Data Structures for Approximate Proximity and Range Searching

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Introduction

Computational Geometry: The study of efficient algorithms and data structures for discrete geometric structures: finite point sets, polyhedra, spatial subdivisions.

Spatial Data Retrieval: Given a finite set of objects (points) preprocess these objects into a data structure that supports efficient processing of some given class of queries.

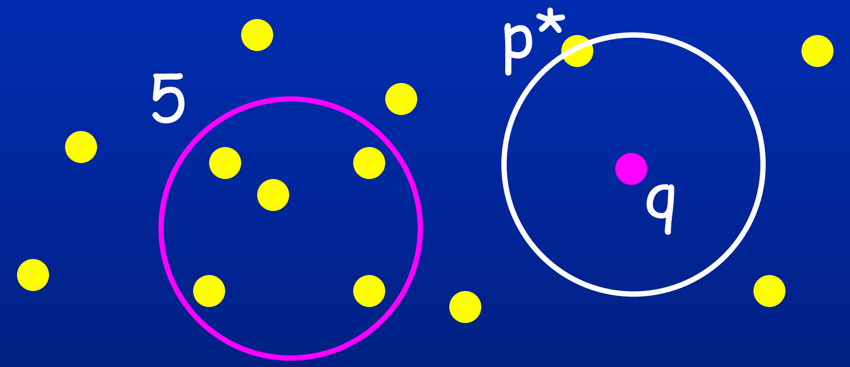
Efficiency: is measured in terms of the resources used as a function of the number of entities in the structure:

- Query time
- Space (for the data structure)
- Preprocessing time (usually of secondary importance)
- Update time (for dynamic applications)

1-Dimensional Example: Binary search and binary search trees. Improves O(n) brute-force search to O(log n) time.

Proximity Searching

Nearest Neighbor: Given a point set $S \subseteq R^d$ and $q \in R^d$, find the point $p^* \in S$ that is closest to q.



(Spherical) Range Queries: Given a query ball, report all the points that lie within, or the total weight of points within.

Throughout we assume Euclidean distances.

Challenges

Curse of Dimensionality: Many methods for geometric retrieval problems have query times that grow exponentially in dimension (assuming a fixed amount of space).

Lower bounds: Many retrieval problems have known lower bounds, which imply that more efficient methods cannot exist.

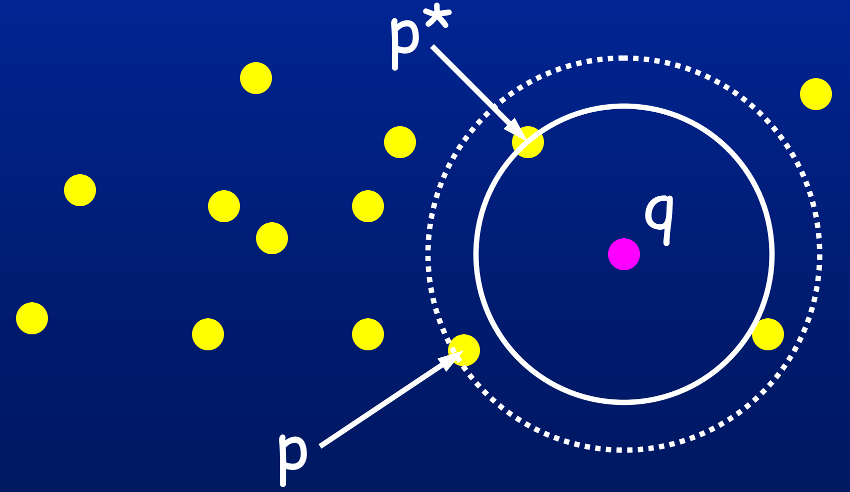
For example, given a parameter m, where $n \le m \le n^d$, if $O(m)$ space is used, then spherical range queries cannot be answered in substantially less than $O(n/(m^{1/d}))$ time [Brönnimann, et. al, 1993].

