Computation of Scattering Clusters of Spheres Using the Fast Multipole Method

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Outline

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- Method of Solution
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  - Iterative Methods
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Introduction

Multiple Scattering Problems

- Sound propagation in disperse media (particles, bubbles, etc.)
- Modeling of scattering from environment (humans, animals, fish, etc.)
- Electromagnetic scattering problems (microwaves, optics, etc.)
- Efficient parametrization in inverse problems (tomography, etc.)

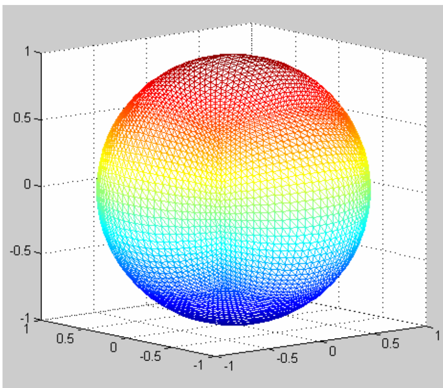
Why Multipole Methods?

Scattering computation with BEM for a single object

BEM Mesh
5402 nodes
10800 elements
Discretization # Dn=30

Can be used
to compute the field
only for ka < 25
(for human head < 16.5 kHz)

Formal Requirement:
ka << Dn



Run Time for one frequency
on Dual Processor
1 GHz Pentium III ~ 1 day.

Required Maximum Frequency
to Compare with Experimental
HRIR (22 or 44 kHz), 200 frequencies

Why Multipole Methods?

Multipole Methods

- Meshless for spherical scatterers
- Fast

BEM

- Needs Mesh
- Relatively Slow

Problem Formulation

Equations and Boundary Conditions

Wave Equation

Fourier Transform

Helmholtz Equation

$$\nabla^2 \psi + k^2 \psi = 0,$$

Impedance Boundary Conditions

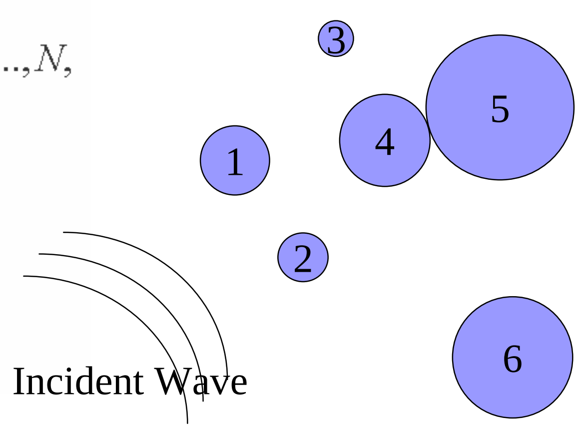
$$\left(\frac{\partial \psi}{\partial n} + i\sigma_p \psi \right) \Big|_{S_p} = 0, \qquad p = 1, ..., N,$$

Field Decomposition

$$\psi(\mathbf{r}) = \psi_{in}(\mathbf{r}) + \psi_{scat}(\mathbf{r}),$$

Sommerfield Radiation Condition

$$\lim_{r \to \infty} r \left(\frac{\partial \psi_{scat}}{\partial r} - ik\psi_{scat} \right) = 0.$$



Multipole Reexpansion (T-Matrix) Method

Scattered Field Decomposition

$$\psi_{scat}(\mathbf{r}) = \sum_{p=1}^{N} \psi_p(\mathbf{r}), \quad \lim_{r\to\infty} r\left(\frac{\partial \psi_p}{\partial r} - ik\psi_p\right) = 0, \qquad p = 1,\ldots,N.$$

Singular Basis Functions

Hankel Functions

$$\psi_p(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_n^{(p)m} S_n^m(\mathbf{r}-\mathbf{r}_p'), \qquad S_n^m(\mathbf{r}) = h_n(kr) Y_n^m(\theta,\varphi).$$

Expansion Coefficients

Spherical Harmonics

$$\mathbf{A} = \left(A_0^0, A_1^{-1}, A_1^0, A_1^1, A_2^{-2}, A_2^{-1}, A_2^0, A_2^1, A_2^2, \ldots\right)^T,$$

Vector Form:

$$\psi_p(\mathbf{r}) = \overline{\mathbf{A}^{(p)}} \cdot \mathbf{S}(\mathbf{r}-\mathbf{r}_p').$$

dot product

Incident Field Decomposition and T-matrix for a Single Sphere

$$\psi_{in}(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} E_n^{(in)m}(\mathbf{r}'_p)R_n^m(\mathbf{r}-\mathbf{r}'_p) = \overline{\mathbf{E}^{(in)}(\mathbf{r}'_p)}\cdot\mathbf{R}(\mathbf{r}-\mathbf{r}'_p).$$

Regular Basis Functions

Bessel Functions

$$\mathbf{R}(\mathbf{r}) = \{R_n^m(\mathbf{r})\}, \quad R_n^m(\mathbf{r}) = j_n(kr)Y_n^m(\theta,\varphi).$$

Analytical Solution of the Problem:

$$A_n^{(p)m} = -\frac{kj'_n(ka_p) + i\sigma_p j_n(ka_p)}{kh'_n(ka_p) + i\sigma_p h_n(ka_p)}E_n^{(in)m}(\mathbf{r}'_p),$$

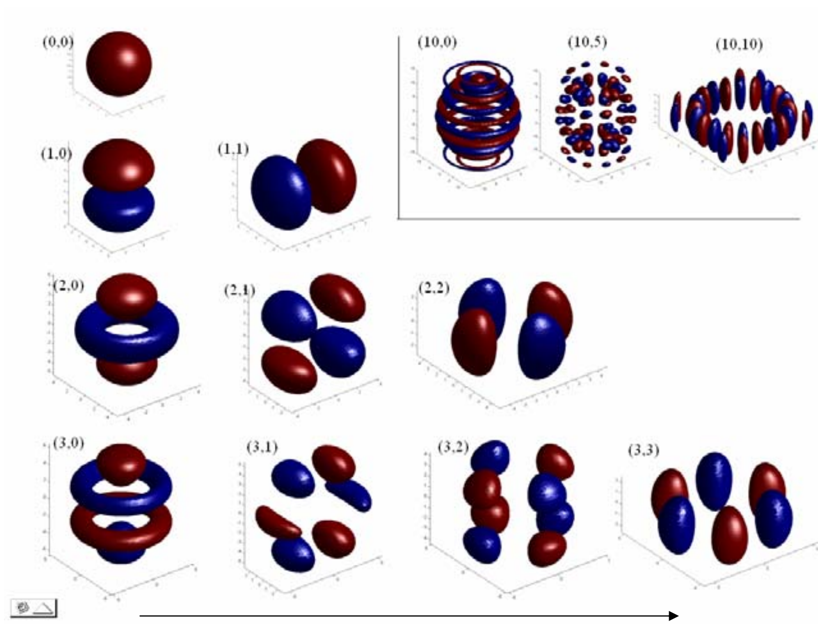
$$n = 0,1,2,...; \quad m = -n,...,n.$$

T-matrix

$$\mathbf{A}^{(p)} = \mathbf{T}^{(p)}\mathbf{E}^{(in)}, \qquad p = 1,...,N.$$

Isosurfaces For Regular Basis Functions

$$\mathrm{Re}\{R_n^m(\mathbf{r})\} = const$$

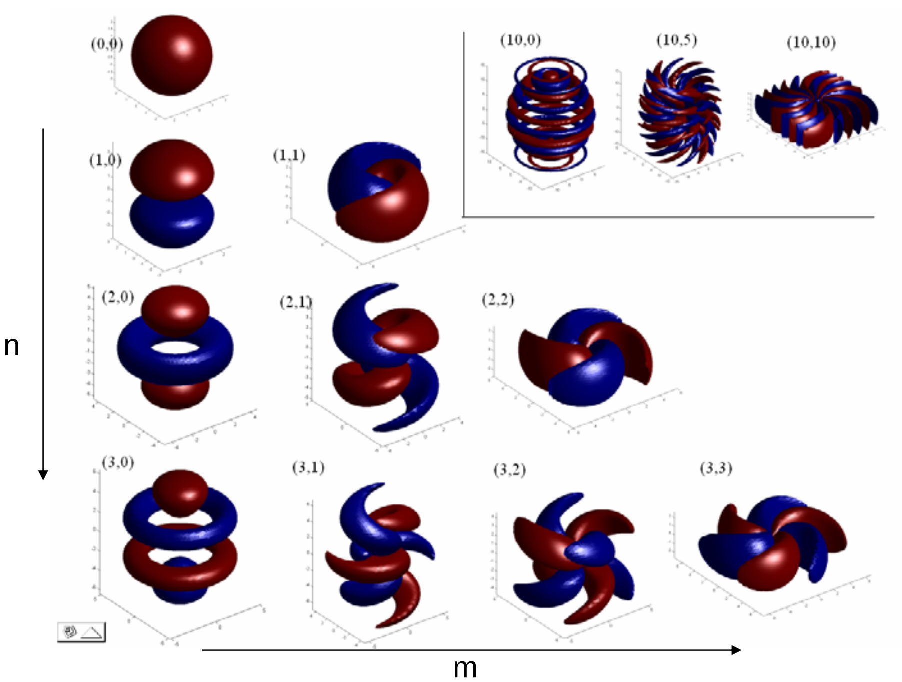


n

m

Isosurfaces For Singular Basis Functions

$$\mathrm{Re}\{S_n^m(\mathbf{r})\} = const$$



Solution of Multiple Scattering Problem

"Effective" Incident Field

$$\psi(\mathbf{r}) = \psi_q(\mathbf{r}) + \psi_{in}(\mathbf{r}) + \psi_{other}^{(q)}(\mathbf{r}) = \psi_q(\mathbf{r}) + \psi_{eff}^{(q)(in)}(\mathbf{r}),$$

$$\psi_{other}^{(q)}(\mathbf{r}) = \sum_{p\neq q} \overline{\mathbf{A}^{(p)}} \cdot \mathbf{S}(\mathbf{r}_p) = \overline{\mathbf{B}^{(q)}} \cdot \mathbf{R}(\mathbf{r}_q), \quad \psi_{eff}^{(q)(in)}(\mathbf{r}) = \overline{\mathbf{E}_{eff}^{(q)}} \cdot \mathbf{R}(\mathbf{r}_q).$$

