FMM Code Libraries for Computational Electromagnetics

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Applications

- Stealth
- Electromagnetic interference
- Antennas on complex platforms (ships/aircraft)
- Packaging, mixed-signal analysis
- Wideband antennas
- etc.

Core Algorithms

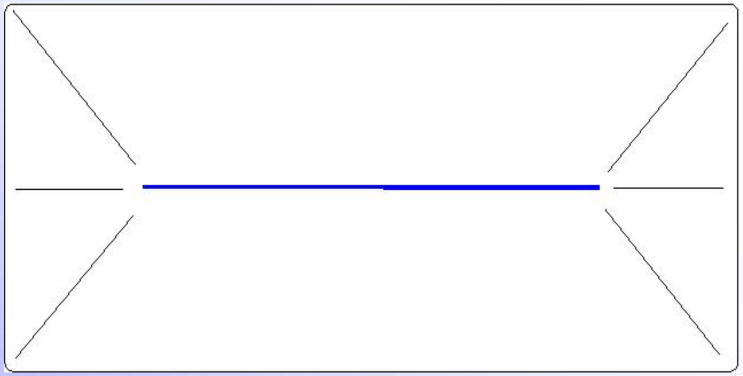
- Fast Multipole Methods
  - O(N) or O(N log N) techniques for applying dense "method of moment" matrices
- Discretization schemes
  - High order accuracy, even with geometric singularities
- Integral equation formulations
  - Closed and open surfaces, complex materials
- Fast, Direct Solvers for Ill-Conditioned Problems
  - handle near resonances, modal analysis

Design Goals

- Wideband performance
- Tunable precision
- Automatic adaptivity
- Geometric flexibility
- Modular software construction

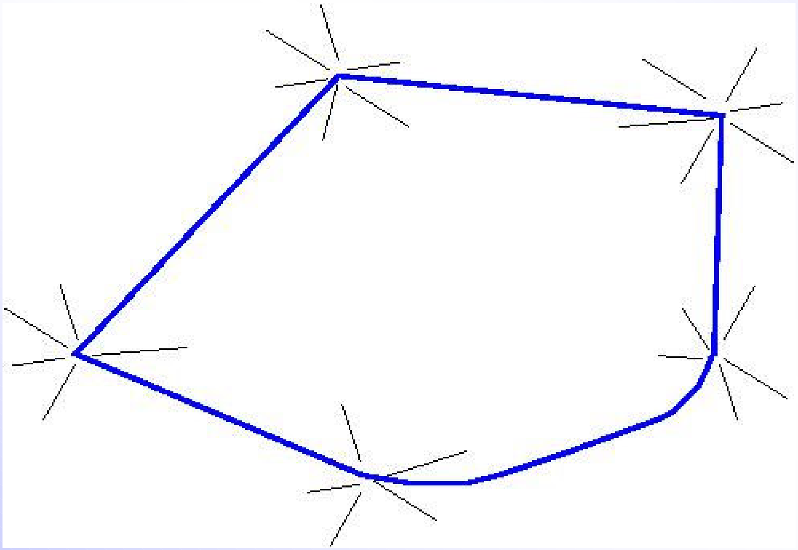
Quadratures

- When solving integral equations with FMM acceleration, an expensive step is the calculation of quadratures for near neighbor interactions
- We have developed robust universal rules:
  - Space divided into regions
  - Generalized Gaussian quadratures derived for each



Quadratures (cont.)

- These rules can incorporate families of singularities to achieve high order accuracy even in the presence of corners