Approximate Proximity Search

Approx Nearest Neighbor: Given $\varepsilon > 0$ and $q \in R^d$, a point $p \in S$ is an ε-nearest neighbor of q if,

$$\mathrm{dist}(q,p) \le (1+\varepsilon)\,\mathrm{dist}(q,p^*),$$

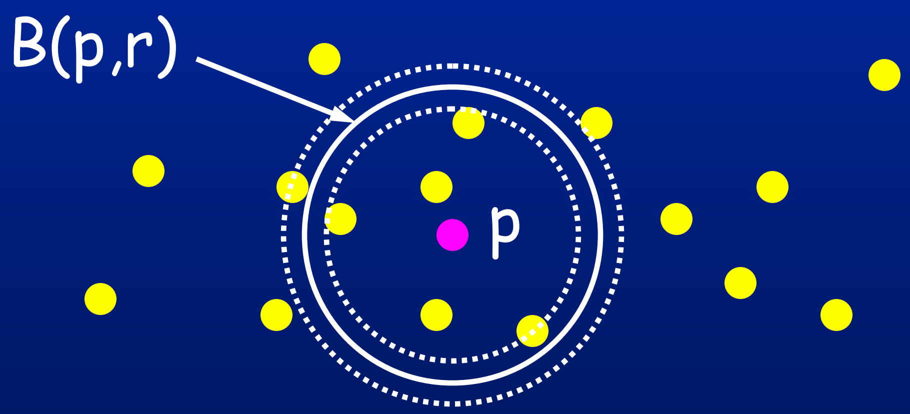
where $p^* \in S$ is the nearest neighbor of q.



Approximate Proximity Search

Approx Range Searching: Let B(q,r) denote the Euclidean ball of radius r centered at q. A set S′ is an admissible solution to an ε-approximate range query if

$$S \cap B(p, r(1-\varepsilon)) \subseteq S' \subseteq S \cap B(p, r(1+\varepsilon))$$



Goal: Return the weight of any admissible solution.

Exact/Approx NN Search

The best exact methods are based on data structures for range searching.

	Query Time	Space
Exact	$\log n$	$n^{d/2}$
AMNSW'98	$(1/\varepsilon)^d \log n$	n
Clarkson '97 Chan '98	$(1/\varepsilon)^{\frac{d-1}{2}} \log n$	$(1/\varepsilon)^{\frac{d-1}{2}} n\log n$

Any convex body in R^d can be ε-approximated by a polyhedron with $(1/\varepsilon)^{(d-1)/2}$ facets [Dudley 74].

Our Results on ε-NN Search

Theorem: (AMM02) Given a point set S in R^d, and $0 < \varepsilon \leq \frac{1}{2}$, $2 \leq \gamma \leq 1/\varepsilon$, it is possible to build a data structure of space $O(n\gamma^{d-1}\log \gamma)$ that can answer ε-NN queries in $O(\log (n\gamma) + 1/(\varepsilon\gamma)^{(d-1)/2})$ time.

γ	**Query Time**	**Space**
$1/\varepsilon$	$\log n + \log (1/\varepsilon)$	n/ε^d
2	$\log n + 1/\varepsilon^{(d-1)/2}$	n

Note: For low-space version, space is independent of ε, and query time is additive, not multiplicative.

Exact/Approx Range Search

Best exact approaches are based on cuttings, but results scale poorly with dimension.

	Query Time	Space
Exact	$\log n$	$n^d/\log^d n$
	$n^{(1 - 1/d)}$	n
	$\log n + n/m^{1/d}$	m
Approx [AM'00]	$\log n + (1/\varepsilon)^{d-1}$	n

The approximate solution of AM'00 is based on kd-trees.

Our Results on ε-Range Search

Theorem: (AMM04) Given a point set S in R^d, and $0 < \varepsilon \le \frac{1}{2}$, $2 \le \gamma \le 1/\varepsilon$, it is possible to build a data structure of space $O(n\gamma^d \log(1/\varepsilon))$ that can answer ε-range queries in $O(\log(n\gamma) + 1/(\varepsilon\gamma)^{d-1})$ time.