Coupled System of Equations:

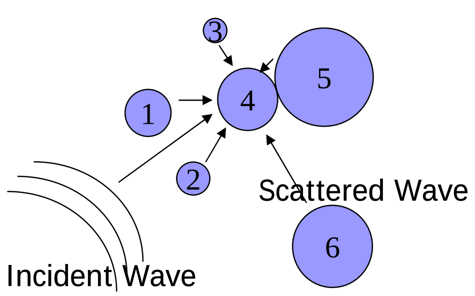
$$\mathbf{A}^{(q)} = \mathbf{T}^{(q)}\mathbf{E}_{eff}^{(q)},$$

$$\mathbf{B}^{(q)} = \sum_{p\neq q} (\mathbf{S}|\mathbf{R})(\mathbf{r}_q' - \mathbf{r}_p')\mathbf{A}^{(p)},$$

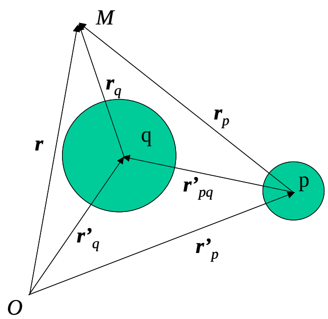
$$\mathbf{E}_{eff}^{(q)} = \mathbf{E}^{(in)}(\mathbf{r}_q') + \mathbf{B}^{(q)},$$

$$q = 1, ..., N$$

(S|R)-Translation Matrix



Reexpansions/Translations



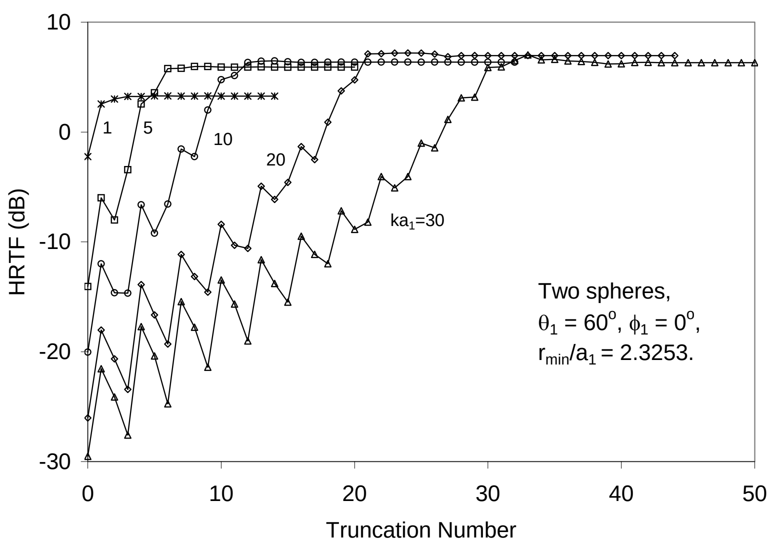
$$\psi_p(\mathbf{r}) = \begin{cases} \overline{\mathbf{A}^{(p)}_{(R,q)}} \cdot \mathbf{R}(\mathbf{r} - \mathbf{r}'_q), & |\mathbf{r} - \mathbf{r}'_q| < |\mathbf{r}'_q - \mathbf{r}'_p| \\ \overline{\mathbf{A}^{(p)}_{(S,q)}} \cdot \mathbf{S}(\mathbf{r} - \mathbf{r}'_q), & |\mathbf{r} - \mathbf{r}'_q| > |\mathbf{r}'_q - \mathbf{r}'_p| \end{cases}$$

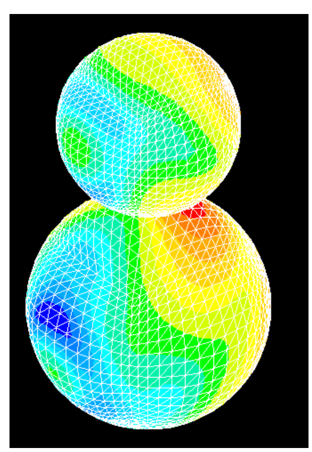
$$\mathbf{A}^{(p)}_{(R,q)} = (\mathbf{S}|\mathbf{R})(\mathbf{r}'_q - \mathbf{r}'_p)\mathbf{A}^{(p)}, \quad \mathbf{A}^{(p)}_{(S,q)} = (\mathbf{S}|\mathbf{S})(\mathbf{r}'_q - \mathbf{r}'_p)\mathbf{A}^{(p)}.$$

$$\mathbf{S}(\mathbf{r} - \mathbf{r}'_p) = \begin{cases} (\mathbf{S}|\mathbf{R})^T(\mathbf{r}'_q - \mathbf{r}'_p)\mathbf{R}(\mathbf{r} - \mathbf{r}'_q), & |\mathbf{r} - \mathbf{r}'_q| < |\mathbf{r}'_q - \mathbf{r}'_p| \\ (\mathbf{S}|\mathbf{S})^T(\mathbf{r}'_q - \mathbf{r}'_p)\mathbf{S}(\mathbf{r} - \mathbf{r}'_q), & |\mathbf{r} - \mathbf{r}'_q| > |\mathbf{r}'_q - \mathbf{r}'_p| \end{cases},$$

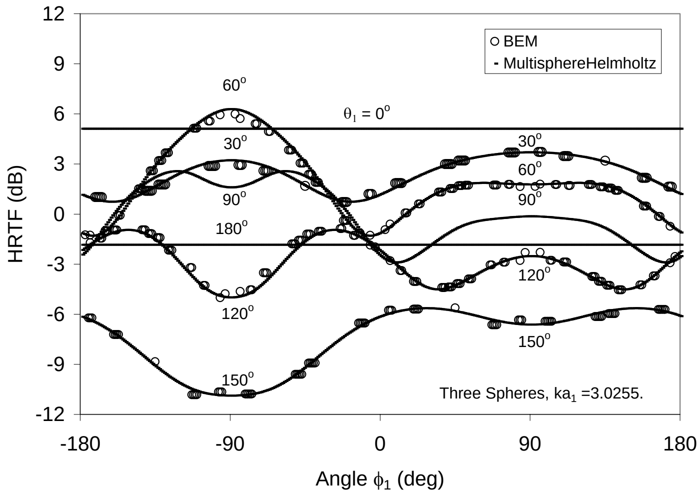
$$\mathbf{R}(\mathbf{r} - \mathbf{r}'_p) = (\mathbf{R}|\mathbf{R})^T(\mathbf{r}'_q - \mathbf{r}'_p)\mathbf{R}(\mathbf{r} - \mathbf{r}'_q).$$

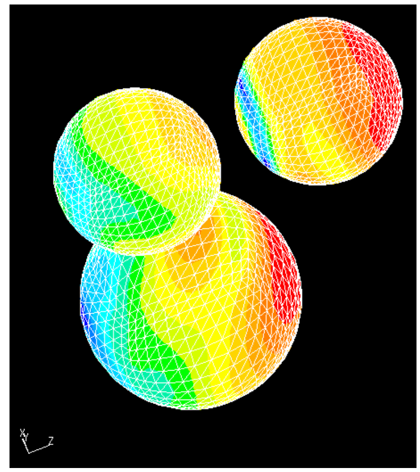
Two Spheres: Convergence with Respect to Truncation Number





Three Spheres Comparisons of BEM & MultisphereHelmholtz





BEM: 5184 triangular elements
MH: N_{trunc} = 9 (100 coefficients for each sphere)

Conclusions on T-matrix Method

- We used recursive computation of translation matrices (Chew, 1992; Gumerov & Duraiswami, 2001).
- In some cases speed up of computations 10^3-10^4 times compared to BEM.
- But… Computational Complexity is $O(N^3P^3)$= $O(N^3 p^6)$, where $P= p^2$ is the total length of the vector of expansion coefficients. Method is not suitable for large N and ka.
- Details can be found in our paper JASA 112(6), 2002, 2688-2701.