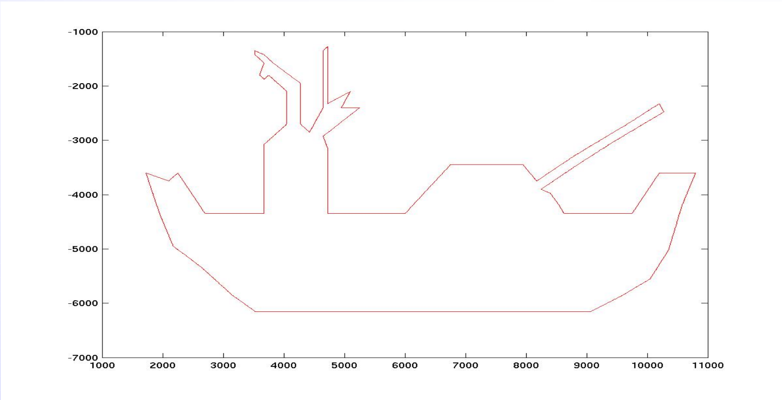


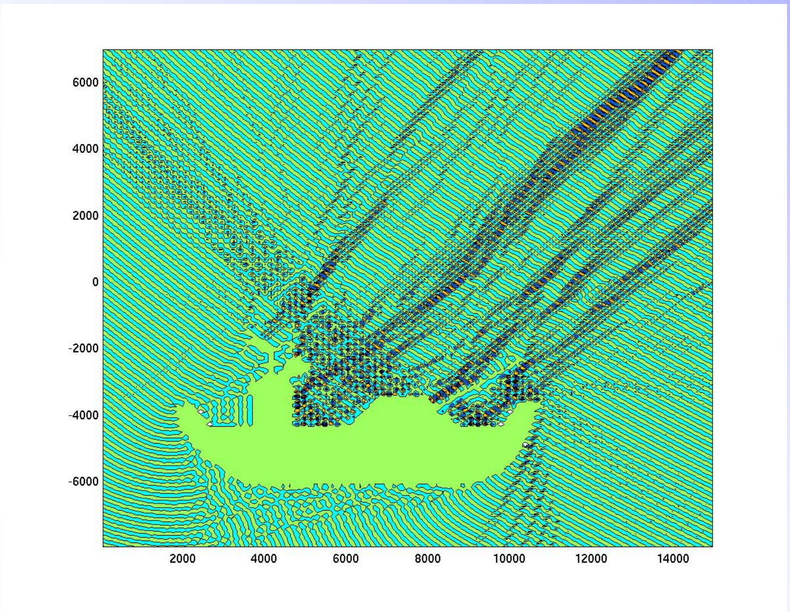
2D Wideband FMM

N	k	T_{FMM}	Storage	T_{DIR}	Error
50,000	10^{-5}	19	1.1 MW	1,700	10^{-7}
50,000	10	14	1.1 MW	1,300	10^{-7}
50,000	1,000	14	1.2 MW	1,400	10^{-7}
50,000	10,000	42	4.8 MW	2,400	10^{-7}
500,000	10^{-5}	143	15 MW	170,000	10^{-7}
500,000	10	107	16 MW	130,000	10^{-7}
500,000	1,000	110	18 MW	140,000	10^{-7}
500,000	10,000	218	24 MW	240,000	10^{-7}

Uniform distribution, unit box

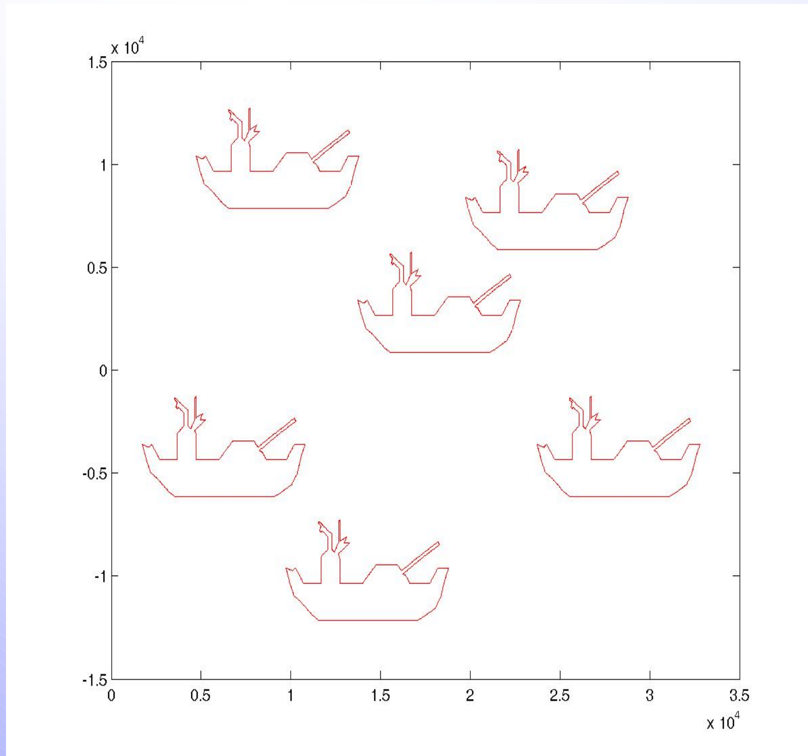
2D Wideband FMM

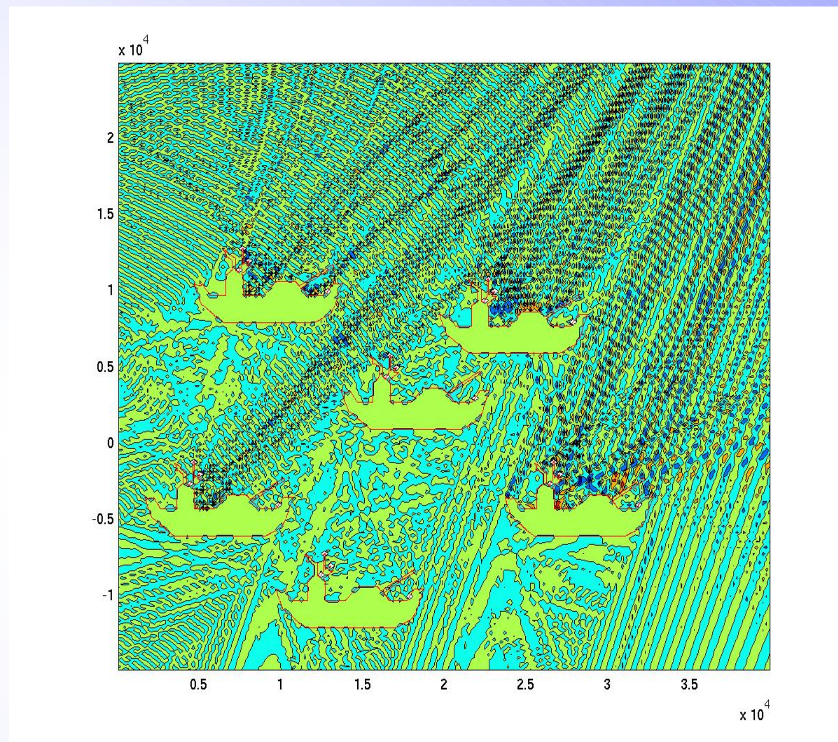




2,520 points, 10th order accuracy, 50λ / ship
22 seconds (1GHz Pentium 3 Laptop)

