Filtering Turbulent Signals Using Gaussian and non-Gaussian Filters with Model Error

June 13, 2013

Nan Chen
Center for Atmosphere Ocean Science (CAOS)
Courant Institute of Mathematical Sciences
New York University

I. Introduction

Outline

Use test models to study the skills of filtering the turbulent signals.

1. These test models have exactly solvable statistics.
2. They mimic the atmosphere and ocean behaviors.

Main features of Atmosphere and Ocean Sciences:

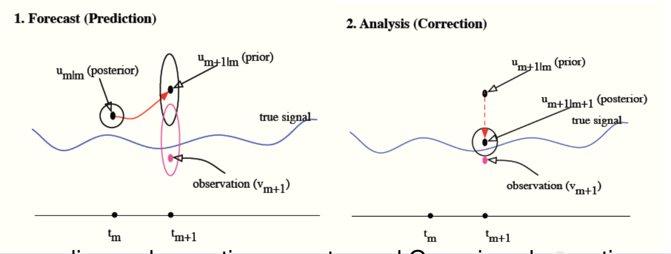
1. Intermittent instability.
2. Unresolved/unobserved variable/process.
3. Model error.

Goal

Generate the signals from the system with **intermittent instability** and **unresolved process**.

1. Study the filtering skill of using **imperfect forecast models**.
2. Compare the filtering skill using **Gaussian** and **non-Gaussian filters** in a perfect model setting.

Different types of filter (w.r.t. the prior/background distribution).



Assume linear observation operator and Gaussian observation noise.

- Kalman Filter: linear and Gaussian system

$$u_{m+1} = Fu_m + \sigma_{m+1} \quad \text{with constant } F.$$

- Extended Kalman Filter: nonlinear non-Gaussian system with linear tangential approximation

$$u_{m+1} = f(u_m) + \sigma_{m+1} \Longrightarrow u_{m+1} \approx f(\bar{u}_{m|m}) + \nabla f(\bar{u}_{m|m})(u_m - \bar{u}_{m|m}) + \sigma_{m+1}.$$

- Two moments filter (or Nonlinear Extended Kalman Filter): nonlinear non-Gaussian system with exactly solvable mean and covariance. Note that model error still exists when using two moments filter to filter the non-Gaussian signal!
- Non-Gaussian filter: The information in the higher order moments is included in filtering the signals.

II. Models and Mathematical Tools

A. Test Models

$$\begin{aligned}\frac{du}{dt} &= r(\gamma,t)u + l(\gamma,t) + \sigma_u(\gamma,t)\dot{W}_u(t),\\ \frac{d\gamma}{dt} &= F(\gamma,t) + \sigma_\gamma(\gamma,t)\dot{W}_\gamma(t),\end{aligned} \tag{1}$$

1. Resolved/unresolved variable:
   - u is the resolved variable.
   - γ is the unresolved variable.
2. Intermittent instability in u:
   - Positive $r(\gamma,t)$ corresponds to instability.
   - Negative $r(\gamma,t)$ corresponds to stability.
3. Model error:
   - A complicated dynamics is used to generate the true signal.
   - Some simplified dynamics are used as the forecast model.

To obtain the statistics of (1), there's no need to solve the 2-D Fokker-Planck equation nor use the Monte Carlo simulation. Instead, **the statistics can be solved with some cheap ways using the conditional moments.**

Distribution and Moment

- Joint distribution: $p(u,\gamma)$.
- Marginal distribution:

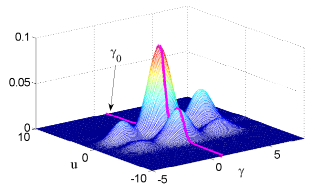
$$\pi(\gamma) = \int p(u,\gamma)du, \qquad \pi(u) = \int p(u,\gamma)d\gamma.$$

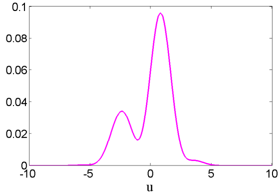
- Conditional distribution: $p(u|\gamma_0)$.
- Conditional Moment

$$\text{At one fixed } \gamma_0: \; M_N(\gamma_0) = \int u^N p(\gamma_0, u)du,$$

$$\text{As a function of } \gamma: \; M_N(\gamma) = \int u^N p(\gamma, u)du.$$

Remark: $M_0(\gamma) = \int p(u,\gamma)du = \pi(\gamma)$.