Iterative Methods

Reflection Method &
Krylov Subspace Method (GMRES)

Reflection (Simple Iteration) Method:

$$\mathbf{A}_j^{(q)} = \mathbf{T}^{(q)}\left[\mathbf{E}^{(in)}(\mathbf{r}'_q) + \mathbf{B}_j^{(q)}\right],$$

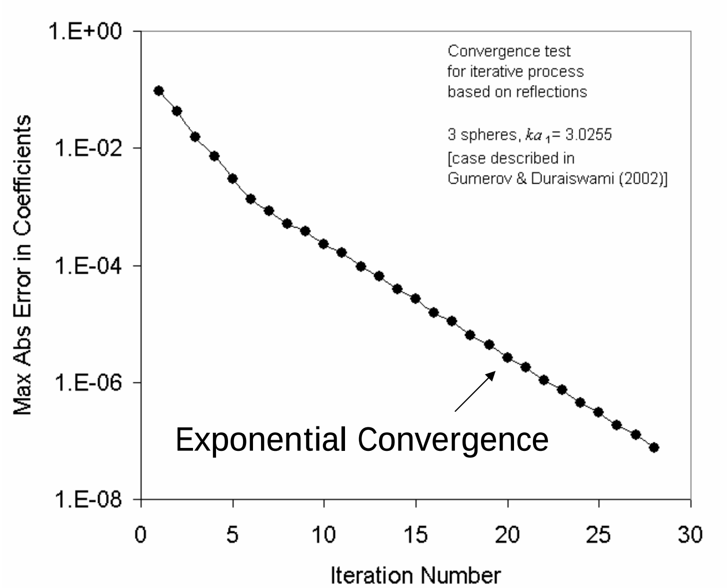
$$\mathbf{B}_{j+1}^{(q)} = \sum_{p \neq q} (\mathbf{S}|\mathbf{R})(\mathbf{r}'_q - \mathbf{r}'_p)\mathbf{A}_j^{(p)},$$

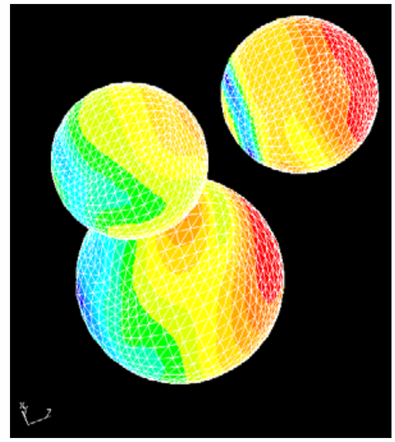
$$\left|\mathbf{A}_j^{(q)} - \mathbf{A}_{j+1}^{(q)}\right| < \epsilon, \quad q = 1, ..., N.$$

General Formulation (used in GMRES)

$$\left[\mathbf{I} - \mathbf{T}^{(q)} \sum_{p \neq q} (\mathbf{S}|\mathbf{R})(\mathbf{r}'_q - \mathbf{r}'_p)\right]\mathbf{A}^{(q)} = \mathbf{T}^{(q)}\mathbf{E}^{(in)}(\mathbf{r}'_q).$$

Convergence of Reflection Iteration Method





Conclusions on Iterative Methods

- Both the Reflection Method (RM) and the GMRES converge well, while the RM is simpler and faster;
- Some problems in convergence were found for larger ka and regular spacing of the scatterers;
- In iterative methods fast translation algorithms can be used (we used $O(p^3)=O(P^{3/2})$ fast translation based on sparse matrix decomposition of translation operators). This cost potentially can be reduced further (we are working on *O(PlogP)* methods).
- Complexity of Iterative Methods in this case $O(N^2 N_{iter} p^3)$;
- Savings in complexity compared to straightforward T-matrix are $O(p^3 N/N_{iter})$
- For $N\sim200$, $N_{iter}\sim20$, $p\sim10$ $(P\sim100)$ this yields of order 10^4 times savings.

Fast Multipole Method

Some Facts on the Fast Multipole Methods (FMM)

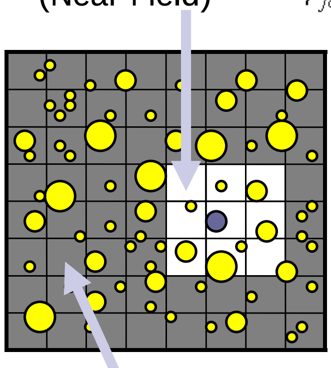
- Introduced by Rokhlin & Greengard (1987,1988) for computation of 2D and 3D fields for Laplace Equation;
- Reduces complexity of matrix-vector product from $O(N^2)$ to $O(N)$ or $O(N \log N)$ (depends on data structure);
- Hundreds of publications for various 1D, 2D, and 3D problems (Laplace, Helmholtz, Maxwell, Yukawa Potentials, etc.);
- Application to acoustical scattering problems (Koc & Chew, 1998; JASA);
- We taught the first in the country course on FMM fundamentals & application at the University of Maryland (2002,2003);
- Some technical reports are available online;
- A book on the FMM for the 3D Helmholtz equation submitted to Academic Press.

Far and Near Fields

$$\psi(\mathbf{r}) = \psi_q(\mathbf{r}) + \psi_{in}(\mathbf{r}) + \psi_{near}^{(q)}(\mathbf{r}) + \psi_{far}^{(q)}(\mathbf{r}) = \psi_q(\mathbf{r}) + \psi_{eff}^{(q)(in)}(\mathbf{r}),$$

$$\psi_{near}^{(q)}(\mathbf{r}) = \sum_{\mathbf{r}_p \in Neighborhood(\mathbf{r}_q)} \overline{\mathbf{A}^{(p)}} \cdot \mathbf{S}(\mathbf{r}_p) = \overline{\mathbf{N}^{(q)}} \cdot \mathbf{R}(\mathbf{r}_q),$$

$$\psi_{far}^{(q)}(\mathbf{r}) = \sum_{\mathbf{r}_p \notin Neighborhood(\mathbf{r}_q)} \overline{\mathbf{A}^{(p)}} \cdot \mathbf{S}(\mathbf{r}_p) = \overline{\mathbf{F}^{(q)}} \cdot \mathbf{R}(\mathbf{r}_q),$$



$$\begin{aligned}
\mathbf{A}^{(q)} &= \mathbf{T}^{(q)}\mathbf{E}_{eff}^{(q)},\\
\mathbf{N}^{(q)} &= \sum_{\mathbf{r}_p \in Neighborhood(\mathbf{r}_q)} (\mathbf{S}|\mathbf{R})(\mathbf{r}'_q - \mathbf{r}'_p)\mathbf{A}^{(p)},\\
\mathbf{F}^{(q)} &= MLFMM(\mathbf{A}^{(p)}),\\
\mathbf{E}_{eff}^{(q)} &= \mathbf{E}^{(in)}(\mathbf{r}'_q) + \mathbf{N}^{(q)} + \mathbf{F}^{(q)},\\
q &= 1, \ldots, N.
\end{aligned}$$

Max Level of Space Subdivision

$$D_{l_{\max}} > \frac{4}{3-\sqrt{3}} a_{\max}, \quad l_{\max} < \log_2\left(\frac{3-\sqrt{3}}{4}\frac{D_0}{a_{\max}}\right),$$

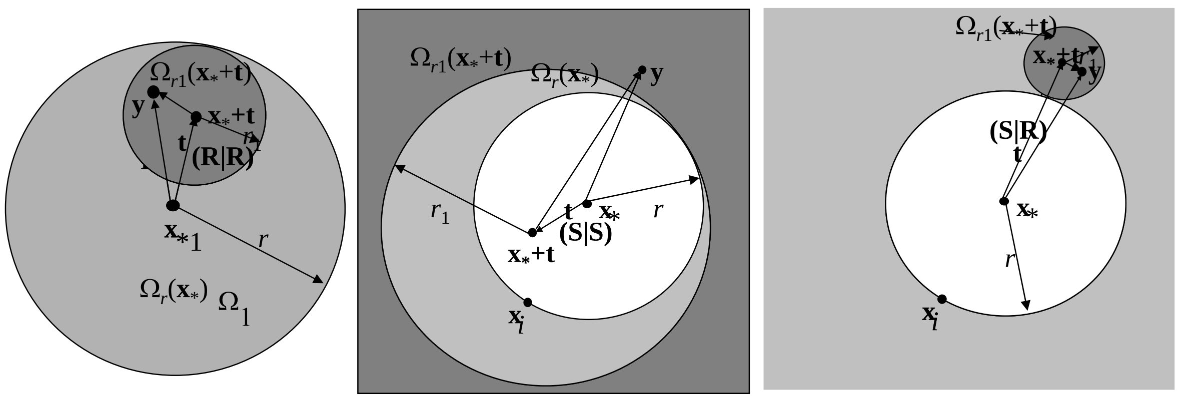
Translations

$$\psi(\mathbf{y}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} C_n^m(\mathbf{x}_{*1}) E_n^m(\mathbf{y}-\mathbf{x}_{*1}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} C_n^m(\mathbf{x}_{*2}) F_n^m(\mathbf{y}-\mathbf{x}_{*2}),$$

$$C_n^m(\mathbf{x}_{*2}) = \sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} (E|F)_{nn'}^{mm'}(\mathbf{t}) C_{n'}^{m'}(\mathbf{x}_{*1}), \quad \mathbf{t} = \mathbf{x}_{*2} - \mathbf{x}_{*1}$$

