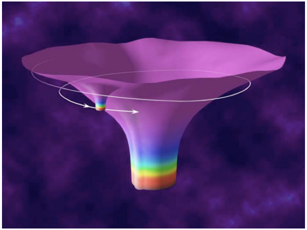
Time Domain Schemes for Gravitational Self Force



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Talk Outline

1. **Motivation**
   - Why compute GSF on Kerr?
2. **Formulation**
   - Linearized eqns and gauge choice.
   - Mass and angular momentum
   - Regularization / Puncture scheme
   - m-mode decomposition
3. **Results: Circular orbits on Kerr**
   - Radiative modes
   - Problem: $m = 0, 1$: linear-in-t modes
   - Solution: generalized Lorenz gauge
   - Comparison of ΔU with Friedman *et al.*
4. **Prospects**

Motivation: Lorenz-gauge time-domain calc. on Kerr

Q1. Why consider Kerr?

- Galactic BHs are **rotating**, $a/M \sim 0.5 - 0.99$.
- Rotation breaks symmetry and leads to physical effects, e.g. ergodic geodesics, frame-dragging, light-cone caustics become 'tubes', etc.
- **Resonances:** Generic orbits may pass through resonance when $\omega_r/\omega_\theta \sim n_1/n_2$ [Hinderer & Flanagan].

Q2. Why work in Lorenz-gauge?

- Hyperbolic (wave-like) formulation of equations.
- S-field has 'symmetric' singular part $\bar{h}_{ab} \sim 1/r$
  $\Rightarrow$ regularization is well-understood.

Q3. Why work in time-domain?

- Lorenz-gauge MP not separable on Kerr
  $\Rightarrow$ no *ordinary* diff. eq. formulation in freq. domain.
- **Self-consistent evolutions** are most naturally handled with time-domain scheme.

Formulation: Linearized equations

Linearized Einstein Eqs for Ricci-flat background:

$$\Box\bar{h}_{ab} + 2R^{c\ d}_{\ a\ b}\bar{h}_{cd} + Z^{c}_{\ ;c} - Z_{a;b} - Z_{b;a} = -16\pi T_{ab},$$

$Z_a \equiv \bar{h}_{ab}{}^{;b}$, where $\bar{h}_{ab}$ is the trace-reversed metric perturbation:

$$\bar{h}_{ab} = h_{ab} - \tfrac{1}{2}g_{ab}h, \qquad \text{and} \qquad h = h^{a}{}_{a}.$$

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Z4 system and gauge choice

Use Generalized Lorenz gauge with gauge-driver $H_a(h_{bc}, x)$:

$$Z_a = H_a(x, h_{bc}) \quad (= 0 \quad \text{for Lor. gauge})$$

Z4 system: 10 eqns with 4 constraints,

$$\begin{aligned} \Box\bar{h}_{ab} + 2R^{c\ \ d}_{\ a\ \ b}\bar{h}_{cd} + H^{c}_{\ ;c} - H_{a;b} - H_{b;a} &= -16\pi T_{ab}, \\ c_a \equiv Z_a - H_a &= 0 \end{aligned}$$

Formulation: Linearized equations

Z4 with constraint damping

$$\Box\bar{h}_{ab} + 2R^{c}{}_{a}{}^{d}{}_{b}\bar{h}_{cd} + H^{c}{}_{;c} - H_{a;b} - H_{b;a} + \kappa\left(n_a c_b + n_b c_a\right) = -16\pi T_{ab},$$

where $\kappa(x)$ is a scalar function and n_a is a vector, and

$$c_a = Z_a - H_a.$$

- Choose κ, n_a so that constraints are damped, under

$$\Box c_a = -\left(\kappa(n_a c_b + n_b c_a)\right)^{;b}.$$

- Good choice: n_a = ingoing principal null direction, with $\kappa < 0$.
- h_{ab} is a solution of lin. Einstein eqns iff $c_a = 0$.

Formulation: Mass and angular momentum

- Combine Killing vector X^a and stress-energy T_{ab} to form

$$\textbf{conserved current:} \quad j_a \equiv T_{ab} X^b, \qquad {j_a}^{;a} = 0.$$

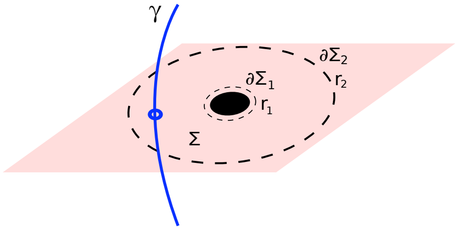
- Poincaré lemma: $\delta j = 0 \Rightarrow j = \delta F$ (where $\delta = *d*$), i.e.

$$j_a = {F_{ab}}^{;b}, \quad \text{where} \quad F_{ab} = F_{[ab]},$$

Abbott & Deser (1982): Conserved two-form

$$F_{ab} \equiv -(8\pi)^{-1} \left(X^c \bar{h}_{c[a;b]} + {X^c}_{;[a} \bar{h}_{b]c} + X_{[a} Z_{b]} \right),$$