2D Wideband FMM





15,120 points, 10^{th} order accuracy, 50λ / ship
5 minutes (1GHz Pentium 3 Laptop)

3D Low Frequency FMM

N	k	T_{FMM}	T_{DIR}	Error
28,000	10^{-8}	32	800	10^{-3}
28,000	10	33	800	10^{-3}
154,000	10^{-8}	186	24,000	10^{-3}
154,000	10	169	27,000	10^{-3}

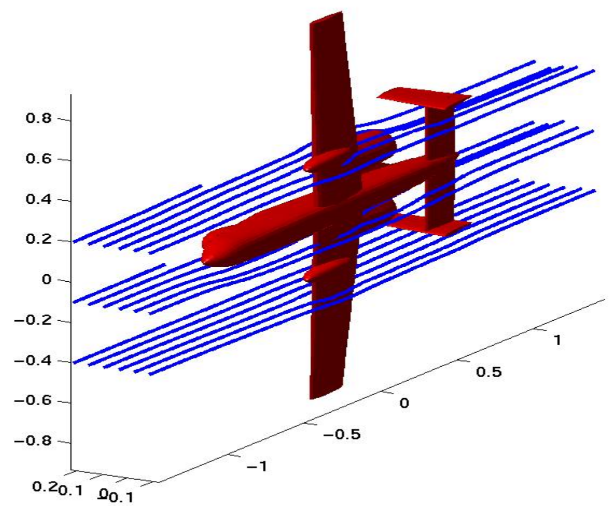
Uniform distribution, unit box

3D Low Frequency FMM

N	k	T_{FMM}	T_{DIR}	Error
28,000	10^{-8}	60	800	10^{-6}
28,000	10	61	800	10^{-6}
154,000	10^{-8}	286	24,000	10^{-6}
154,000	10	256	27,000	10^{-6}