Test Models

$$\frac{du}{dt} = r(\gamma,t)u + l(\gamma,t) + \sigma_u(\gamma,t)\dot{W}_u(t),$$
$$\frac{d\gamma}{dt} = F(\gamma,t) + \sigma_\gamma(\gamma,t)\dot{W}_\gamma(t), \quad (1)$$

Proposition

The vector of conditional moments $M_N(\gamma,t)$ of order N associated with the probability density of (1) satisfies the coupled generalized Fokker-Planck equations (CGFPE)

$$\frac{\partial}{\partial t}M_N(\gamma,t) = L_{FP}M_N(\gamma,t) + r(z,t)NM_N(\gamma,t) + Nl(\gamma,t)M_{N-1}(\gamma,t) + \frac{1}{2}N(N-1)\sigma_u^2(\gamma,t)M_{N-2}(\gamma,t), \quad (2)$$

where $M_0 = \pi(\gamma,t)$ and we prescribe $M_{-1} \equiv 0$ and $M_{-2} \equiv 0$ and

$$L_{FP}M_N(\gamma,t) = -\frac{\partial}{\partial\gamma}(F(\gamma,t)M_N(\gamma,t)) + \frac{1}{2}\frac{\partial^2}{\partial\gamma^2}(\sigma_\gamma^2(\gamma,t)M_N(\gamma,t)).$$

The moment equation (2) can be solved recursively.

With the conditional moments,

$$M_0(\gamma,t)=\int p(\gamma,u,t)du,\ \ M_1(\gamma,t)=\int up(\gamma,u,t)du,\ \ M_2(\gamma,t)=\int u^2p(\gamma,u,t)du,$$

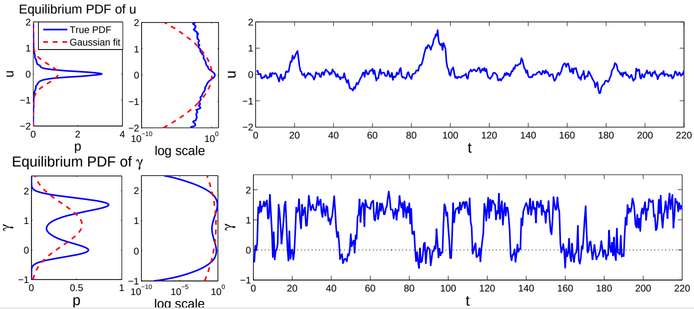
the mean and covariance of system (1) are solved via

$$\begin{aligned}
\bar{u}(t) &= \iint up(u,\gamma,t)dud\gamma = \int\left(\int up(u,\gamma,t)du\right)d\gamma = \int M_1(\gamma,t)d\gamma,\\
\bar{\gamma}(t) &= \iint \gamma p(u,\gamma,t)dud\gamma = \int \gamma\left(\int p(u,\gamma,t)du\right)d\gamma = \int \gamma M_0(\gamma,t)d\gamma,\\
Var(u)(t) &= \iint u^2p(u,\gamma,t)dud\gamma - \bar{u}^2(t)\\
&= \int\left(\int u^2p(u,\gamma,t)du\right)d\gamma - \bar{u}^2(t) = \int M_2(\gamma,t)d\gamma - \bar{u}^2(t),\\
Var(\gamma)(t) &= \iint \gamma^2p(u,\gamma,t)dud\gamma - \bar{\gamma}^2(t)\\
&= \int \gamma^2\left(\int p(u,\gamma,t)du\right)d\gamma - \bar{\gamma}^2(t) = \int \gamma^2M_0(\gamma,t)d\gamma - \bar{\gamma}^2(t),\\
Cov(u,\gamma)(t) &= \iint u\gamma p(u,\gamma,t)dud\gamma - \bar{u}\bar{\gamma}\\
&= \int \gamma\left(\int up(u,\gamma,t)du\right)d\gamma - \bar{u}\bar{\gamma} = \int \gamma M_1(\gamma,t)d\gamma - \bar{u}\bar{\gamma}.
\end{aligned}$$

III. Filtering the turbulent signals

Assume the true signal is generated from the following system,

$$du = (-\gamma u + f_u)dt + \sigma_u dW_u,$$
$$d\gamma = (-a\gamma + b\gamma^2 - c\gamma^3 + f_\gamma)dt + (A - B\gamma)dW_c + \sigma dW_\gamma, \quad (3)$$



- Assume linear observations with $g = [1, 0]$, and therefore $v_{m+1} = u_{m+1} + \sigma^o_{m+1}$.
- Observation time step: $\Delta t = 0.5$, less than the averaged decorrelation time $\tau^u_{corr} = 1.2362$ of u and decorrelation time $\tau^\gamma_{corr} = 3.64$ of γ.
- Observation noise level: R^o equals 12.5% of the averaged energy of the true signal.