Formulation: Mass and angular momentum



Apply Stokes' theorem to get 'quasi-local' definitions:

$$\begin{aligned}\int_\Sigma j^a d\Sigma_a &= \int_\Sigma F^{ab}{}_{;b}\, d\Sigma_a \\ &= \frac{1}{2}\left[\int_{\partial\Sigma} F^{ab} dS_{ab}\right]_{r_1}^{r_2} \\ &= \begin{cases}\mu X^a u_a, & r_1 < r_0 < r_2, \\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Formulation: Mass and angular momentum

$$\text{Quasi-local quantity:} \quad Q(X, \partial\Sigma) \equiv \frac{1}{2}\int_{\partial\Sigma} F^{ab} dS_{ab}.$$

Is Q a useful definition of the mass/ang.mom. in a given homogeneous metric perturbation h_{ab}?

Property 1: Q is **gauge-invariant**

- If $h_{ab} = 2\xi_{(a;b)}$ then $F_{ab} \propto \eta_{abc}{}^{;c}$, where

$$\eta_{abc} \propto X_{[a}\xi_{b;c]} + X_{[a;b}\xi_{c]}.$$

- It follows that $Q \propto \int (b_{\phi,\theta} - b_{\theta,\phi}) d\theta d\phi = [b_\phi]_0^\pi = 0$, where $b = *\eta$.

Formulation: Mass and angular momentum

$$\boxed{\text{Quasi-local quantity:}\quad Q(X,\partial\Sigma) \equiv \frac{1}{2}\int_{\partial\Sigma} F^{ab} dS_{ab}.}$$

Is Q a useful definition of the mass/ang.mom. in a given homogeneous metric perturbation h_{ab}?

Property 2: Q gives correct mass/ang. mom. for Kerr pert.

- $X^a_{(t)} = [1,0,0,0] \Leftrightarrow Q_{(t)}$ and $X^a_{(\phi)} = [0,0,0,1] \Leftrightarrow Q_{(\phi)}$
- Mass (M) and ang. mom ($J \equiv aM$) perturbations:

$$h_{ab} = \mu\mathcal{E}\,\frac{\partial}{\partial M} g^{\text{Kerr}}_{ab}\bigg|_J \quad \Rightarrow \quad Q_{(t)} = \mu\mathcal{E}, \quad Q_{(\phi)} = 0.$$

$$h_{ab} = \mu\mathcal{L}\,\frac{\partial}{\partial J} g^{\text{Kerr}}_{ab}\bigg|_M \quad \Rightarrow \quad Q_{(t)} = 0, \quad Q_{(\phi)} = \mu\mathcal{L}.$$

Formulation: Puncture scheme

- **Problem:** $\bar{h}_{ab}$ is divergent $\sim 1/\epsilon$ towards worldline
- **Solution:** Introduce **puncture** $\bar{h}^P_{ab}$: a local approximation to Detweiler-Whiting singular field $\bar{h}^S_{ab}$.
- Covariant expansion of $\bar{h}^S_{ab} \Rightarrow$ power-series in coordinate differences,

$$\delta x^a = x^a - \bar{x}^a, \quad \text{where} \quad x = \text{field pt}, \quad \bar{x} = \text{worldline pt}$$

- **Classification:** nth order puncture iff

$$h^P_{ab} - h^S_{ab} \sim \mathcal{O}\left(|\delta x|\, \delta x^{n-2}\right)$$

- 2nd-order in Barack et al ’07, 4th+ order from Wardell.
- Local $\rightarrow$ Global definition: let $\bar{x}$ become a function of x, e.g. set $\bar{t} = t$, $\bar{\mathbf{x}} = \mathbf{x}_p(t)$.
- Global continuation is arbitrary, but should be smooth around circle, except at worldline
- Use a periodic definition φ, e.g. $\delta\varphi^2 \rightarrow 2(1 - \cos\delta\varphi) = \delta\varphi^2 + \mathcal{O}(\delta\varphi^4)$

Formulation: Puncture scheme

Introduce a **worldtube $\mathcal{T}$** surrounding the worldline:

- Outside worldtube $\mathcal{T}$, evolve *retarded* field $\bar{h}_{ab}$.
- Inside worldtube $\mathcal{T}$, evolve *residual* field $\bar{h}^{\mathcal{R}}_{ab}$, i.e.

$$\begin{cases} \hat{\mathcal{D}} h_{ab} = 0, & \text{outside } \mathcal{T}, \\ \hat{\mathcal{D}} h^{\mathcal{R}}_{ab} = -16\pi T^{\text{eff}}_{ab}, & \text{inside } \mathcal{T}, \\ h^{\mathcal{R}}_{ab} = h_{ab} - h^{\mathcal{P}}_{ab}, & \text{across } \partial\mathcal{T}. \end{cases}$$

  where $T^{\text{eff}}_{ab} \equiv T_{ab} - (-16\pi)^{-1}\hat{\mathcal{D}} h^{\mathcal{P}}_{ab}$, and $\hat{\mathcal{D}}$ is wave operator.