<u>Uniform distribution, unit box</u>

Example: Potential Flow

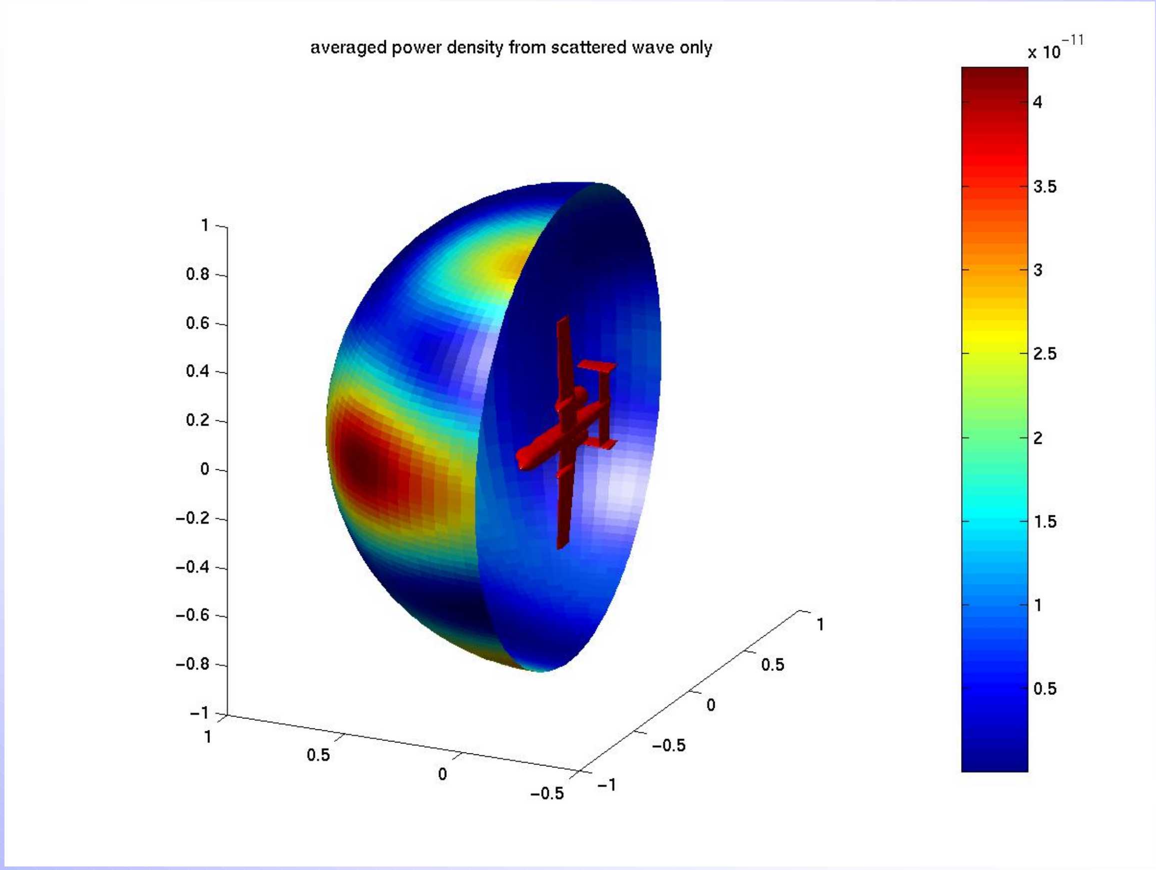


30,000 panels

60 iterations, 100 matrix-vector multiplications

Solution time: 5 minutes (Pentium IV, 1.6GHz)

Example: Acoustic Scattering



Part 1 Summary

- Core FMM technology is reasonably mature from zero to hundreds of wavelengths
  - Remaining issues are software issues: Code optimization, Supportability, Parallel platforms, etc.
- Integral equation formulations are reasonably mature for piecewise isotropic materials with closed surfaces
  - Open surfaces still active area of research
- Geometric singularities in three dimensions
  - Active area of research

Part 1 Summary (cont.)

- Anisotropic materials
  - Volume integral equations vs. FEM/BEM hybrids
  - Core library tools to be developed include fast volume integral techniques (FFT and FMM based)
- Resonant cavities, Modal Analysis
  - Not suitably addressed by existing algorithms
  - Iterative methods (frequency domain) converge poorly. Marching methods (time domain) take excessively many steps.

Part 2: Fast Direct Solvers

- Can one construct FMM-type schemes which will yield “sparse” factorizations of the solution operator for an integral equation?

- This would overcome the difficulties with iterative methods and allow more efficient solution of problems with multiple right-hand sides

Brief History

- 1989: Chew - $O(n^2)$ direct solver for n “small” scatterers in two dimensions
- 1991: G- and Rokhlin - $O(n)$ direct solver for dense linear systems that arise from one-dimensional integral equations
- 1993 Canning - matrix compression
- 1993 Alpert, Beylkin, Coifman, Rokhlin - Wavelet-based compression
- 1995 Lu and Chew – $O(n^2)$ direct solver for volume integrals in three dimensions

Brief History

- 1996: Michielssen, Boag, Chew:

  $O(n \log^2 n)$ direct solver for elongated objects
- 1997 Lee and G-

  Automatic mesh refinement for SKIE
- 2001 Gope and Jandhyala

  $O(n^{2.3})$ solver for non-oscillatory boundary integral equations (capacitance extraction, etc.)
- 2002 Chen

  $O(n^{1.5})$ solver for volume integral equations in two dimensions

Simplest Example

- Consider the ODE

$$u''(x) + p(x)\, u'(x) + q(x)\, u(x) = f(x)$$

$$u(0) = u(1) = 0$$

- Seek representation in the form

$$u(x) = \int G(x,t)\, s(t)\, dt$$

where $G(x,t)$ is the Green's function for the 1D Laplace operator with zero Dirichlet conditions.

Simplest Example

- Obtain Fredholm integral equation of second kind

$$s(x) + p(x) \int G_x(x,t)\, s(t)\, dt + q(x) \int G(x,t)\, s(t)\, dt = f(x)$$

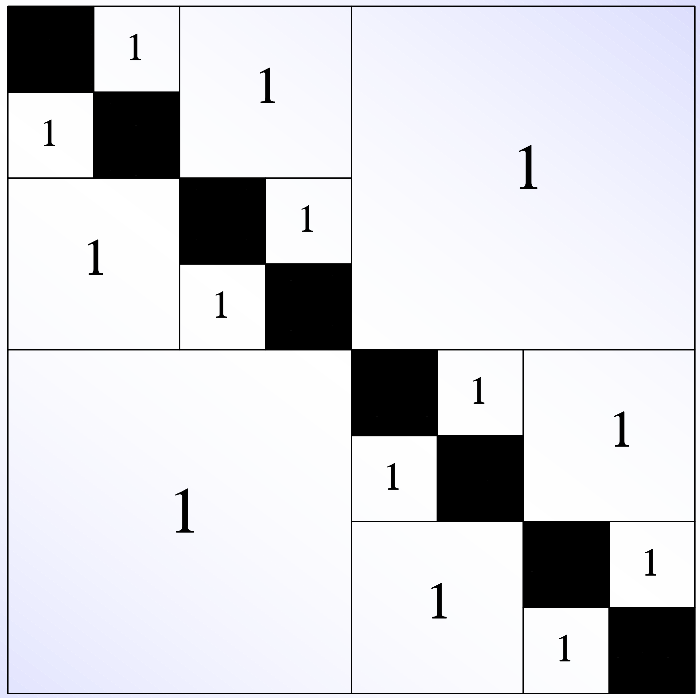
or $$P\, s = f$$

where P is a dense matrix.