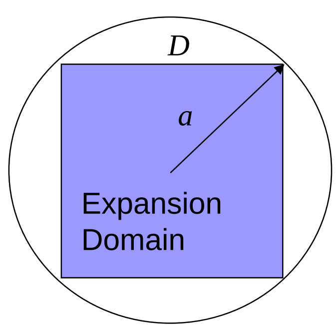
$$E,F = S,R, \quad n = 0,1,\ldots, \quad m = -n,\ldots,n.$$

R|R S|S S|R



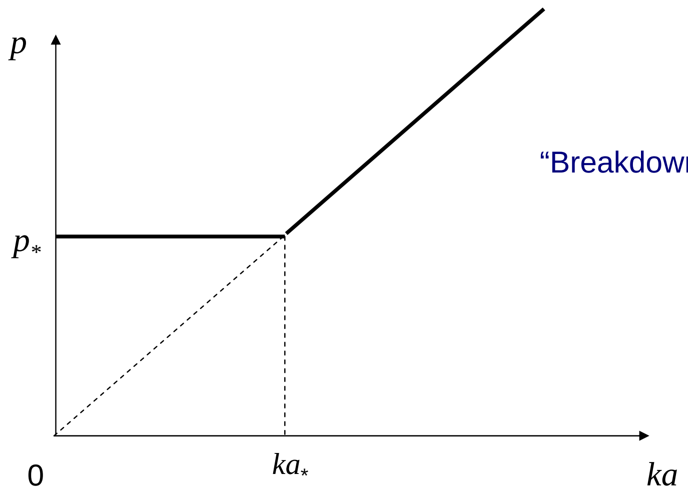
Problem:

- For the Helmholtz equation absolute and uniform convergence can be achieved only for $p > ka$. For large ka the FMM with constant p is
  - very expensive (comparable with straightforward methods);
  - inaccurate (since keeps much larger number of terms than required, which causes numerical instabilities).



$2a=3^{1/2}D$

Model of Truncation Number Behavior for Fixed Error



In the multilevel FMM we associate its own p_l with each level l:

$$P_l = p_l^2$$

"Breakdown level"

$$kD_{l_{b*}} = \kappa_b, \quad l_{b*} = \log_2 \frac{kD_0}{\kappa_b}$$

$$P_l = A(kD_l)^2, \quad l \leqslant l_{b*}$$

$$P_l = A\kappa_b^2 = P_*, \quad l \geqslant l_{b*}$$

Box size at level l

Complexity of Single Translation

Translation exponent

$$CostTrans(P_l) = CP_l^{\nu} = Cp_l^{2\nu}, \quad l = 2, \ldots, l_{\max}.$$

Spatially Uniform Data Distributions

$$N_l \sim 8^{-l} N, \quad l_{\max} \sim \frac{1}{3} \log N$$

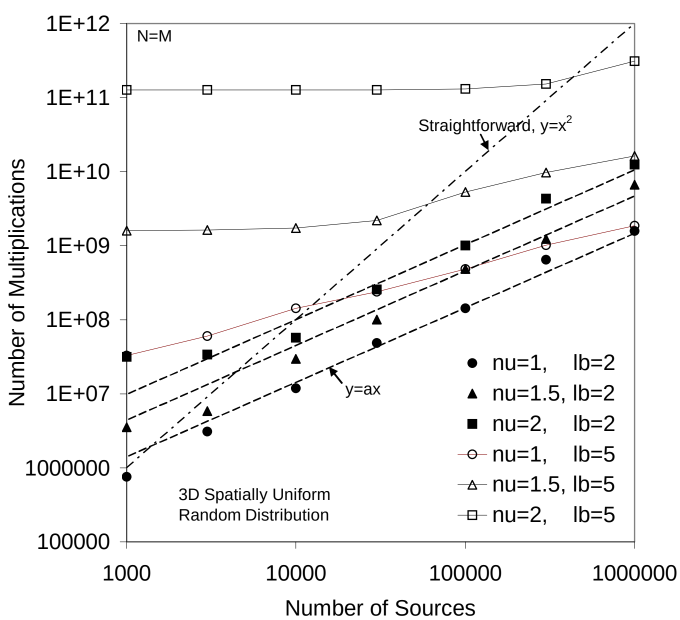
$$p_l \sim 2^{-l} k D_0,$$

$$N_{oper} \sim (kD_0)^{2\nu} \sum_{l=2}^{l_{\max}} 2^{-2\nu l} 8^l = (kD_0)^{2\nu} \sum_{l=2}^{l_{\max}} 2^{(3-2\nu)l}.$$

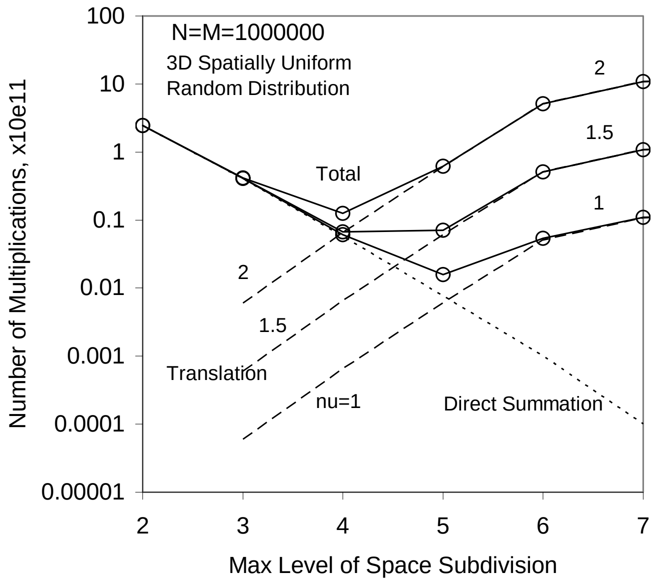
- $\nu < 1.5: \quad ComplexityFMM \sim (kD_0)^{2\nu} 2^{(3-2\nu) l_{\max}} \sim (kD_0)^{2\nu} N^{1-2\nu/3}$
- $\nu = 1.5: \quad ComplexityFMM \sim (kD_0)^{2\nu} l_{\max} \sim (kD_0)^{2\nu} \log N$
- $\nu > 1.5: \quad ComplexityFMM \sim (kD_0)^{2\nu}$

Constant!

Complexity of the Optimized FMM for Fixed kD_0 and Variable N



Optimum Level for Low Frequencies



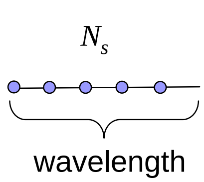
Volume Element Methods

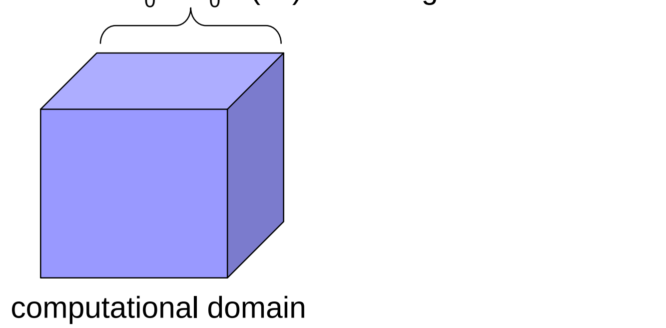
$$N = \left(\frac{N_s}{2\pi} kD_0\right)^3, \quad kD_0 \sim N^{1/3}$$

- $\nu < 1.5$: $ComplexityFMM \sim (kD_0)^{2\nu} 2^{(3-2\nu)l_{\max}} \sim (kD_0)^{2\nu} N^{1-2\nu/3} \sim N$
- $\nu = 1.5$: $ComplexityFMM \sim (kD_0)^{2\nu} l_{\max} \sim (kD_0)^{2\nu} \log N \sim N \log N$
- $\nu > 1.5$: $ComplexityFMM \sim (kD_0)^{2\nu} \sim N^{2\nu/3} \gg N \log N.$

Critical Translation Exponent!

$D_0 = D_0\, k/(2\pi)$ wavelengths = $N^{1/3}$ sources





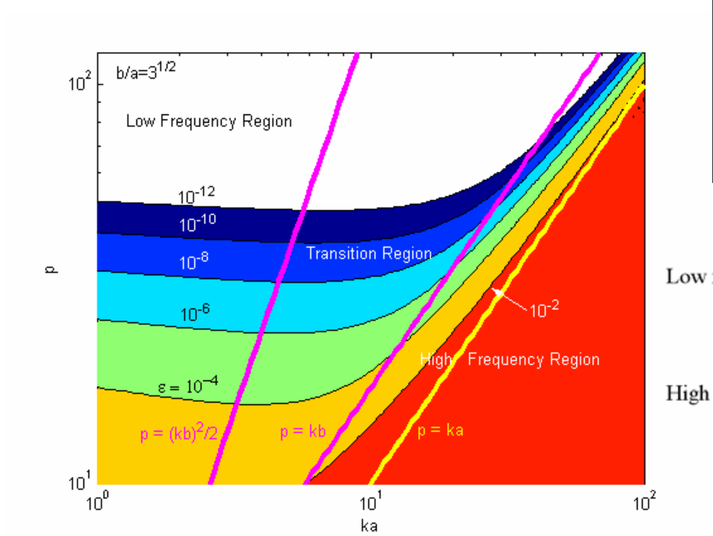
What Happens if Truncation Number is Constant for All Levels?