- Compute self-force F_a and gauge-invariant H from residual $\mathcal{R}$ field:

$$\mu^{-1} F_a = \lim_{x \to z(\tau)} k^{abcd} \bar{h}^{\mathcal{R}}_{bc;d}, \quad H = \tfrac{1}{2} u^a u^b \lim_{x \to z(\tau)} h^{\mathcal{R}}_{ab}.$$

Formulation: m-mode decomposition

- Exploit the **axial symmetry**: decompose MP in m-modes $\Rightarrow$ 2+1D eqns:

$$\bar{h}_{ab} = \sum_m \bar{h}_{ab}^{(m)} e^{im\varphi}.$$

- Real field $\Rightarrow \bar{h}_{ab}^{(m)*} = \bar{h}_{ab}^{(-m)}$
- Reconstruct self-force, field, etc. from mode sums, e.g.

$$\bar{h}_{ab}^{R} = \lim_{x\to z} \left(\bar{h}_{ab}^{(m=0)} + 2\sum_{m=0}^{\infty} \mathrm{Re}\left[\bar{h}_{ab}^{\mathcal{R}(m)} e^{im\varphi_0(t)} \right] \right)$$

- Convergence-with-m depends on order of puncture:
  1. Second-order $\Rightarrow F_r^{(m)}, \bar{h}_{ab}^{\mathcal{R}(m)} \sim \mathcal{O}(m^{-2})$
  2. Fourth-order $\Rightarrow F_r^{(m)}, \bar{h}_{ab}^{\mathcal{R}(m)} \sim \mathcal{O}(m^{-4})$
  3. . . . etc . . .

Implementation: Circular orbits on Kerr

- Particle on circular orbit with frequency $\omega = \sqrt{M}/(r_0^{3/2} + a\sqrt{M})$
- Define $\bar{h}_{ab}$ w.r.t. Boyer-Lindquist coordinate system (t, r, θ, ϕ)
- Introduce tortoise coords: $r_* = \int \frac{r^2+a^2}{\Delta} dr$, $\varphi = \phi + \int \frac{a}{\Delta} dr$

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- Second-order puncture $\bar{h}^{\mathcal{P}}_{ab} \sim 4\mu\chi_{ab}/\epsilon$ [Barack et al.'07], with

$$\chi_{ab} = \begin{cases} u_a u_b + C_{ab}\delta r & \text{for } ab = tt, t\phi, \phi\phi \\ C_{ab} \sin\delta\phi & \text{for } ab = tr, t\phi. \end{cases}$$

- m-mode decomposition:

$$\bar{h}^{\mathcal{P}(m)}_{ab} = \frac{e^{-im(\omega t + \Delta\phi)}}{2\pi} \int_{-\pi}^{\pi} \bar{h}^{\mathcal{P}}_{ab}(\delta r, \delta\theta, \delta\phi) e^{-im\delta\phi} d(\delta\phi)$$

  Integrals have an elliptic integral representation.

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  Integrals have an elliptic integral representation.

- Use scaled evolution variables $u^{(m)}_{ab}$,

$$\bar{h}^{(m)}_{ab} = \frac{1}{r} \Xi_a \Xi_b u^{(m)}_{ab}(t, r, \theta) \qquad \text{(no sum)}$$

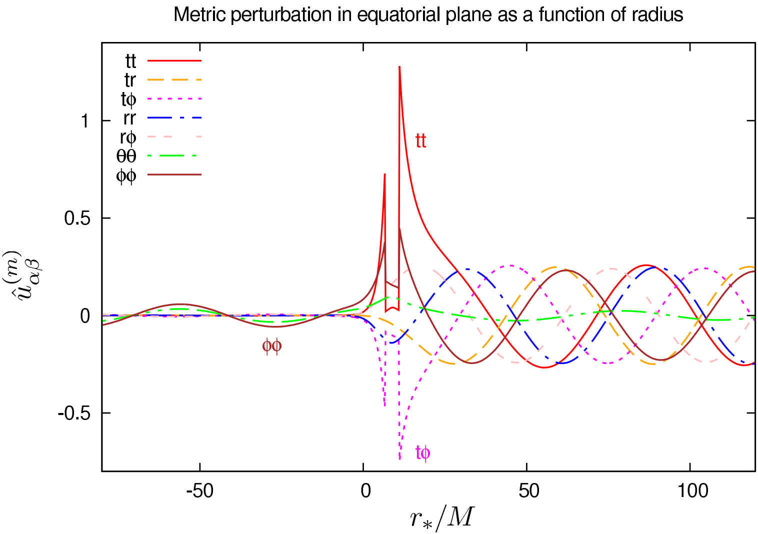
  where $\Xi_a = [1, 1/(r - r_h), r, r\sin\theta]$.