<u>How can one solve this directly in less than $O(N^3)$ time?</u>

Simplest Example

$P =$



Simplest Example

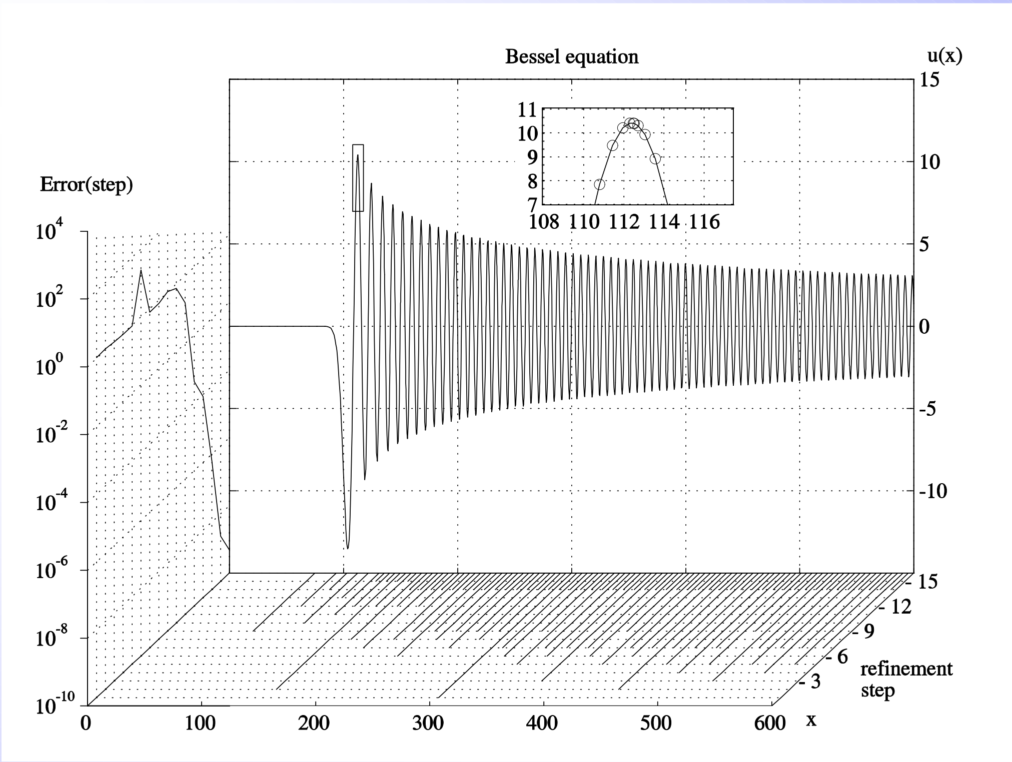
- Simple recursive scheme leads to O(N log N) algorithm which requires
  - Compressed representation of all low rank submatrices
  - Sherman-Morrison-Woodbury formula

- More complex scheme leads to $O(N p^2)$ algorithm where p is the desired order of accuracy.

