Wave propagation and caustics in curved spacetimes

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Wave equations

Many fields in classical physics are modelled as satisfying wave equations of various kinds:

$$-\partial_t^2 \psi + c^2 \nabla^2 \psi + (\text{lower order terms}) = (\text{source})$$

1. Electromagnetic fields
2. Spacetime metric
3. Acoustic/deformation fields
4. Water waves

In self-force problems, we often introduce a pointlike source and find gradients of the field at that source.

Green functions

For many reasons, it's useful to build up general solutions from a single impulsive solution:

$$(g^{ab}\nabla_a\nabla_b + C^a\nabla_a + D)G_{\mathrm{ret}}(p,p') = -4\pi I\delta(p,p')$$

After finding G_{ret}, sum:

$$\psi(p) = \int_{t>t_0} G_{\mathrm{ret}} j' dV' + \frac{1}{4\pi}\int_{t=0} [G_{\mathrm{ret}}\nabla^{a'}\psi' - \nabla^{a'} G_{\mathrm{ret}}\psi'] dS_{a'}$$

Short-distance behavior of Green functions

If p and p' are sufficiently close,

$$G_{\mathrm{ret}}(p,p') = \Theta(p \geq p')[U\delta(\sigma) + V\Theta(-\sigma)]$$

$U\delta(\sigma)$ describes propagation along null rays.

As in geometric optics,

- U increasing $\Rightarrow$ ray focusing
- U decreasing $\Rightarrow$ ray defocusing

$V\Theta(-\sigma)$ represents a "ringdown" following behind the wavefront.

$V = 0$ only in very special cases.

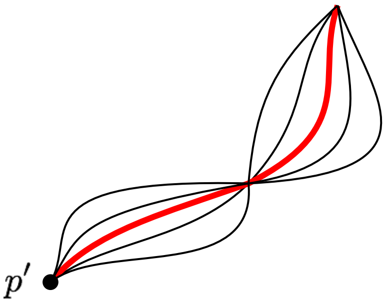
Caustics

This picture can break down at large distances.

Null geodesics starting at a point p' generically refocus.

Then,

- σ and U develop problems (at least at isolated points)
- Timelike curves can "catch up" to light rays.



Global singularity structure

What happens to $G_{\mathrm{ret}}(\cdot, p')$?

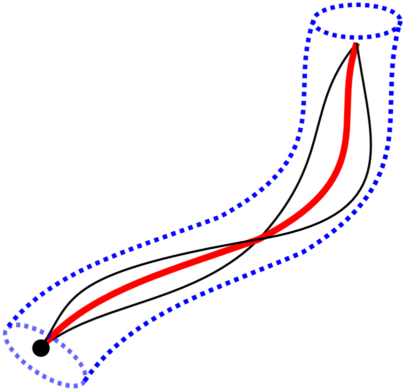
General theorems describe the propagation of singularities in solutions to linear wave equations.

Very roughly, **singularities are propagated on null geodesics**.

Globally, $G_{\mathrm{ret}}(p, p')$ is singular along all future-directed null geodesics starting at p'.

It seems unlikely that anything general can be said about $G_{\mathrm{ret}}(p, p')$ as a whole. Concentrate only on its singular components.

Do this by considering a small neighborhood of some future-directed null geodesic starting at p'.



Penrose limits

Given a null geodesic $z(u)$ in an arbitrary geometry, the metric "sufficiently near" that geodesic is always equivalent to a plane wave.

From null Fermi-like coordinates $(u, v, \mathbf{x})$, boost by λ^{-1} along $\dot{z}^a$, scale all spatial coordinates by λ^{-1}, conformally rescale metric by λ^{-2}, and send $\lambda \to 0$.

Then,

$$ds^2 \to -2dudv + H_{ij}(u)x^i x^j du^2 + dx^2 + dy^2$$

This represents a plane-symmetric wave travelling in the v direction with phase u and the 3-component waveform

$$H_{ij}(u) = -R_{abcd}(z(u))\dot{z}^a(u)e_i^b(u)\dot{z}^c(u)e_j^d(u)$$

Hereditary properties of Penrose limits

Various properties of the original spacetime are preserved by Penrose limits:

- Ricci-flatness (if present)
- Conformal-flatness (if present)
- Conjugate point structure of the reference geodesic

The metric "near" a null geodesic is a plane wave and all caustics/conjugate points are retained in the associated plane wave.

Can something interesting be learned about generic Green function singularities using plane waves?

Leading-order singular structure

Yes!

Consider test functions $\varphi(u,v,x,y) \in C_0^\infty(\mathbb{R}^4)$. Define

$$\varphi_\lambda(u,v,\mathbf{x}) := \varphi(u,\lambda^{-2}v,\lambda^{-1}\mathbf{x})$$

in null Fermi coordinates $(u,v,\mathbf{x})$ associated with $z(u)$.