Implementation: Circular orbits on Kerr

- I used Lorenz-gauge $Z4$ system with constraint damping.
- Cauchy evolution in (t, r_*, φ), with worldtube and effective source.
- Fourth-order-accurate finite-differencing . . . except at worldline where residual field is not smooth.
- Boundary conditions:
  1. Regular MP at the poles
  2. Regular MP on the future horizon
  3. $u_{ab}^{(m)} \sim \mathcal{O}(1)$ as $r \to \infty$
- Trivial initial conditions, $u_{ab}^{(m)} = 0$ ... wait long enough and
- 'Junk' dissipates with time (in radiative sector).
- Gauge-violation is driven to zero.

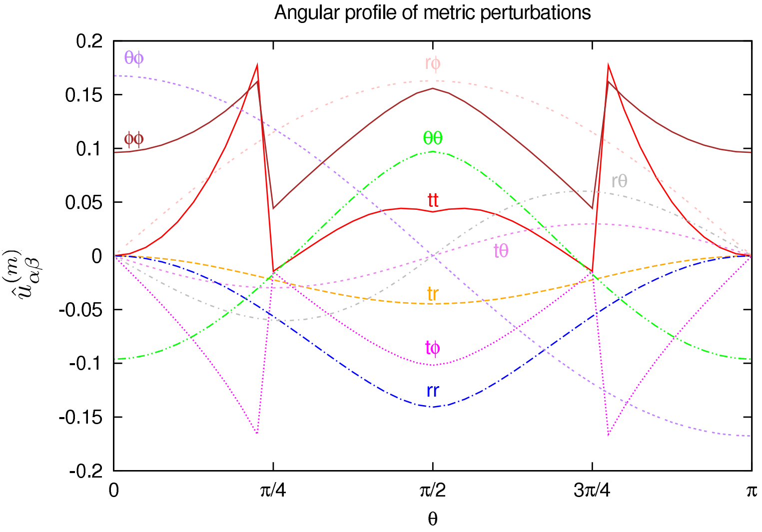
Results: Modal profiles

Slice 1: $t = 250M, \theta = \pi/2$ (and $r_0 = 7M$, $m = 2$)



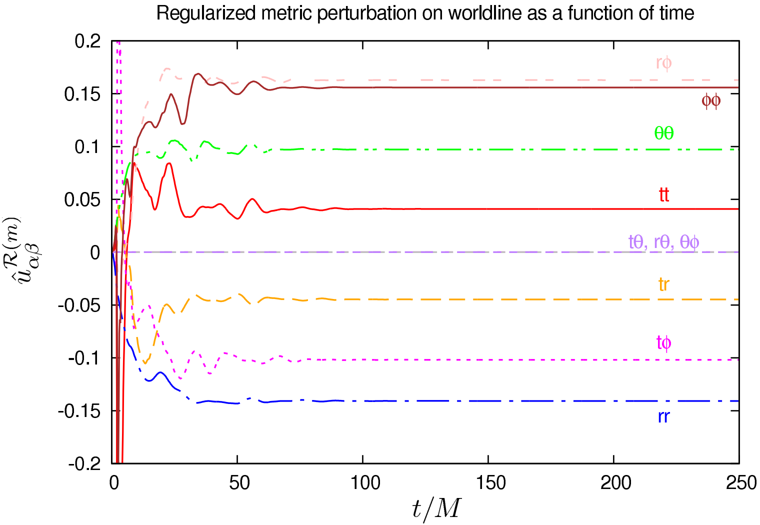
Results: Modal profiles

Slice 2: $t = 250M, r = r_0$ (and $r_0 = 7M$, $m = 2$)

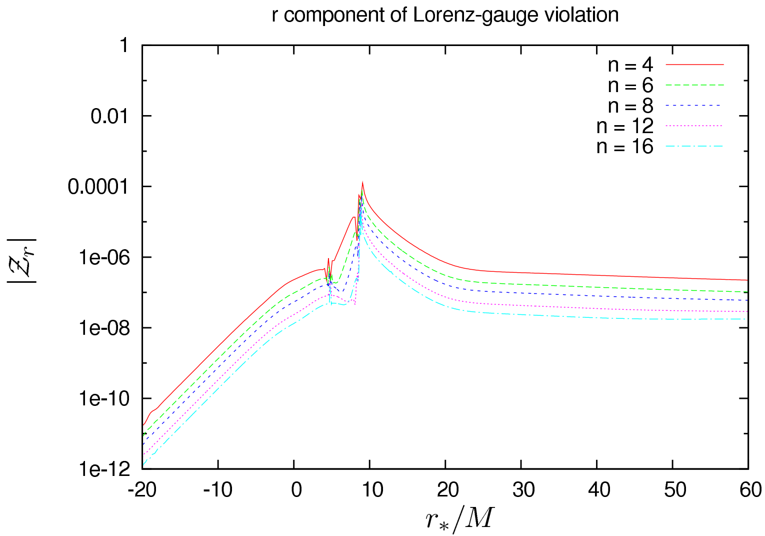


Results: Modal profiles ($r_0 = 7M$, $m = 2$)