γ	**Query Time**	**Space**
$1/\varepsilon$	$\log n + \log(1/\varepsilon)$	$(n/\varepsilon^d)\log(1/\varepsilon)$
2	$\log n + 1/\varepsilon^{(d-1)}$	$n \log(1/\varepsilon)$

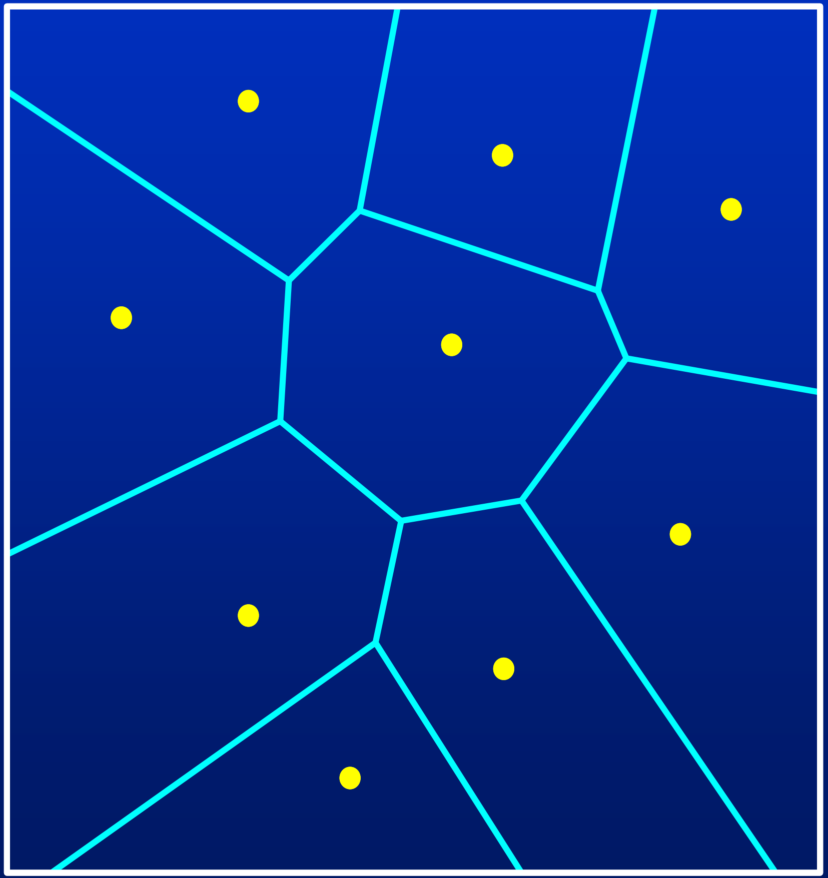
Note: Can be used for answering approximate k-th nearest neighbor queries in similar time bounds.

Remainder of the Talk

- General techniques used in efficient approximate retrieval.
- Approximate Voronoi Diagram (AVD)
- Applying AVDs to range searching
- Implementation (time permitting)

Voronoi Diagrams

- Given a set S of n point **sites** in R^d.
- **Voronoi diagram** is a subdivision of space into regions according to which site is closest.
- Use **point location** to answer NN queries.



Voronoi Diagrams: Difficulties

High Complexity: In dimension d, it may be as high as $\Theta\left(n^{\lceil d/2 \rceil}\right)$.

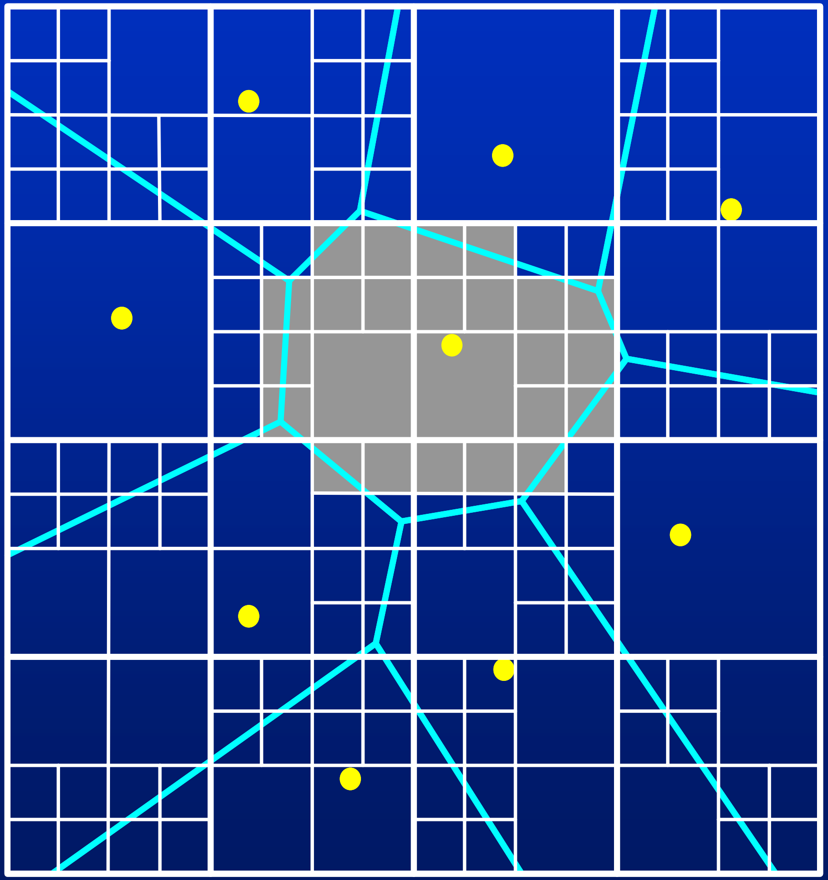
Computational Issues: Geometric degeneracies and topological consistency.