Then,

$$\langle G_{\mathrm{pw}},\varphi\rangle := \lim_{\lambda\to 0}\lambda^{-2}\langle G,\varphi_\lambda\rangle$$

is a Green function for a field propagating on the Penrose limit plane wave spacetime.

This limit zooms up on the reference geodesic and extracts singular parts of G like $\delta(\sigma) \sim 1/\sigma \sim \lambda^2$.

It ignores anything smooth or with a singularity like $\ln|\sigma| \sim \ln \lambda$.

All that remains is to find plane wave Green functions $G_{\rm pw}$. This is relatively simple.

Some quick facts about plane wave spacetimes

- Almost everywhere, pairs of points are connected by exactly one geodesic
- σ and all related functions can be defined almost everywhere.
- Some pairs of points are connected by multiple geodesics. These are connected by an infinite number of geodesics. They are conjugate with respect to all of them.
- Points connected by an infinite number of geodesics are conjugate on all of them.
- Before reaching caustics, the scalar Green function $\Box_{\mathrm{pw}} G_{\mathrm{pw}} = -4\pi\delta$ has no tail: $V = 0$.

Plane wave Green functions

Following p along some future-directed curve starting at p', solutions to $\Box_{\rm pw} G_{\rm pw}(p,p') = -4\pi\delta(p,p')$ do the following at caustics/conjugate points:

1. If a caustic is non-degenerate (multiplicity 1),

$$\left[G_{\rm pw} = |\Delta|^{1/2}\delta(\sigma)\right]_{\rm before} \to \left[G_{\rm pw} = |\Delta|^{1/2}{\rm pv}\left(\frac{1}{\pi\sigma}\right)\right]_{\rm after}$$

$$\left[G_{\rm pw} = |\Delta|^{1/2}{\rm pv}\left(\frac{1}{\pi\sigma}\right)\right]_{\rm before} \to \left[G_{\rm pw} = -|\Delta|^{1/2}\delta(\sigma)\right]_{\rm after}.$$

2. Crossing a caustic with multiplicity 2 is equivalent to two crossings of non-degenerate caustics.

Green functions on generic spacetimes

The most singular parts of a generic Green function follow the same pattern:

Follow a null geodesic starting at a fixed point p'. Near this, $G(\cdot, p')$ switches between $\pm|\Delta|^{1/2}\delta(\sigma)$ and $\pm|\Delta|^{1/2}\mathrm{pv}(1/\pi\sigma)$.

Here, σ and Δ are the world function and van Vleck determinant associated with the Penrose limit plane wave of {spacetime, reference geodesic}.

Causality

Green functions involving $1/\sigma$ might seem to be acausal. They aren't:

- $\sigma(p, p') > 0$ implies that p and p' are causally disconnected only if p and p' are close.
- If p is connected to p' via a future-directed null geodesic segment that includes at least one point conjugate to p', p is in the *chronological* future $I^+(p')$ of p'.
- $I^+(p')$ is open $\Rightarrow$ (small neighborhood of p) $\subset I^+(p')$
- Plane wave-like structure is only valid in a vanishingly small region

An application: self-force

Consider a charged particle moving in a curved spacetime.

What is the effect of this particle's own field on its motion?

How much does this depend on its past history?

Do past caustics contribute anything interesting?

Self force in a plane wave

Consider a scalar charge in a plane wave spacetime where all conjugate points have multiplicity 2 (e.g., spacetime associated with scalar field plane wave).

Green functions exist that are everywhere proportional to $\pm\delta(\sigma)$ with no tail.

Despite this, the R-field includes contributions from the charge's past: Timelike curves intersect (or "almost intersect") the null cone multiple times.

More generally, the field on a charge's timelike worldline $z(u)$ is affected via discrete contributions from every caustic in its past.

For a point charge in 1D motion, mass varies according to

$$m(u) = m_* + 2q^2 \sum_{n=1}^{N(u)} (-1)^n \frac{[\tau_n'(u)]^{\frac{1}{2}}}{\Delta s_n} \left(\frac{dv/ds|_{\tau_n}}{\langle dv/ds \rangle_n} \right)$$

Similar expressions show up for force ($\propto 1/\Delta s_n^2$)

One finds delay differential equations. These can be solved self-consistently if desired.

Summary

- Part of the singular structure of Green functions in generic spacetimes is equivalent to a Green function in an appropriate plane wave spacetime.
- Plane wave Green functions can be computed explicitly.
- Encountering conjugate points on a null geodesic converts $\delta(\sigma)$-type singularities into $1/\sigma$ ones (and vice versa).
- Caustic effects can contribute significantly to an object's self-field.