Slice 3: $\theta = \pi/2$, $r = r_0$ (and $r_0 = 7M$, $m = 2$)

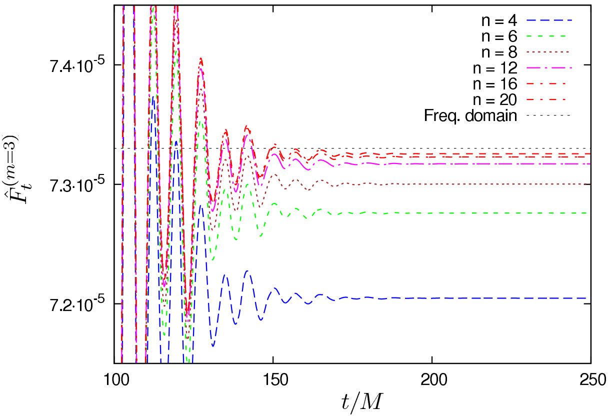


Results: Gauge-constraint violation



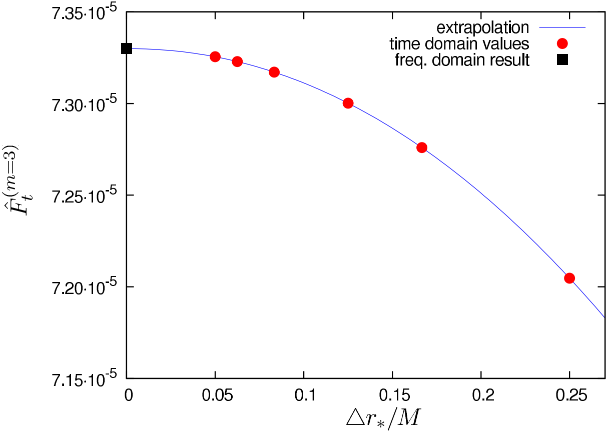
- Constraint violation diminishes with increasing grid resolution

Results: F_t and energy balance



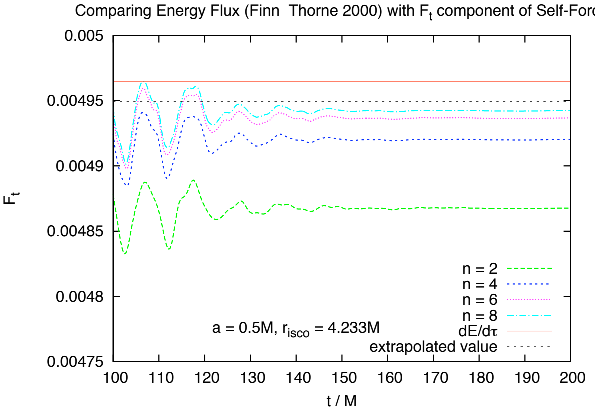
- Showing time-domain value of F_t for various grid resolutions $dr_* = M/n$.
- In principle, $F_t = u_0^t \dot{E}$, where $\dot{E}$ is energy loss rate (from Teuk. ψ_0, ψ_4).

Results: F_t and energy balance



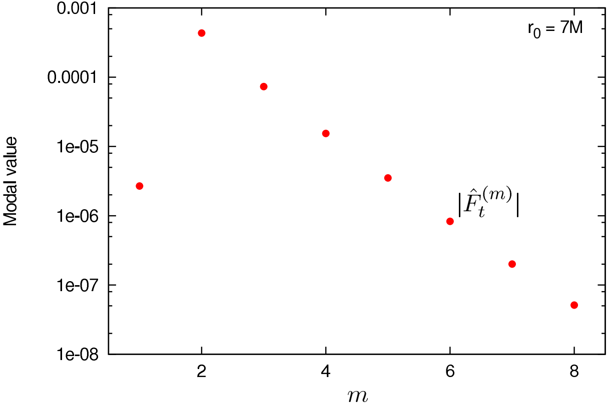
- Extrapolate over grid resolution to obtain best estimate
- Convergence rate only $x^2 \ln x$ with 2nd-order puncture

Results: F_t validation at $a = 0.5M$ $(m = 2 \text{ mode})$



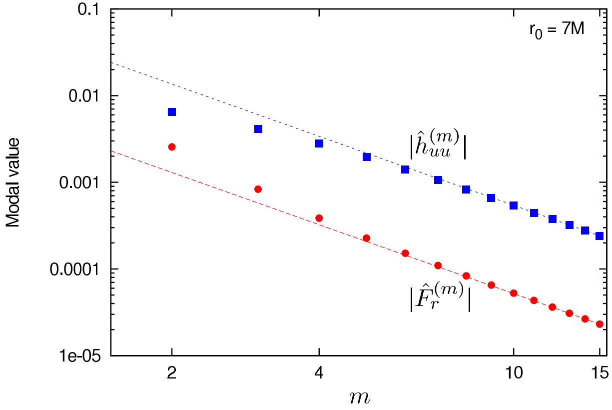
- For each m-mode, validate $\dot{E} = F_t/u_0^t$ against results of Finn & Thorne.
- 0.3% disagreement here because Finn & Thorne give $\dot{E}_\infty$, whereas $\dot{E} = \dot{E}_\infty + \dot{E}_{hor}$.