Measurement of the filtering skill.

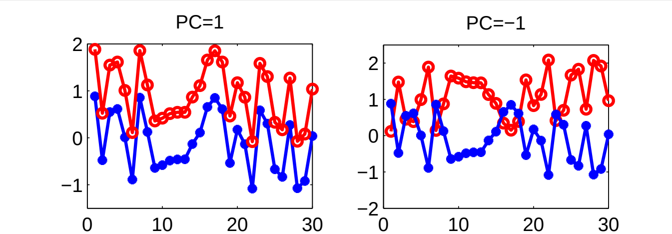
Let u_m be the true signal and $\bar{u}_{m|m}$ be the filter estimate.

- Root mean square (RMS) error

$$RMSE = \sqrt{\frac{1}{T - T_0} \sum_{m=T_0}^{T} (u_m - \bar{u}_{m|m})^2}.$$

- Pattern correlation (PC)

$$PC = \frac{\Big\langle (u_m - \langle u_m \rangle)(\bar{u}_{m|m} - \langle \bar{u}_{m|m} \rangle) \Big\rangle}{\sqrt{\Big\langle (u_m - \langle u_m \rangle)^2 \Big\rangle \Big\langle (\bar{u}_{m|m} - \langle \bar{u}_{m|m} \rangle)^2 \Big\rangle}}$$



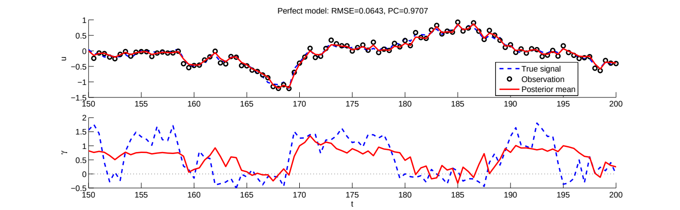
A. Two Moments Filter

1. The perfect forecast model,

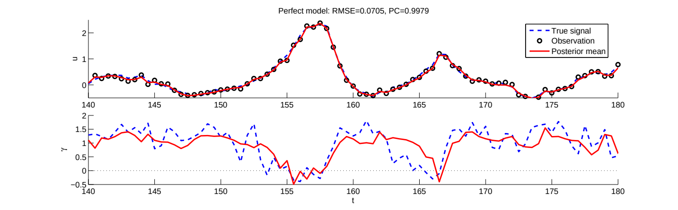
$$du = (-\gamma u + f_u)dt + \sigma_u dW_u,$$

$$d\gamma = (-a\gamma + b\gamma^2 - c\gamma^3 + f_\gamma)dt + (A - B\gamma)dW_c + \sigma dW_\gamma,$$

$$f_u = 0$$



$$f_u = 0.5\sin(0.5t)$$

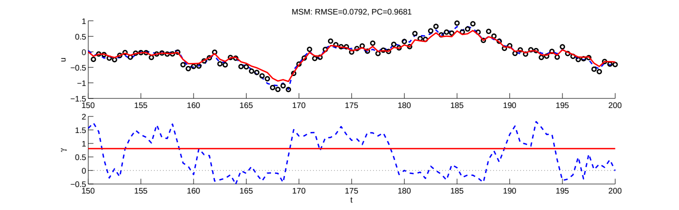


	Perfect in u	Obs error
$f_u = 0$	0.0643 (0.9707)	0.0864
$f_u = 0.5\sin(0.5t)$	0.0706 (0.9979)	0.0864

