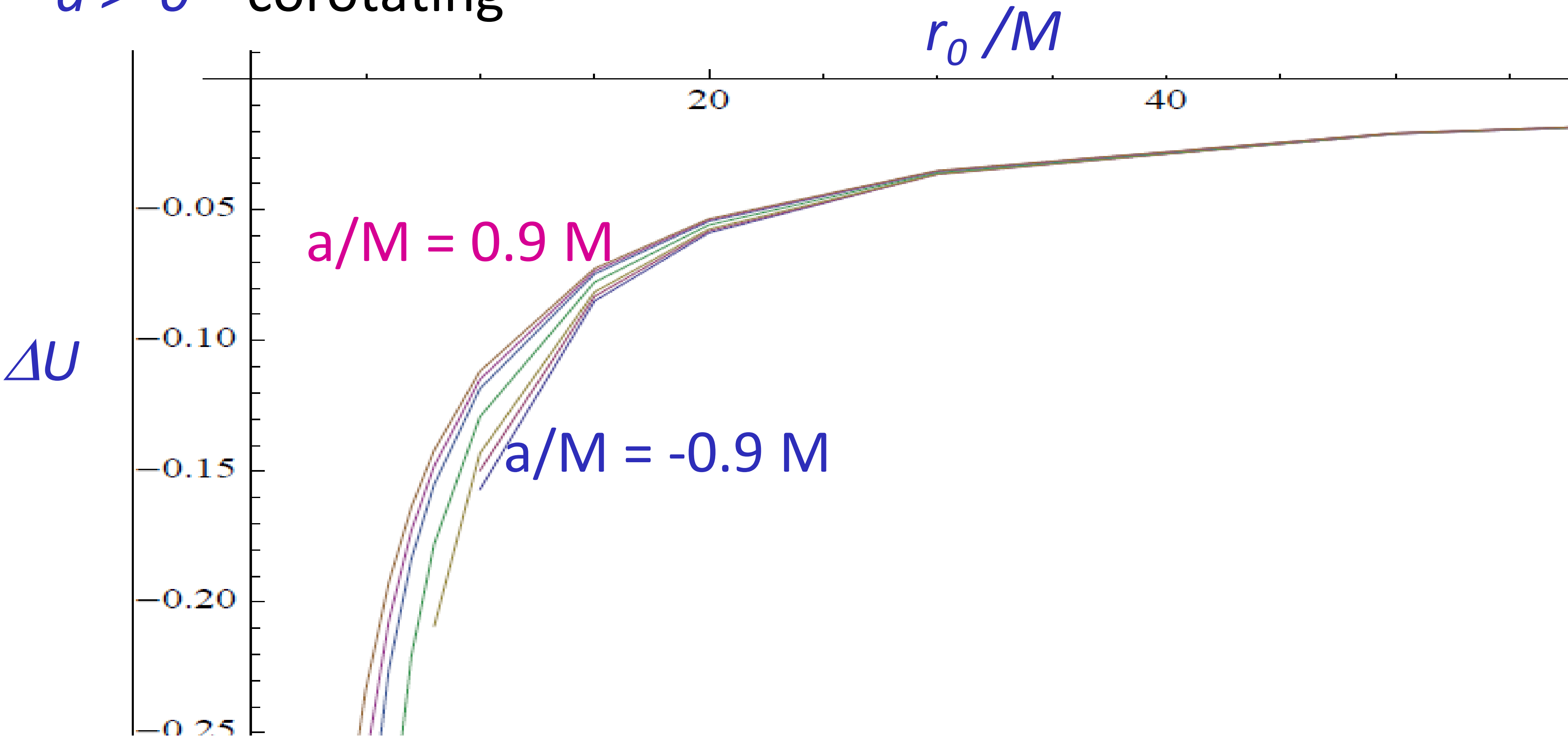
r_0/M	$a = -0.9M$	$a = -0.7M$	$a = -0.5M$	$a = 0.0M$	$a = 0.5M$	$a = 0.7M$	$a = 0.9M$
4	-	-	-	-	-	0.049494757	0.047064792
5	-	-	-	-	0.045714221	0.044118589	0.043175673
6	-	-	-	0.042727891	0.039444628	0.038657945	0.038163269
7	-	-	-	0.036056740	0.034230599	0.033772187	0.033467250
8	-	-	-0.032654832	0.031046361	0.029912954	0.029617108	0.029410780
10	-0.025452677	-0.025047514	-0.024678134	0.023913779	0.023380440	0.023232381	0.023121616
15	-0.014748048	-0.014648207	-0.014556074	0.014359915	0.014213208	0.014168481	0.014131741
20	-0.0099345954	-0.0098961562	-0.0098603936	0.0097828022	0.0097222383	0.0097028068	0.0096861192
30	-0.0055402445	-0.0056086307	-0.0055989040	0.0055772872	0.0055595452	0.0055535368	0.0055481511
50	-0.0026950345	-0.0026929863	-0.0026910361	0.0026865907	0.0026827611	0.0026814019	0.0026801414
70	-0.0016493214	-0.0016486061	-0.0016479203	0.0016463355	0.0016449360	0.0016444281	0.0016439500
100	-0.00097594981	-0.00097571320	-0.00097548493	0.00097495060	0.00097446889	0.00097429076	0.00097412099

TABLE IV: This table presents the numerical values of $\Delta\Omega$ for different values of r_0/M and a.