Results: m-mode convergence: dissipative



- Modes of dissipative component of GSF, F_t, converge exponentially, $F_t^m \sim \exp(-\lambda|m|)$.

Results: m-mode convergence: conservative



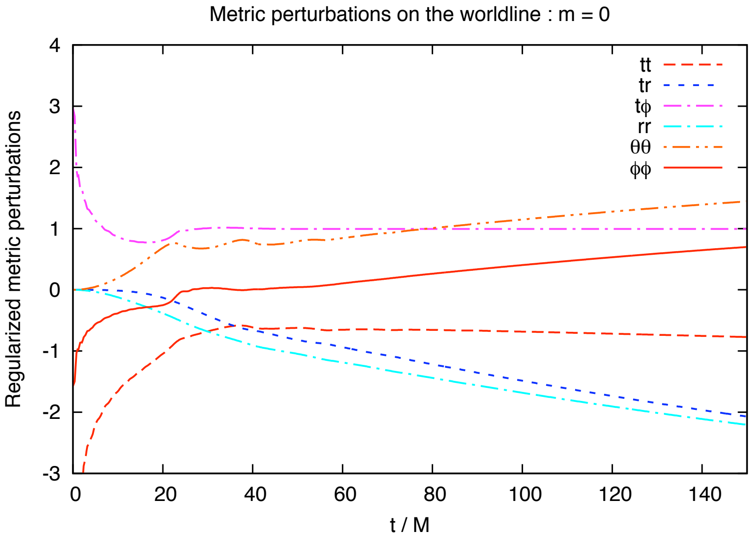
- Modes of conservative component, F_r (and h^R_{uu}) converge with power-law, $F^m_t \sim m^{-2}$ (for 2nd-order puncture).

Problem: Linear-in-t modes in Lorenz gauge

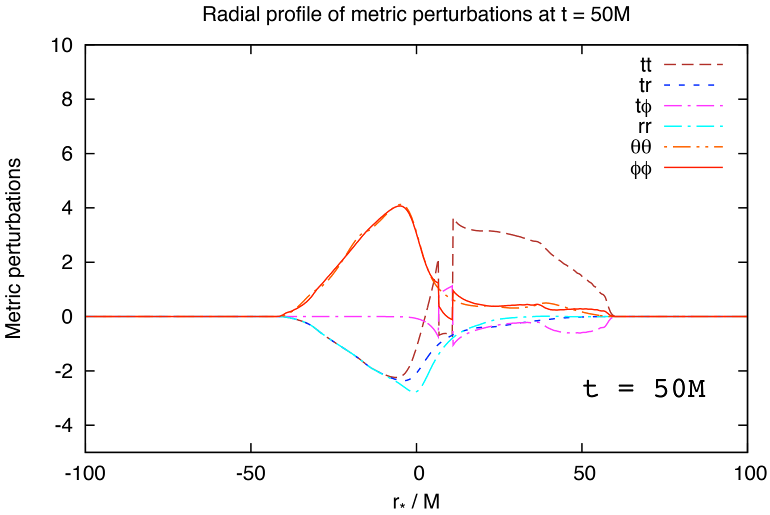
- **Problem:** Modes $m = 0, 1$ suffer from **linear-in-t** instabilities!
- Linear-in-t modes are homogeneous, pure-Lorenz-gauge solutions.
- Linear-in-t modes are regular on future horizon and asymp-flat.
- Linear-in-t modes are excited by generic initial data.
- In Schw., these modes are in $l = 0$, $l = 1$ sectors only.
- N.B. No l-mode time-domain scheme has successfully evolved Schw. $l = 0, 1$ modes in Lorenz gauge.
- **Solution:** Use a **generalized** Lorenz gauge to achieve stable evolutions,

$$\bar{h}_{ab}{}^{;b} = H_a(h_{bc}, x).$$

Problem: Time Evolution of $m = 0$ mode



Radial Profile : $m = 0$ mode



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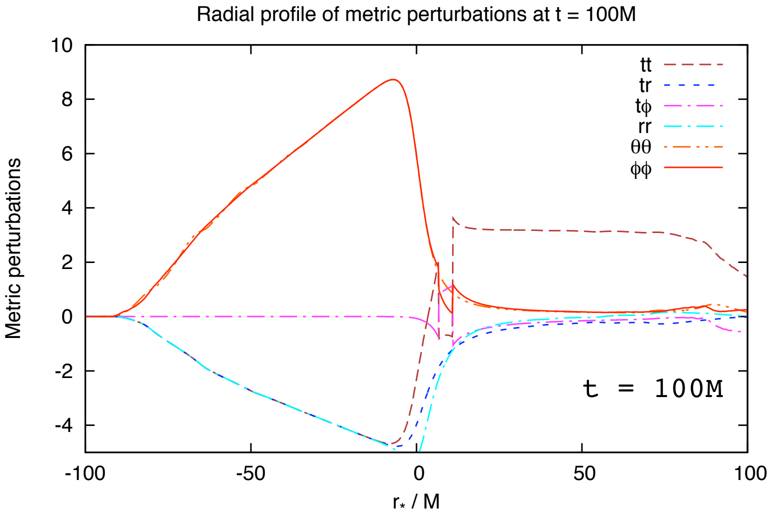