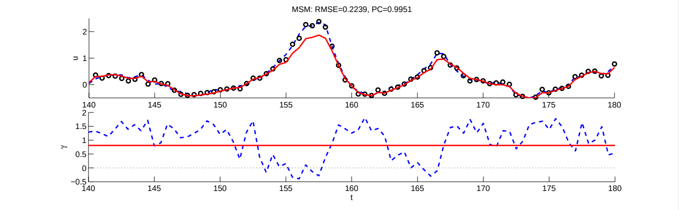
2. Mean stochastic model (MSM), in which the damping in u is set to be the averaged value of γ and therefore no unresolved dynamics is included,

$$du = (-\bar{\gamma}^M u + f_u)dt + \sigma_u dW_u.$$

$f_u = 0$



$f_u = 0.5\sin(0.5t)$



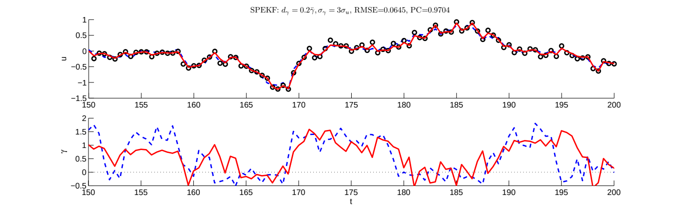
	MSM in u	Perfect in u	Obs error
$f_u = 0$	0.0792 (0.9681)	0.0643 (0.9707)	0.0864
$f_u = 0.5\sin(0.5t)$	0.2239 (0.9951)	0.0706 (0.9979)	0.0864

3. The imperfect forecast model of γ is a linear Gaussian model, such that the forecast model becomes the simplified SPEKF-type model,

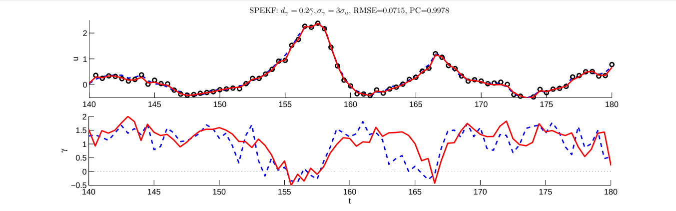
$$du = (-\gamma u + f_u)dt + \sigma_u dW_u,$$
$$d\gamma = d_\gamma^M(\gamma - \hat{\gamma}^M)dt + \sigma_\gamma^M dW_\gamma.$$

Remark: The evolution of the mean and covariance of SPEKF model can be solved analytically.

$f_u = 0$



$f_u = 0.5\sin(0.5t)$



	SPEKF in u	Perfect in u	Obs error
$f_u = 0$	0.0645 (0.9704)	0.0643 (0.9707)	0.0864
$f_u = 0.5\sin(0.5t)$	0.0715 (0.9978)	0.0706 (0.9979)	0.0864

Robustness of the parameters in SPEKF model with $f_u = 0$.

$$du = (-\gamma u + f_u)dt + \sigma_u dW_u,$$
$$d\gamma = d_\gamma^M(\gamma - \hat{\gamma}^M)dt + \sigma_\gamma^M dW_\gamma.$$

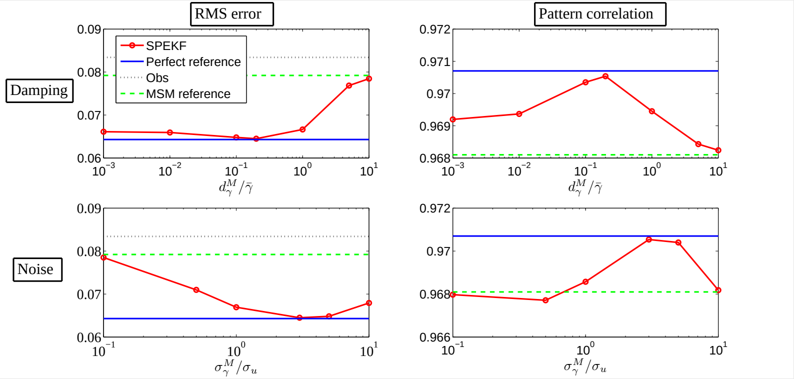


Figure 1: Robustness of the parameters in SPEKF. Top: filtering skill dependence of d_γ^M with fixed ratio $\sigma_\gamma^M/\sigma_u = 3$. Bottom: filtering skill dependence of σ_γ^M with fixed ratio $d_\gamma^M/\bar{\gamma} = 0.2$.