ΔU for circular orbits in a Kerr background

a < *0* counter-rotating

a > *0* corotating



Comparisons underway with Alexandre Le Tiec (PN) and Sam Dolan (time-domain calculation).

To find the self-force itself, one needs two final pieces:

$h^{\mathrm{ret}}_{\alpha\beta}[\delta J_{\perp}]$ the part of $\delta \boldsymbol{J}$ orthogonal to the background $\boldsymbol{J}$

$h^{\mathrm{ret}}_{\alpha\beta}[CM]$ the change in the center of mass

Each is pure gauge outside the source, but the gauge transformation is discontinuous across the source.

2^{o} $$h_{\alpha\beta}[\delta J_{\perp}], \quad h_{\alpha\beta}[CM]$$

If they are pure gauge, how can they have a source?

$h^{g}_{\alpha\beta} = \pounds_{\xi} g_{\alpha\beta}\, \Theta(r - r_0)$ is not pure gauge at $r=r_0$

($h^{g}_{\alpha\beta} = \pounds_{\xi\Theta(r-r_0)} g_{\alpha\beta}$ is pure gauge)

For Schwarzschild these are $l=1$ perturbations, with axial and polar parity, respectively.

How do we identify them in Kerr?

The idea is to find the part of the source that has not contributed to $h^{\rm ret}_{\alpha\beta}[\psi]+h_{\alpha\beta}[\delta m]+h_{\alpha\beta}[\delta J]$

One could in principle simply subtract from $\delta T^{\alpha\beta}$ the contribution from these three terms. Writing

$$\mathscr{E} h_{\alpha\beta} := \delta G_{\alpha\beta}$$

we have

$$\mathcal{E} h^{\mathrm{ret}}{}_{\alpha\beta} = 8\pi\delta T_{\alpha\beta},$$

$$8\pi\delta T^{\mathrm{remaining}}_{\alpha\beta} = 8\pi\delta T_{\alpha\beta} - \mathcal{E}\left(h^{\mathrm{ret}}[\psi] + h[\delta m] + h[\delta J]\right)_{\alpha\beta}$$

Find ξ at r_0 from the jump condition

$$\int_{r_0-\varepsilon}^{r_0+\varepsilon} (\mathcal{E}\, h^{\mathrm{gauge}})_{\alpha\beta} = \int_{r_0-\varepsilon}^{r_0+\varepsilon} 8\pi\delta T^{\mathrm{remaining}}_{\alpha\beta}$$

For h^{gauge} continuous, the jump in $\mathcal{E}\, h^{\mathrm{gauge}}$ involves only the few terms in $\mathcal{E}$ with second derivatives in the radial direction orthogonal to u^{α}.

But

Now we're back to the old difficulty of handling terms that are singular at the particle.

Instead of trying directly to evaluate

$$8\pi\delta T_{\alpha\beta} - \mathscr{E}\left(h^{\rm ret}[\psi] + h[\delta m] + h[\delta J]\right)_{\alpha\beta}$$

use the fact that $h^{\rm sing}$ has source $\delta T_{\alpha\beta}$:

$$\mathscr{E} h^{\rm gauge}{}_{\alpha\beta} = \mathscr{E}\left(h^{\rm sing}\right)_{\alpha\beta} - \mathscr{E}\left(h^{\rm ret}[\psi] + h[\delta m] + h[\delta J]\right)_{\alpha\beta}$$

$$\int_{r_0-\varepsilon}^{r_0+\varepsilon} \mathscr{E} h^{\rm gauge}{}_{\alpha\beta} = -\int_{r_0-\varepsilon}^{r_0+\varepsilon} \mathscr{E}\left(h^{\rm ren}[\psi] + h[\delta m] + h[\delta J]\right)_{\alpha\beta}$$

• Future problems: Key problems involving conservative part of self-force are not yet done

- Self-force on particle in circular orbit in Kerr (underway in modified radiation gauge and Lorenz gauge) and orbital evolution.
- Self-force on particle in generic orbit in Kerr and orbital evolutions.
- Identify and include relevant 2^{nd}-order corrections. Include particle spin (some calculations already done).
- In our (Abhay Shah's) mode-sum computation, form of singular field agrees with Lorenz. Why? (What happens to a logarithmic divergence of the gauge vector at the position of the particle?)
- Analytically find renormalization coeffs in radiation gauge.