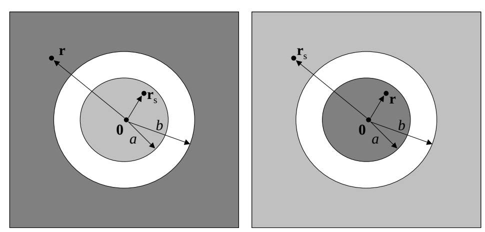
$$N_{oper} \sim (kD_0)^{2\nu} \sum_{l=2}^{l_{\max}} 8^l = (kD_0)^{2\nu} \sum_{l=2}^{l_{\max}} 2^{3l} \sim (kD_0)^{2\nu} 2^{3l_{\max}} \sim (kD_0)^{2\nu} N \sim N^{1+2\nu/3}.$$

- $\nu < 1.5: \quad N \ll ComplexityFMM \ll N^2$
- $\nu = 1.5: \quad ComplexityFMM \sim N^2$
- $\nu > 1.5: \quad ComplexityFMM \sim N^{1+2\nu/3} \gg N^2$

"Catastrophic Disaster of the FMM"

Source Expansion Errors





Low frequencies:

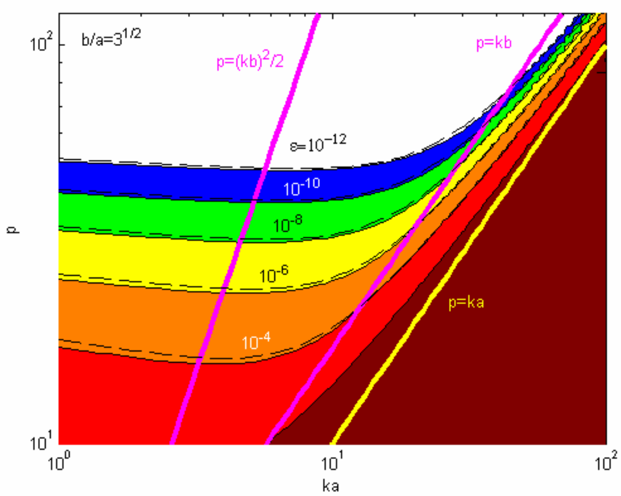
$$p = -\frac{\ln\left[\epsilon ka(1-\sigma^{-1})^{3/2}\right]}{\ln \sigma} - 1.$$

High frequencies:

$$p = ka + \frac{1}{2}\left(3 \ln \frac{1}{\epsilon\sigma}\right)^{2/3} (ka)^{1/3}.$$

Approximation of the Error

$$p = \left\{ \left[\frac{1}{\ln\sigma} \ln \frac{1}{\epsilon ka(1-\sigma^{-1})^{3/2}} + 1 \right]^4 + \left[ka + \frac{1}{2}\left(3 \ln \frac{1}{\epsilon\sigma}\right)^{2/3} (ka)^{1/3} \right]^4 \right\}^{1/4}$$

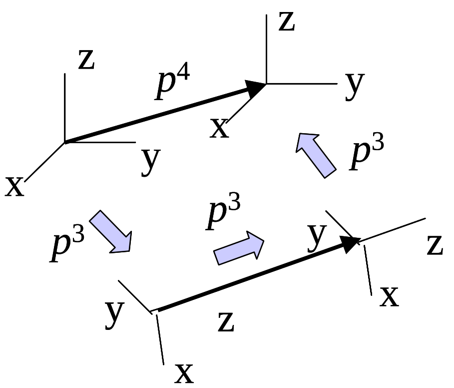


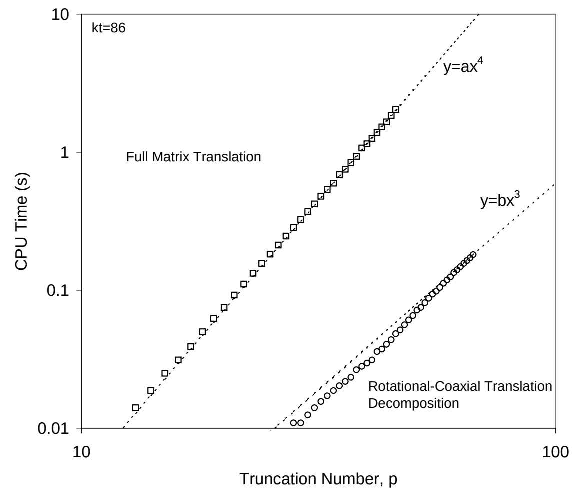
O(p^3) Translation Methods

Rotation - Coaxial Translation Decomposition (Complexity O(p³))

From the group theory follows that general translation can be reduced to

$$(\mathbf{F}|\mathbf{E})(\mathbf{t}) = \mathbf{Rot}(Q^{-1})(\mathbf{F}|\mathbf{E})_{(coax)}(t)\mathbf{Rot}(Q), \quad F,E = S,R.$$





Sparse Matrix Decomposition

$$(\mathbf{R}|\mathbf{R})(\mathbf{t}) = (\mathbf{S}|\mathbf{S})(\mathbf{t}) = \sum_{n=0}^{\infty} \frac{(kt)^n}{n!}\mathbf{D}_{\mathbf{t}}^n = e^{kt\mathbf{D}_{\mathbf{t}}} = \Lambda_r(kt, -i\mathbf{D}_{\mathbf{t}})$$

$$(\mathbf{S}|\mathbf{R})(\mathbf{t}) = \Lambda_s(kt, -i\mathbf{D}_{\mathbf{t}})$$

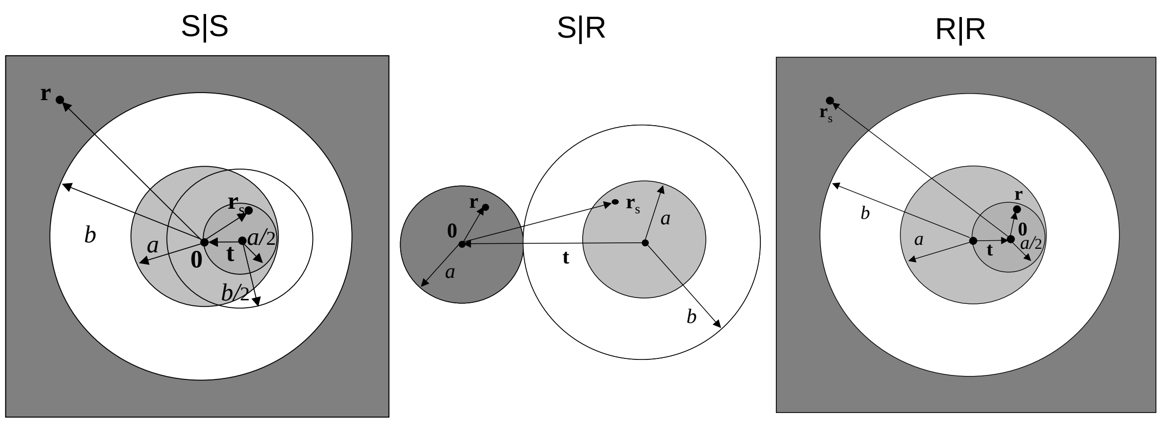
Matrix-vector products with these matrices computed recursively

$$\Lambda_r(kt, -i\mathbf{D}_{\mathbf{t}}) = \sum_{n=0}^{\infty}(2n+1)i^n j_n(kt)P_n(-i\mathbf{D}_{\mathbf{t}})$$

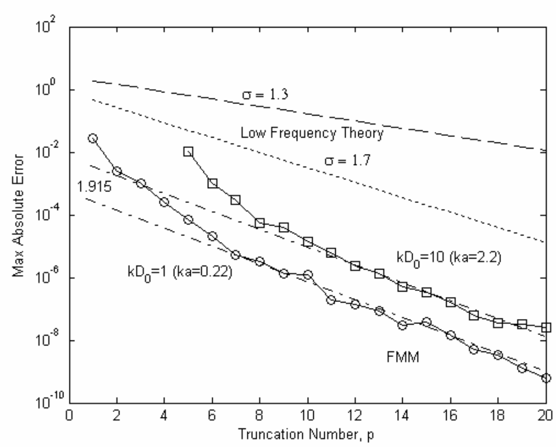
$$\Lambda_s(kt, -i\mathbf{D}_{\mathbf{t}}) = \sum_{n=0}^{\infty}(2n+1)i^n h_n(kt)P_n(-i\mathbf{D}_{\mathbf{t}}).$$

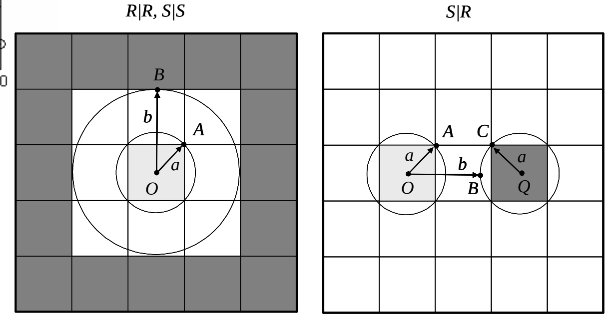
$$(\mathbf{D}_{\mathbf{t}}\mathbf{C})_n^m = \frac{1}{2t}\Big[(t_x + it_y)\big(C_{n-1}^{m+1}b_n^m - C_{n+1}^{m+1}b_{n+1}^{-m-1}\big) + (t_x - it_y)\big(C_{n-1}^{m-1}b_n^{-m} - C_{n+1}^{m-1}b_{n+1}^{m-1}\big)\Big]$$
$$+ \frac{t_z}{t}(a_n^m C_{n+1}^m - a_{n-1}^m C_{n-1}^m), \qquad m = 0, \pm1, \pm2, \ldots, \quad n = |m|, |m|+1, \ldots$$