B. Special type of non-Gaussian filter for the test model.

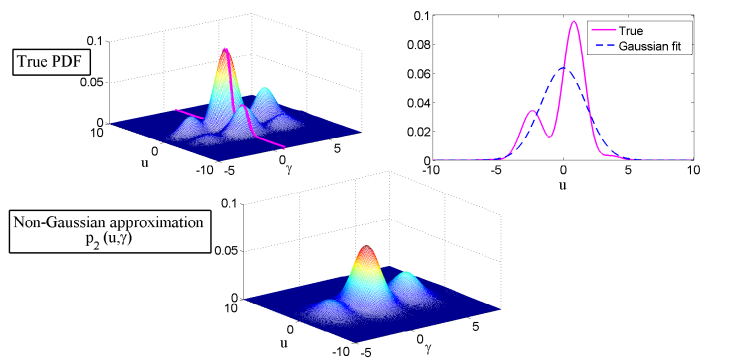
Approximation of the prior distribution $p_2(u,\gamma,t)$:

- Complete recovery of the marginal distribution in γ via

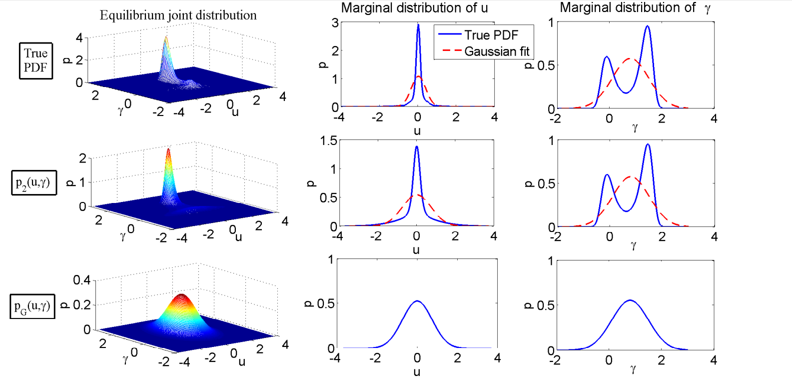
$$M_0(\gamma,t) = \int p(\gamma,u,t)du.$$

- Conditional Gaussian for each fixed γ by making use of

$$M_1(\gamma,t) = \int up(\gamma,u,t)du, \qquad M_2(\gamma,t) = \int u^2 p(\gamma,u,t)du.$$

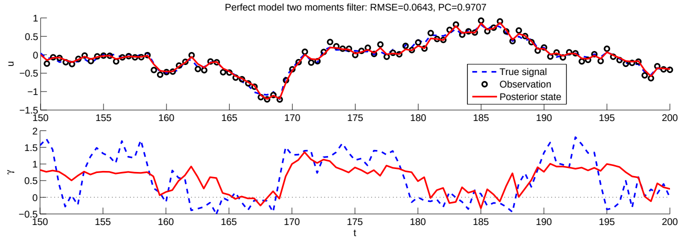


Comparison of the full PDF and its non-Gaussian and Gaussian approximation at the equilibrium in the dynamical regime of filtering turbulent signals.

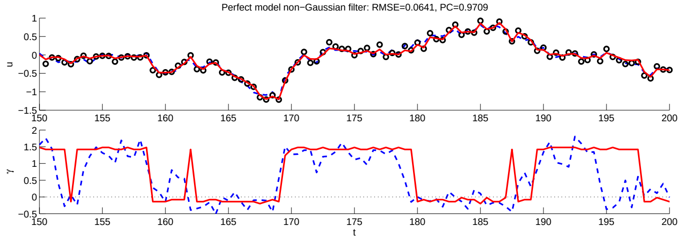


Comparison of two moments filter and non-Gaussian filter.