Bessel equation

10 pts/ λ

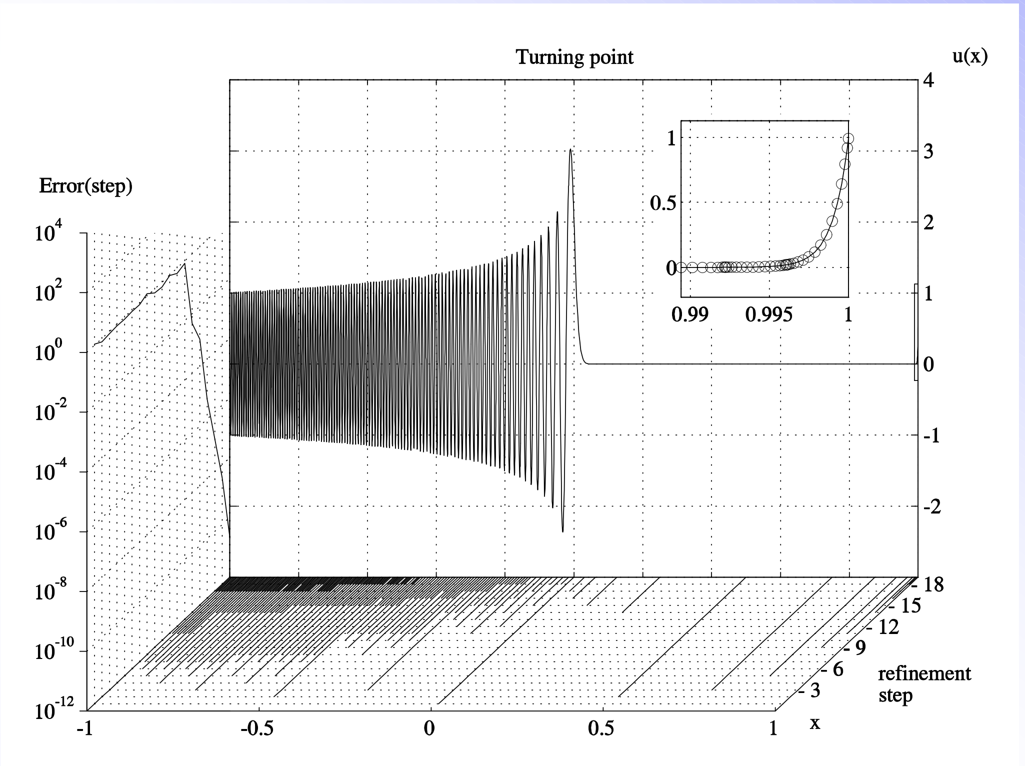
11 digit accuracy



Turning Point

$\varepsilon u'' - x u = 0$

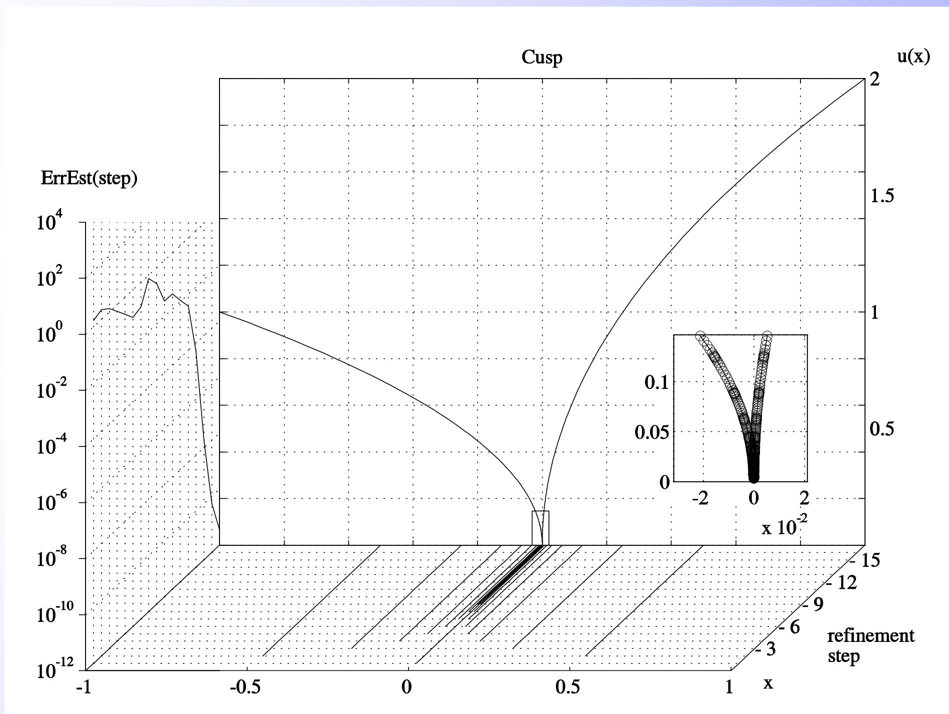
Solution is linear combination of Airy functions



Cusp

$\varepsilon u'' + x u' - u/2 = 0$

Solution is linear combination of parabolic cylinder functions



Two Dimensions

- Currently $O(n^{3/2})$ for oscillatory problems (like the algorithms of Chew and Chen) but applicable to boundary integral equations as well as volume integral equations
- Sparse factorization can be updated when geometry is perturbed

Model Problem (2D)

- Solve $\nabla^2 u + k^2 (1 + q(x))\, u = 0$

  with Sommerfeld condition at infinity.

- Letting $G(x,y) = H_0(||x-y||)$, we obtain Lippman-Schwinger equation

$$u(x) + k^2 \int G(x,t)\, q(t)\, u(t)\, dt = f(x)$$

Analytical Fact

- Given an (m x n) matrix A of ε-rank k, there exists a (k x k) submatrix of A denoted by $A_{k,k}$ and mappings $proj: C^m \rightarrow C^k$ and $eval : C^k \rightarrow C^n$ such that the condition numbers of *proj* and *eval* are less than ($2\sqrt{k}$) and