It can be proved that for source summation problems the truncation numbers can be selected based on the above chart when using translations with rectangularly truncated matrices

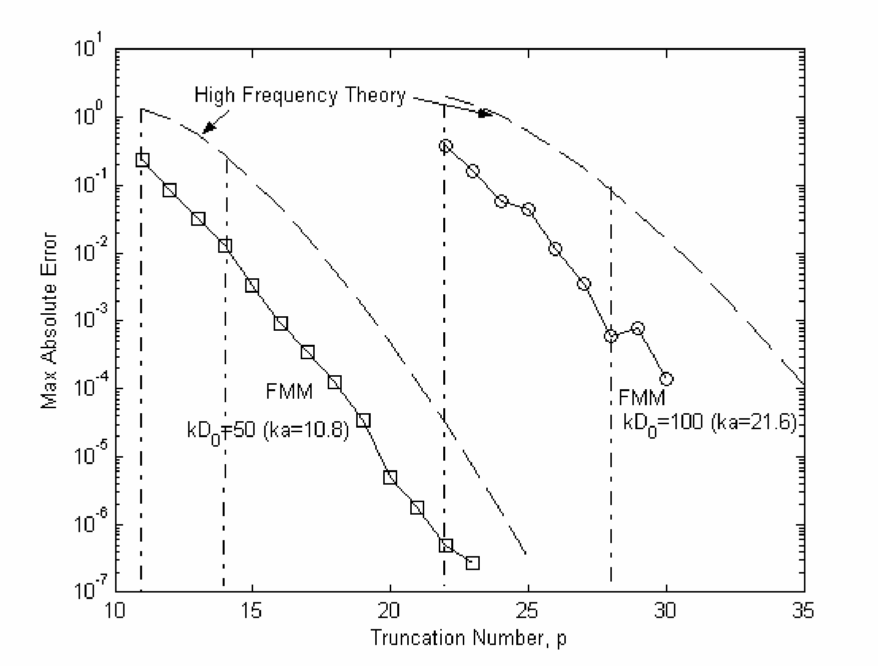


Low Frequency FMM Error





High Frequency FMM Error



Results of Computations

Range of Parameters

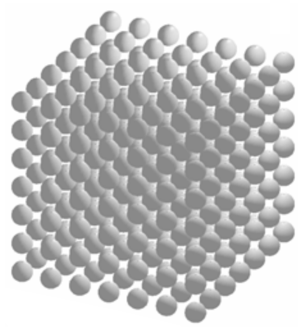
- Number of Spheres: *1-10*^{4};
- *ka: 0.1-10*; kD_0: 1-100;
- Random and regularly spaced grids of spheres;
- Polydispersity: *0.5-1.5* (ratio to the mean radius);
- Volume fractions: *0.01-0.2;*

Advantages and Defficiencies of Our FMM Implementation

This implementation is not perfect, but works!

- O(NlogN) "On fly" computation of neighbor lists, using bit interleaving;
  - Low memory: one can trade memory for speed;
- Rotation-Coaxial Translation Decomposition, Operations with Multipole Expansion Coefficients;
  - For high frequencies some other methods (diagonal forms, asymptotic methods) can be used; Some additional complexity: conversion to the space of expansion coefficients;
- No precomputation of translation and rotation matrices;
  - Low memory: one can trade memory for speed;
- For larger problem size the GMRES is more efficient than the Reflection Method;
  - User can switch, but the GMRES can be used as default.
  - Krylov subspace dimensionalities usually low (of order 10-30).

Some Configurations



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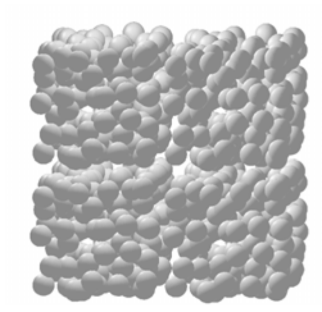
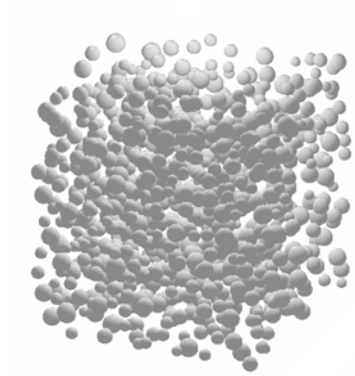
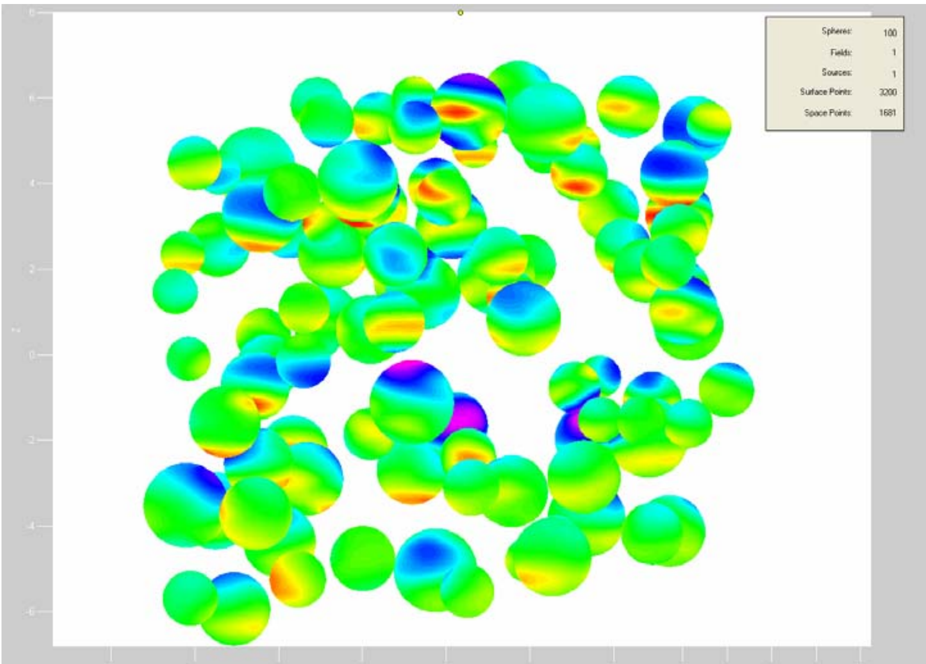


640

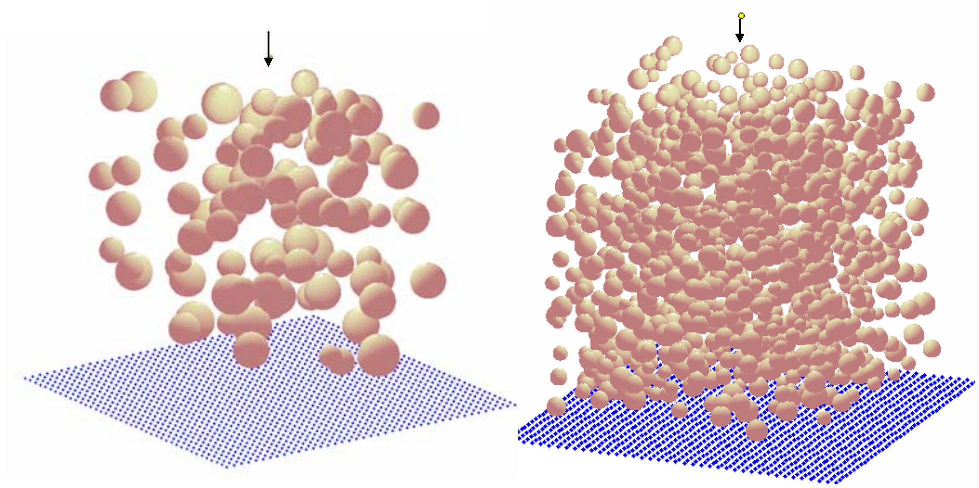


1000

Surface Potential Imaging



Some Pictures



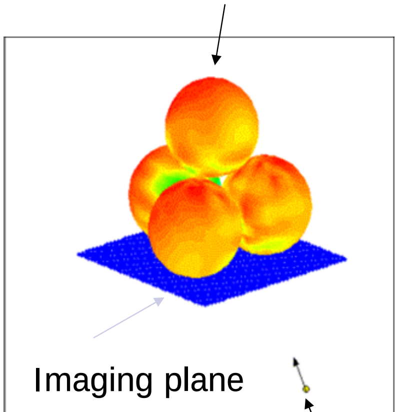
100 random spheres

1000 random spheres

USE the FMM for Spatial Imaging!