Two moments filter

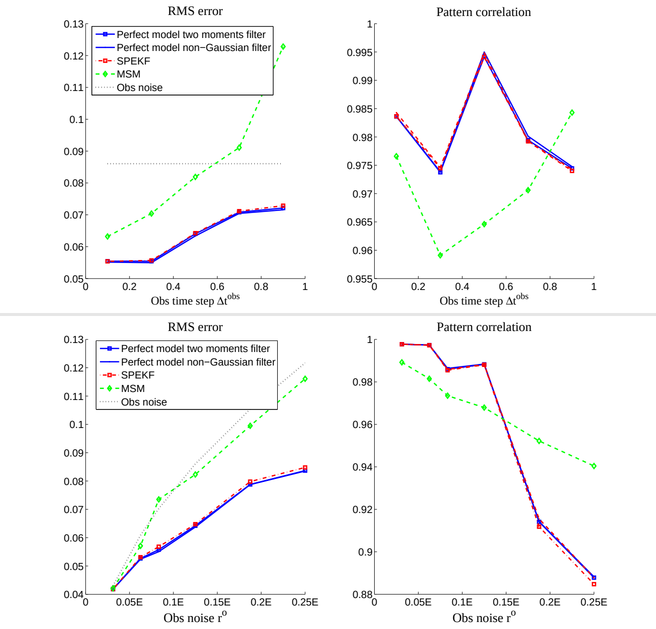


Non-Gaussian filter

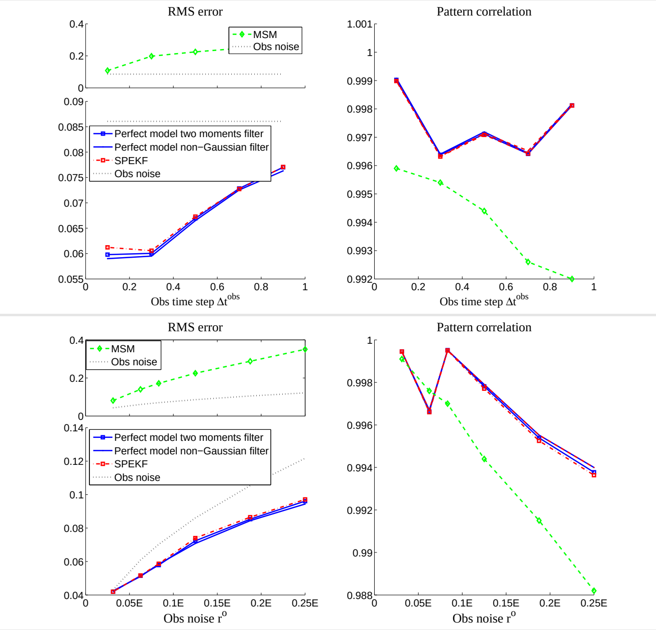


	RMSE (PC) in u	Obs error
Two Moments	0.0643 (0.9707)	0.0864
Non-Gaussian	0.0641 (0.9709)	0.0864

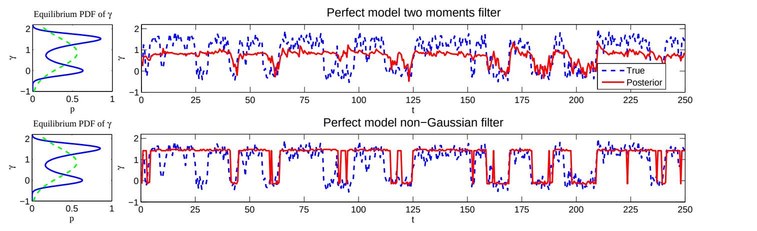
Filtering skill as a function of observation time step (top) and observation noise (bottom) using different filters with respect to the resolved variable u (unforced case: $f_u = 0$).



Filtering skill as a function of observation time step (top) and observation noise (bottom) using different filters with respect to the resolved variable u (unforced case: $f_u = 0.5\sin(0.5t)$).



Comparison of the unresolved variable γ.



1. Path-wise solution.

	Two moments filter	Non-Gaussian filter
RMS error	**0.6184**	0.6853
Pattern correlation	0.5345	**0.5713**

2. Information criteria.

The Shannon entropy, $S(\mathcal{U}_m)$, of the residual $\mathcal{U}_m \equiv u_m - \bar{u}_{m|m}$, $\mathcal{U} \sim p$, is given by

$$S(\mathcal{U}_m) = -\int p(\mathcal{U}_m) \ln p(\mathcal{U}_m) d\mathcal{U}_m.$$

It expresses the uncertainty in the filter estimate $\bar{u}_{m|m}$ about the true state u_m at time t_m.

	Two moments filter	Non-Gaussian filter
Shannon entropy	0.7901	**0.7356**

IV. Summary

In this work, we studied

1. Filtering skill using two moments filter with different prediction models.
2. Filtering skill using two moments and non-Gaussian filter in a perfect model setting.

Main conclusions are summarized as follows:

1. For the two moments filter, SPEKF has comparable filtering skill with the perfect model filter while MSM filter has low filtering skill.
2. The special non-Gaussian filter has very little improvement of the filtering skill with respect to the resolved variable u.
3. Filter estimates of the unresolved variable γ are quite different using two moments filter and non-Gaussian filter.

Thank you