$$A \approx eval \circ A_{k,k} \circ proj$$

- Similar to SVD
- Cheng, Gimbutas, Martinsson, Rokhlin
- Gu and Eisenstadt

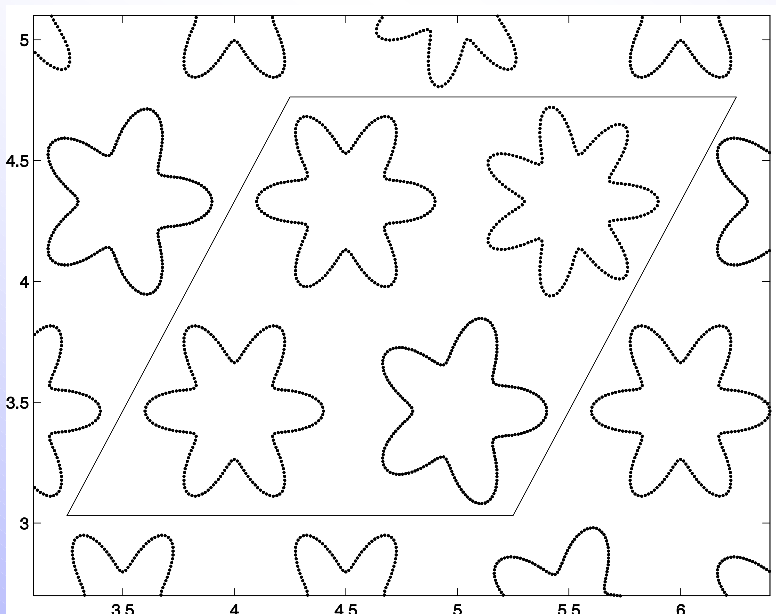
Physical Interpretation

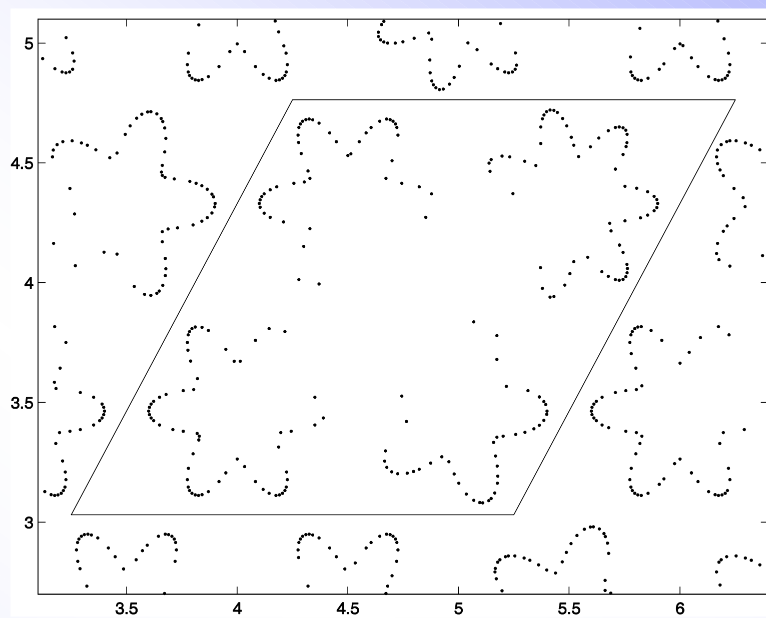
- Suppose one has a collection of m charges in S and the matrix A describes the field induced at n target points in T.
- Then there is a *subset* of those same m charges of dimension k that can be used to represent the field in T to precision ε.
- Likewise, there is a *subset* of the n targets in T of dimension k from which the field at all n targets can be generated to precision ε.

Skeletons

- The choice of the k-dimensional subset, called a *skeleton*, is not unique.
- Moreover, incoming and outgoing skeletons can be the same
- We define *scattering matrix* for region D as mapping from incoming field at all points in D to “charge” strengths at all points in D which describe the outgoing field

Example





Benefit

D	E		
F	G		

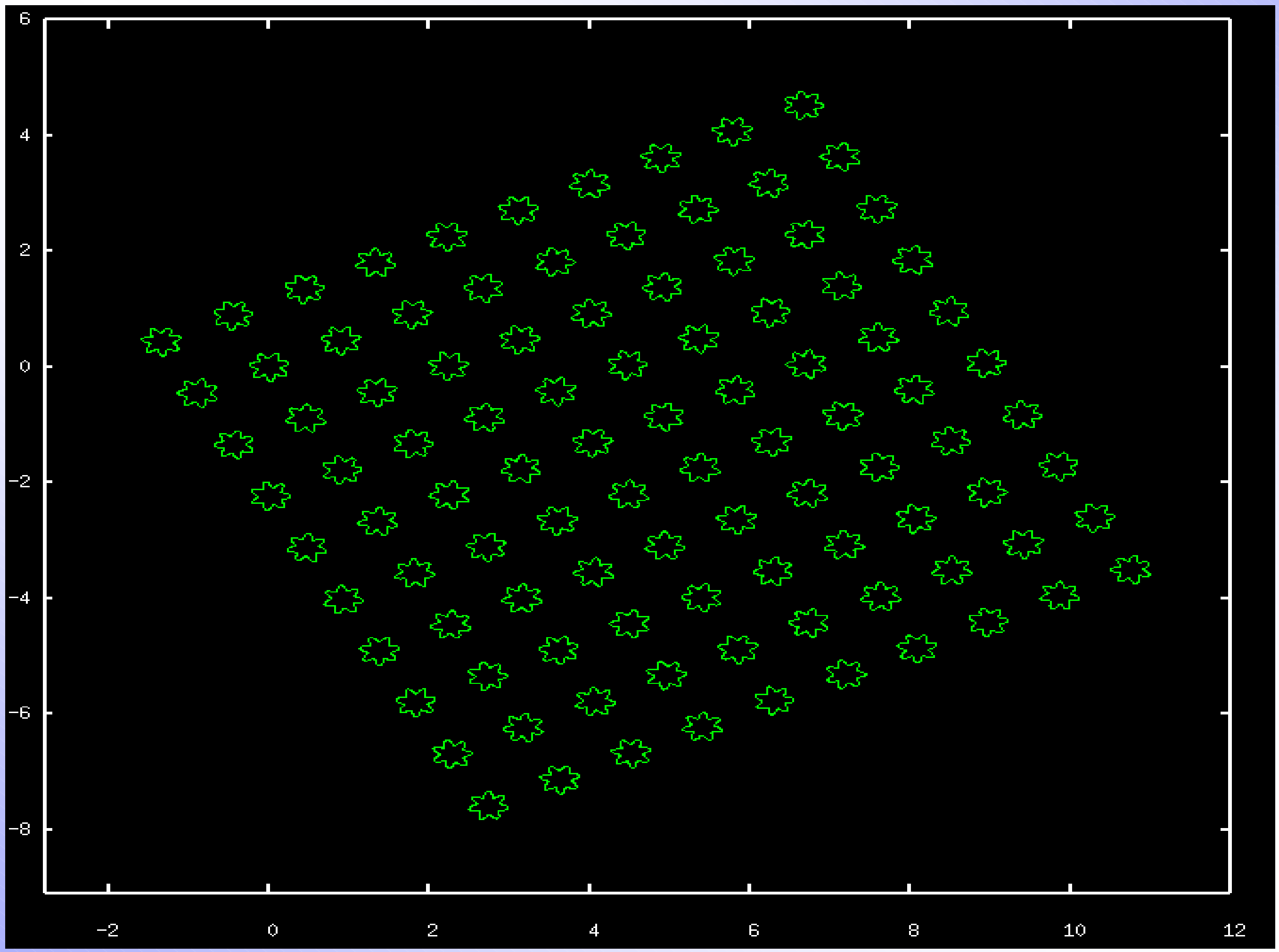
- Interaction matrices can be compressed using “skeletonized” scattering matrices
- Allows recursion