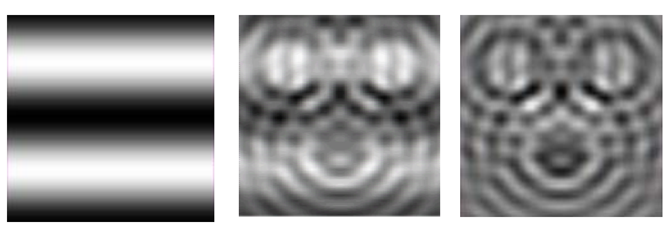
4 spheres (T-matrix straightforward)

Scatterers



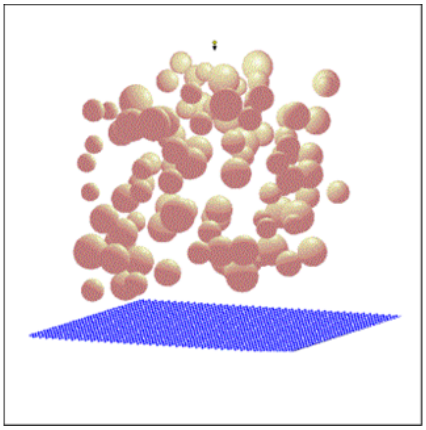
Vector of the incident plane wave

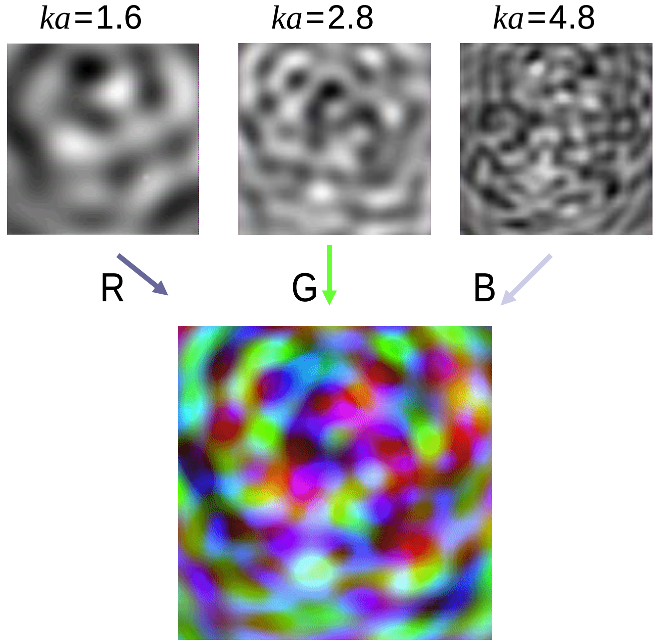
ka=15.2



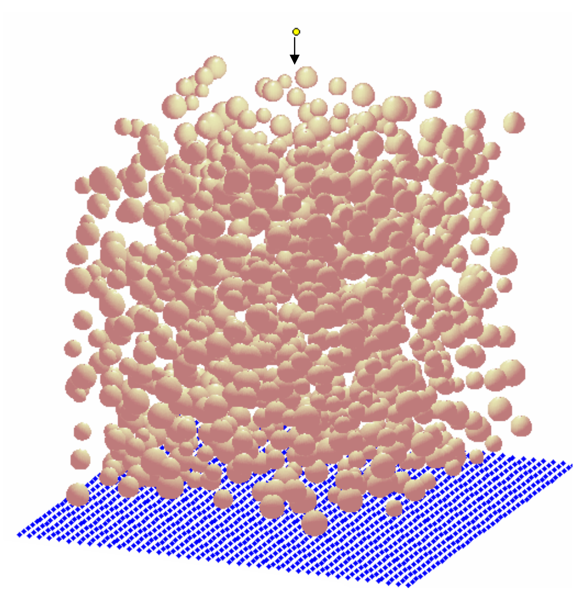
Incident Field   Total Field   Scattered Field

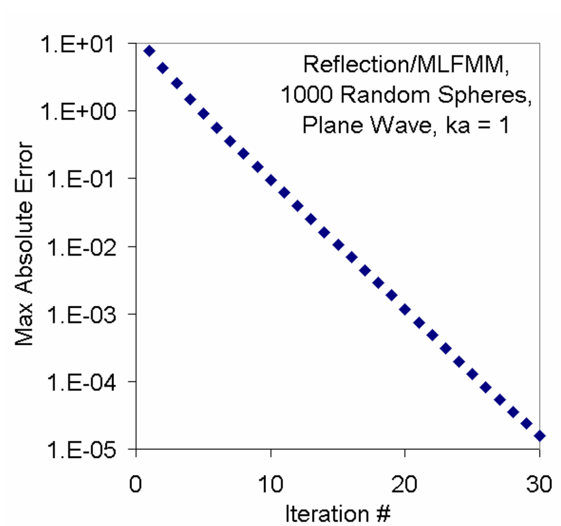
100 random spheres (MLFMM)





1000 random spheres

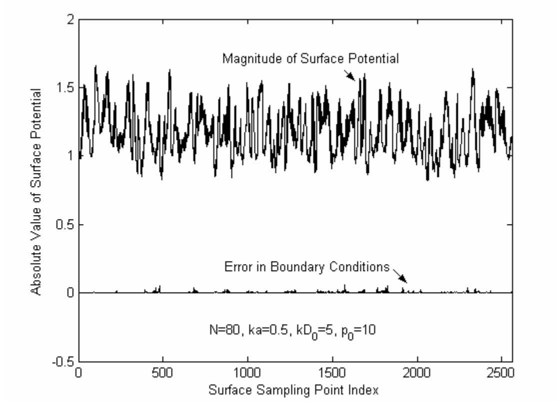




Aposteriory Error Control

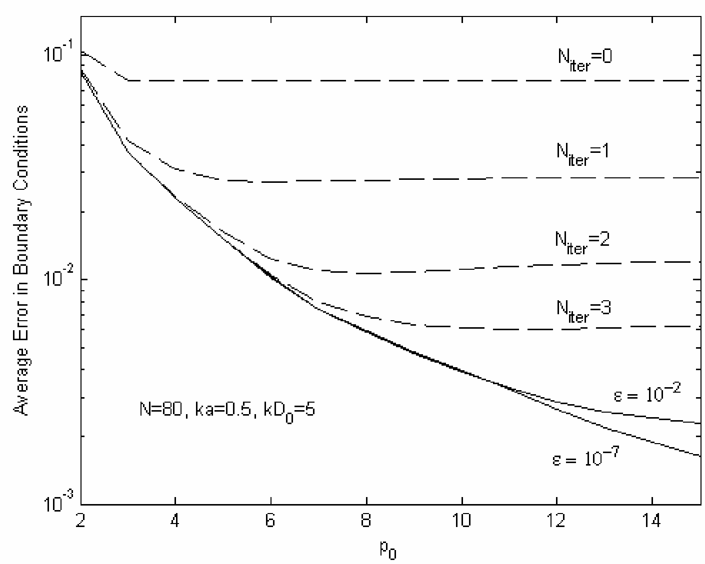
$$\epsilon_{bc}^{(1)} = \max_m \left| \frac{\Delta_{bc}(\mathbf{y}_m)}{\psi|_{S_q}(\mathbf{y}_m)} \right|, \quad \epsilon_{bc}^{(2)} = \left[\frac{\sum_m |\Delta_{bc}(\mathbf{y}_m)|^2}{\sum_m \left|\psi|_{S_q}(\mathbf{y}_m)\right|^2} \right]^{1/2}, \quad \Delta_{bc}(\mathbf{y}_m) = a_q \left(\frac{\partial\psi}{\partial n}(\mathbf{y}_m) + i\sigma_q \psi(\mathbf{y}_m) \Big|_{S_q} \right).$$

$$\frac{1}{k}\mathbf{n} \cdot \nabla S_n^m(\mathbf{r}) = \frac{1}{2}(n_x - in_y)\left[b_{n+1}^{-m-1} S_{n+1}^{m+1}(\mathbf{r}) - b_n^m S_{n-1}^{m+1}(\mathbf{r})\right] + \frac{1}{2}(n_x + in_y)\left[b_{n+1}^{m-1} S_{n+1}^{m-1}(\mathbf{r}) - b_n^{-m} S_{n-1}^{m-1}(\mathbf{r})\right] + n_z[a_{n-1}^m S_{n-1}^m(\mathbf{r}) - a_n^m S_{n+1}^m(\mathbf{r})],$$