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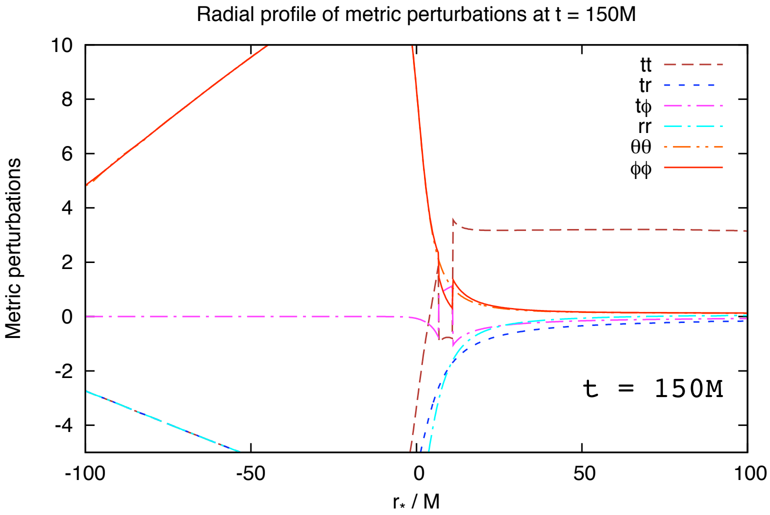


Solution : Generalized Lorenz gauge

- Found that an explicit gauge driver of the form:

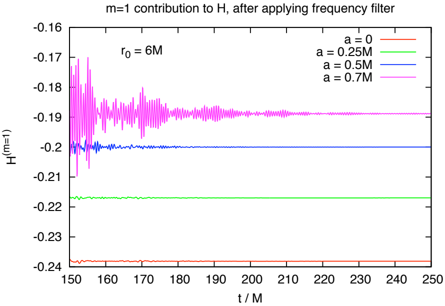
$$H_a \propto n_a \times h_{tr}^{(m=0)}/r^k, \quad \text{where} \quad n_a \text{ is ingoing null vector}$$

  restores stability to $m = 0$ sector.
- For circular orbits, $h_{tr}^S = 0$, so this gauge is non-singular.
- Non-unique stationary solution which depends on initial condition.
- The *static* solution ($h_{ti} = 0$) is also in Lorenz gauge ($Z_a = 0$).
- Take linear combination of solutions to find static soln $h_{tr} = 0$.
  1. Schw.: combine **two** solns in monopole ($l = 0$) sector.
  2. Kerr: combine **three** solns, as mass & ang. mom. pert. are no longer decoupled.
- Unnecessary if we are only interested in gauge-invariant (e.g. ΔU).

Solution : $m = 1$ mode?

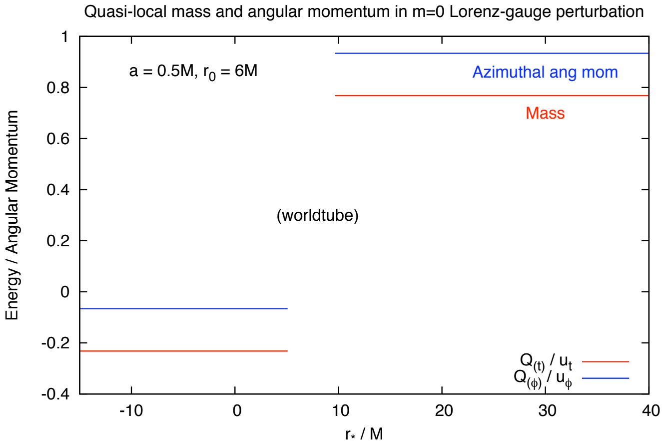
- I have **not** found a generalized Lorenz gauge that stabilizes the $m = 1$ sector.
- Instead, I apply a frequency-filter to eliminate stationary and linear-in-t modes:

$$h_{ab} \to -\frac{1}{\omega^2}\frac{\partial^2}{\partial t^2} h_{ab}$$



- This trick will *not* work for general orbits

Correcting the mass and angular momentum



- Take integrals over two-spheres to find ‘quasi-local’ mass $Q_{(t)}$ and angular momentum $Q_{(\phi)}$ in numerical solution $h_{ab}^{(m=0)}$.