Procedure

- Begin with hierarchical subdivision of scatterer (like the finest level of an FMM data structure).
- Compress interactions between each subregion and "rest of world" $O(N^{3/2})$ work
- Upward pass: merge scattering matrices
- Downward pass: construct splitting and exchange matrices (analogous to FMM translation operators)

(Cheng, Rokhlin)

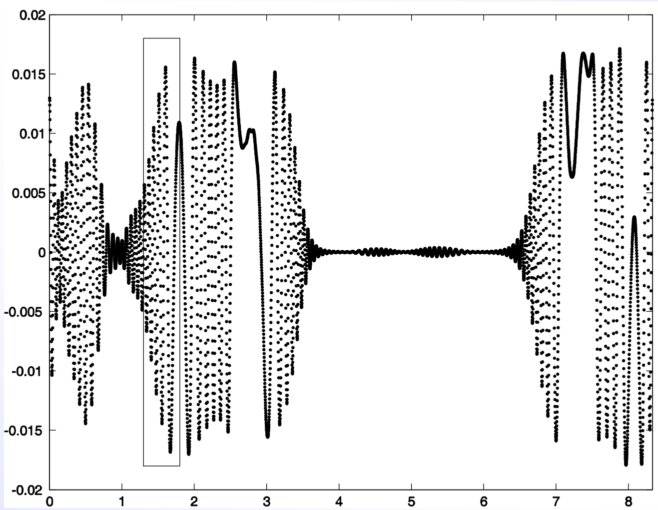
100 dielectric bodies



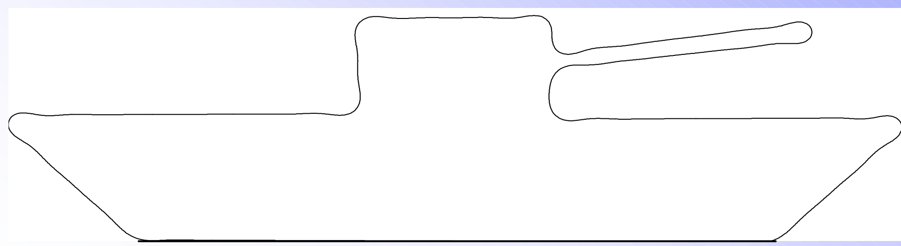
Numerical Results

- 100 Snowflakes, 15 Wavelengths
- TE excitation
- 15,000 unknowns
- 10 Minutes solve time
- 7 digit accuracy

- 400 Ellipses, 50 Wavelengths
- TE excitation
- 60,000 unknowns
- 21 Minutes solve time
- 7 digit accuracy



Martinsson, Rokhlin



N_{INIT}	N_{FINAL}	k	T_{FACTOR}	T_{SOLVE}	Error
800	435	21	15 s	.003 s	10^{-7}
3,200	683	79	53 s	.12 s	10^{-7}
12,800	1,179	316	180 s	.39 s	10^{-7}
25,600	1,753	632	430 s	7.5 s	10^{-7}

Summary (Part 2)

- $O(n^{3/2})$ for space-filling oscillatory problems in 2D
- $O(n \log n)$ for many boundary-value problems in 2D
- Current implementations are memory intensive

- Much work remains for 3D, both analytic and numerical

Conclusions

- Fast Multipole and related techniques have reached a point of maturity where top-down, modular design is feasible.
  - Requires standard interfaces, supportable software infrastructure, and careful library design.
  - Will allow rapid development of application layers, and standardization of training for nonspecialists.
- Significant research still required for anisotropic materials, but there have been promising developments (e.g. Boeing, NGC, UIUC)
- Fast direct solvers are likely to become important for large-scale problems near resonance