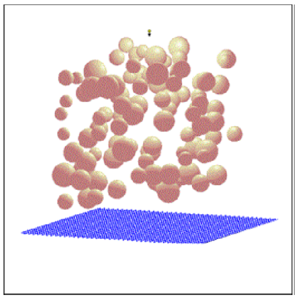


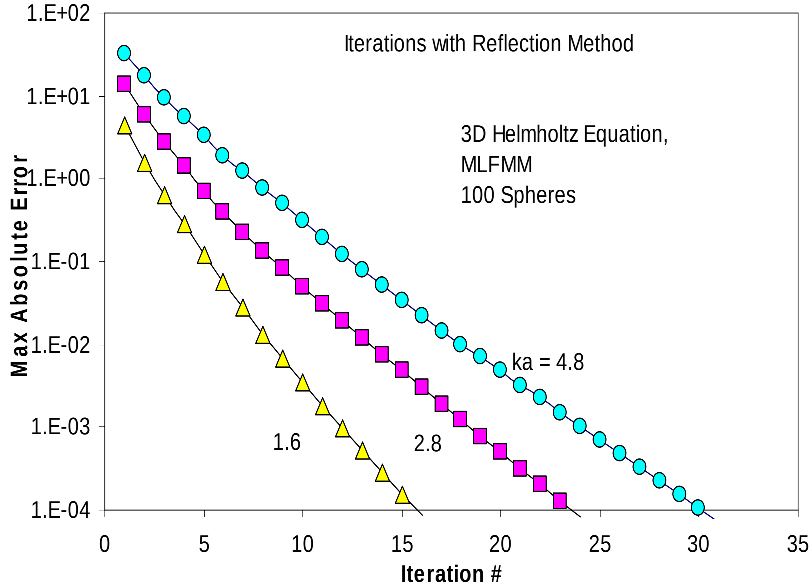
Truncation and Iteration Errors

Reflection Method

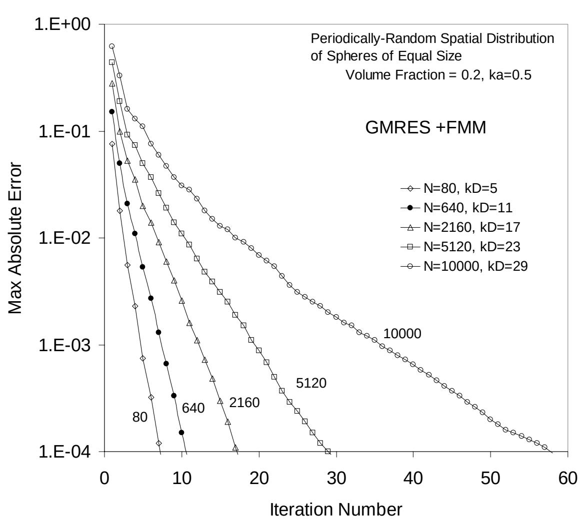


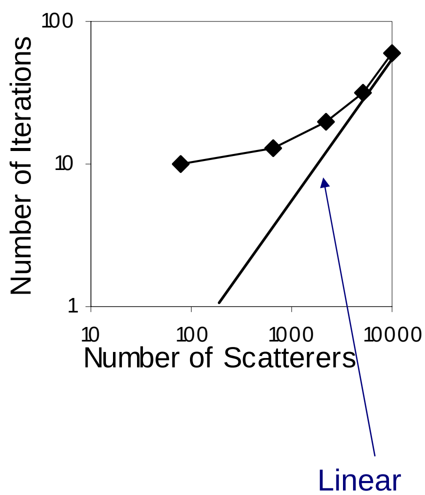
Convergence for 100 spheres (MLFMM)



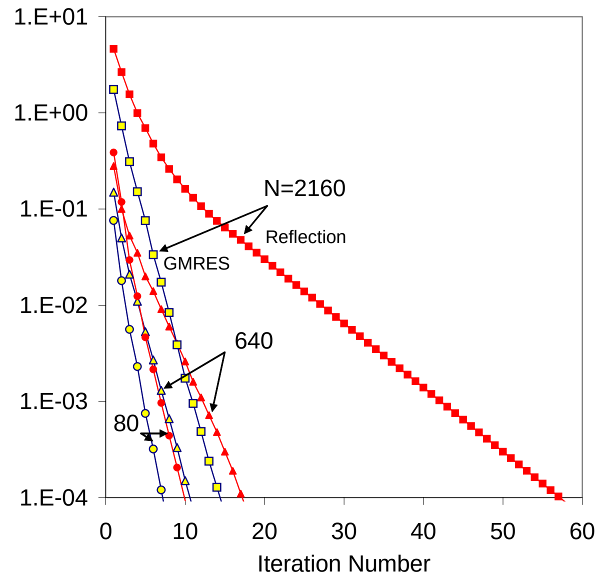


Convergence for Different N



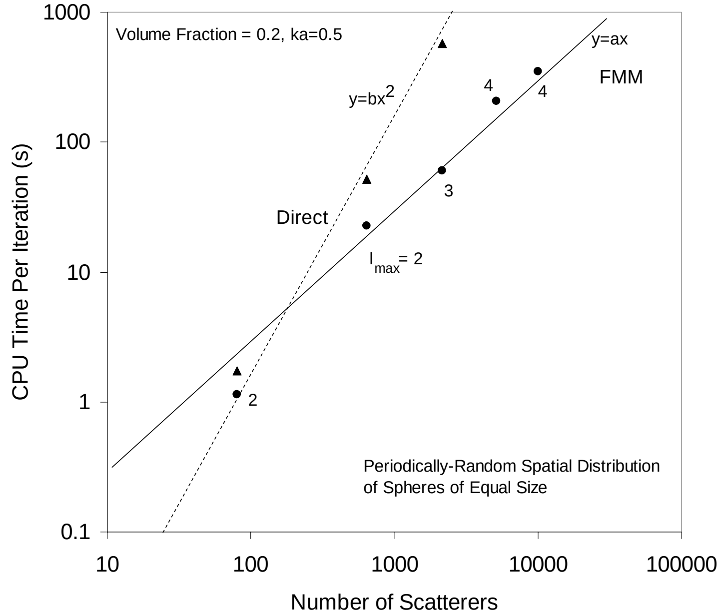


GMRES vs Reflection

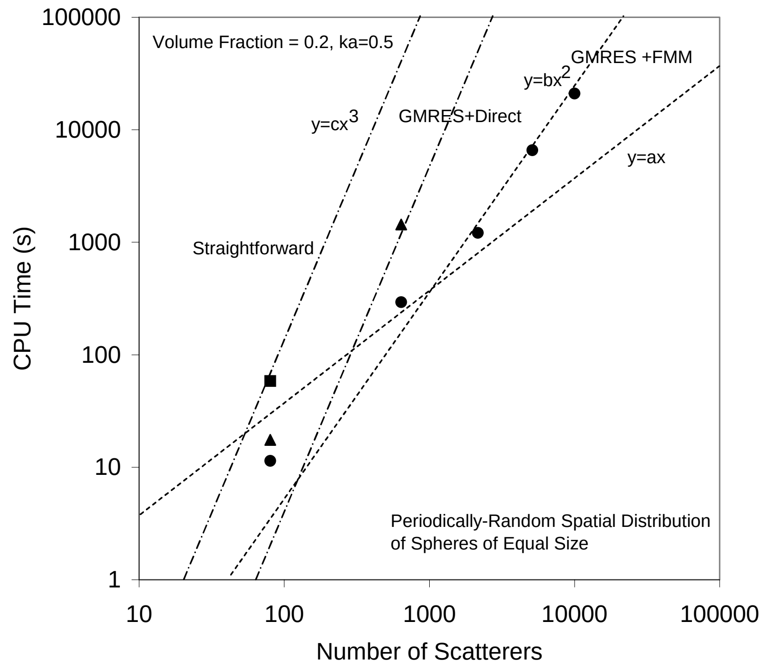


Dual Xeon 3.2GHz,
3.5 GB RAM,
25% resources utilized

CPU Time Per Iteration

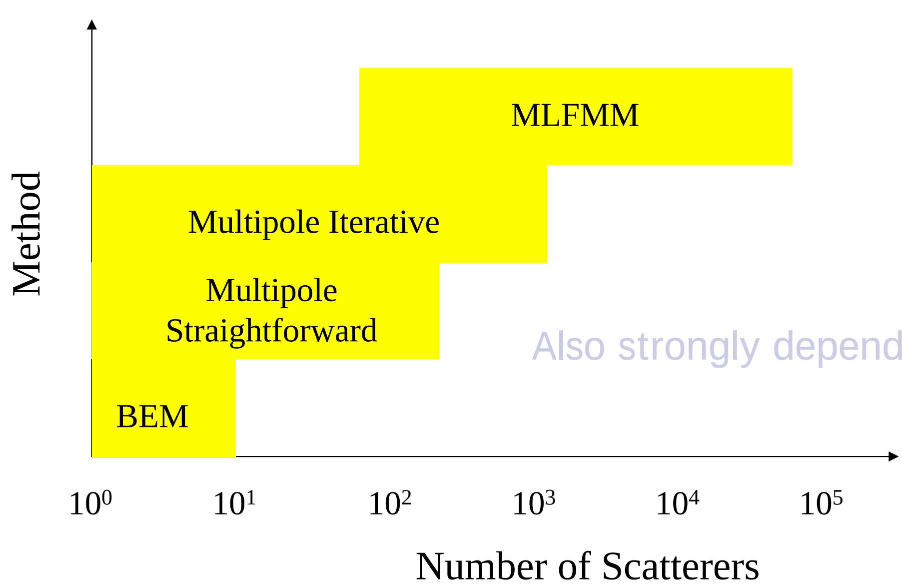


Overall Performance



Dual Xeon 3.2GHz,
3.5 GB RAM,
25% resources utilized

Computable Problems on Desktop PC



Conclusions

- We developed, implemented, and tested the Multilevel Fast Multipole Method for computation of multiple scattering problems.
- Performance of the method depends on a number of controlling parameters. At proper selection of these parameters fast and accurate results can be achieved.
- Some convergence problems in iterative methods were observed for short wave propagation in regularly spaced sphere grids. This may be due to some internal resonances, which should be investigated.

Future work

- Development of faster translation algorithms, covering higher frequencies;
- Extension for non-spherical scatterrers;
- Comparisons with continuum (averaging) theories and theories of wave propagation in random media;
- Computations of acoustic fields in disperse systems (bubbly liquids, particulate systems);
- Comparisons with experimental data.