Correcting the mass and angular momentum

- To correct the mass and ang.mom. I add homogeneous Lorenz-gauge solutions which are regular on the future horizon,

$$h_{ab}^{(\partial M)} = \frac{\partial}{\partial M} g_{ab}\bigg|_J + \text{gauge}, \quad h_{ab}^{(\partial J)} = \frac{\partial}{\partial J} g_{ab}\bigg|_M + \text{gauge}.$$

- ...but these solutions are *not* asymp-flat.
- Recall that in Schw., the static Lorenz-gauge solution with correct mass is *not* asymp-flat: $h_{tt} \to -2\alpha$. (where $\alpha = \mu/\sqrt{r_0(r_0 - 3M)}$ [Sago et al. '08]).
- In Kerr, I find that Lorenz-gauge static solution with correct mass and ang.mom. is not asymp-flat in **two** components:

$$h_{tt} \sim O(1) \quad \text{and} \quad h_{t\phi} \sim O(r^2).$$

- In Schw., azimuthal ang. mom. is in $l = 1, m = 0$ odd-parity mode.
- In Schw., $\partial g_{ab}/\partial J(a = 0)$ is already in Lorenz-gauge – this is not the case in Kerr.

Gauge invariants in asymptotic flat gauge

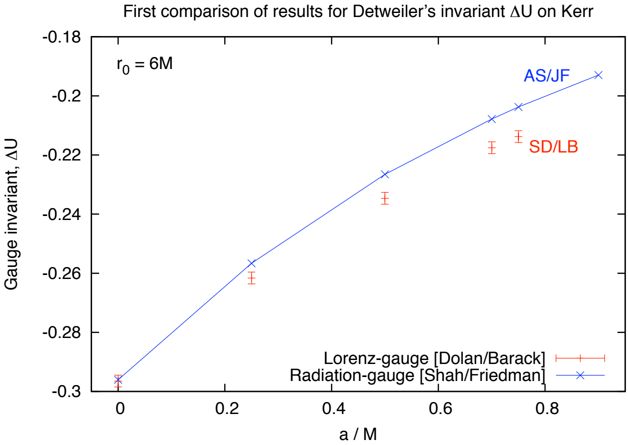
- To compare results with Post-Newtonian expansions, and radiation gauge/Hertz potential approach of Friedman et al., I need the perturbation in an asymptotically-flat gauge.
- $h_{tt} \sim \mathcal{O}(1)$ is fixed with gauge vector $\xi^a = [\tilde{t}, 0, 0, 0]$
- $h_{t\phi} \sim \mathcal{O}(r^2)$ is fixed with $\xi^a = [0, 0, 0, \tilde{t}]$, where $\tilde{t}$ is ingoing time coordinate.
- Detweiler has identified quantities that are **gauge-invariant** in a class of (asymp-flat) gauges sharing helical symmetry of circular orbits:

$$(\partial_t + \omega \partial_\phi)\, \xi^a = 0.$$

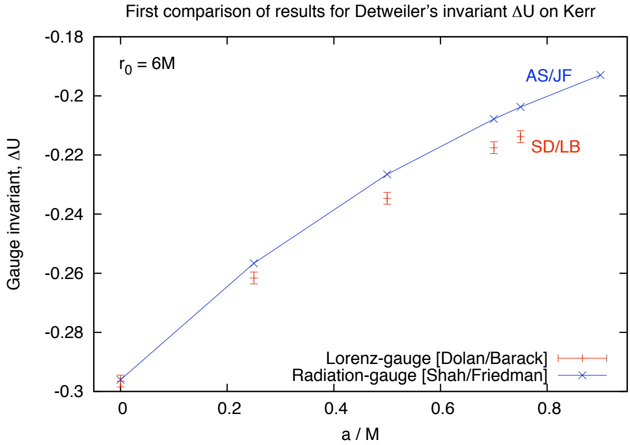
- **Redshift invariant:** $u^t = u^t_0 + \mu \Delta U + \mathcal{O}(\mu^2)$, where

$$\Delta U = H u^t_0, \quad \text{where} \quad H = \tfrac{1}{2} h^R_{ab} u^a u^b.$$

Gauge invariant comparison : ΔU for circular orbits



Gauge invariant comparison : ΔU for circular orbits



a / M	0.0	0.25	0.5	0.7	0.75
Rad. gauge [S/F]	−0.29603	−0.25663	−0.22655	−0.20782	−0.20376
Lor. gauge [D/B]	−0.296(2)	−0.262(2)	−0.235(2)	−0.218(2)	−0.214(2)

Prospects for the future

1. What can we learn from ΔU calculation?
   - Are radiation-gauge and Lorenz-gauge results consistent?
   - If not, why not? Bug or conceptual issue?
   - Does ΔU agree with Post-Newtonian expansion, as $r \to \infty$?
   - $\Delta U \to$ ISCO shift, using method of Le Tiec.
2. Eccentric equatorial orbits:
   - Need to find a stable non-singular gauge for $m = 0$, $m = 1$. What can we learn here from Numerical Relativity?
   - Periastron advance.
   - Calibration of free parameters in EOB theory.
3. Generic orbits: Resonances? [Hinderer & Flanagan]
4. Self-consistent evolutions on Kerr.
5. Second-order-in-μ calculations with worldtube scheme?