Point Location: Optimal solutions only in 2-d.

Question: Are there simpler/faster methods if we are willing to approximate?

Approx Voronoi Diagrams

- **ε-AVD:** (Har-Peled '01) Quadtree-like subdivision of space. Each cell stores a **representative site**, $r \in S$, such that r is an ε-NN of any point q in the cell.
- ε-NN → pt location



Approx Voronoi Diagrams

Har-Peled '01: Size:

$$O\left(\frac{n}{\varepsilon^d}(\log n)\left(\log\frac{n}{\varepsilon}\right)\right).$$

ε-NN Queries: Point location in a compressed quadtree in time

$$O\left(\log\frac{n}{\varepsilon}\right).$$

Arya, Malamatos'02: Multiple representatives

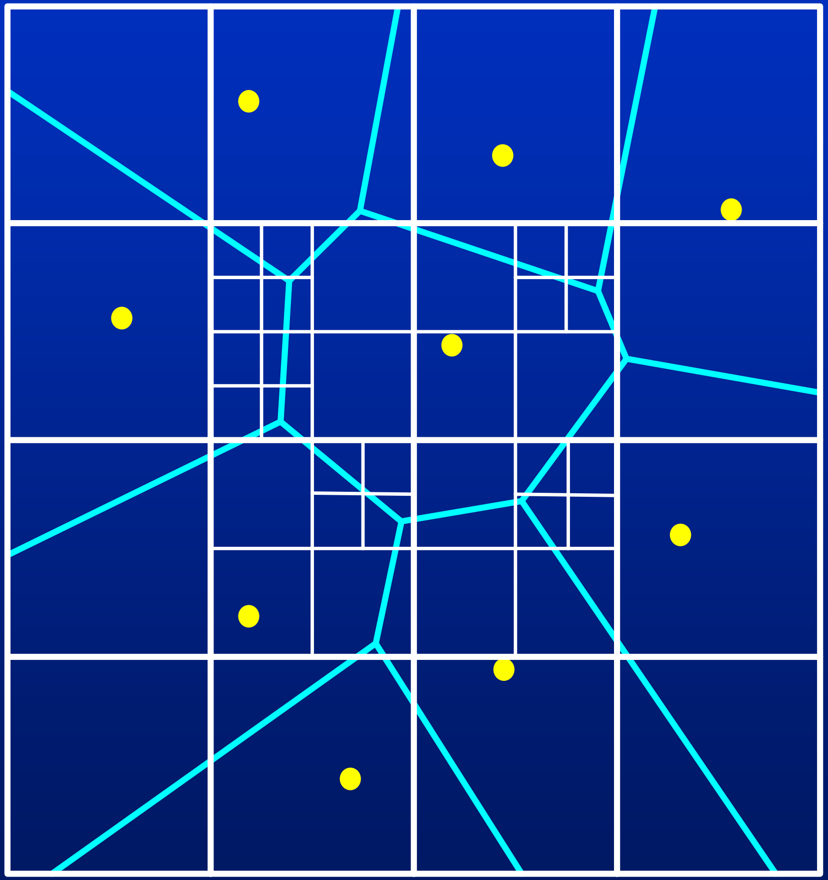
Multiple Representatives

Multi-representatives: Each cell is allowed up to $t \geq 1$ representatives.

Tradeoff: cells vs. representatives.

NN-Query: Pt. Loc. and distance comp.

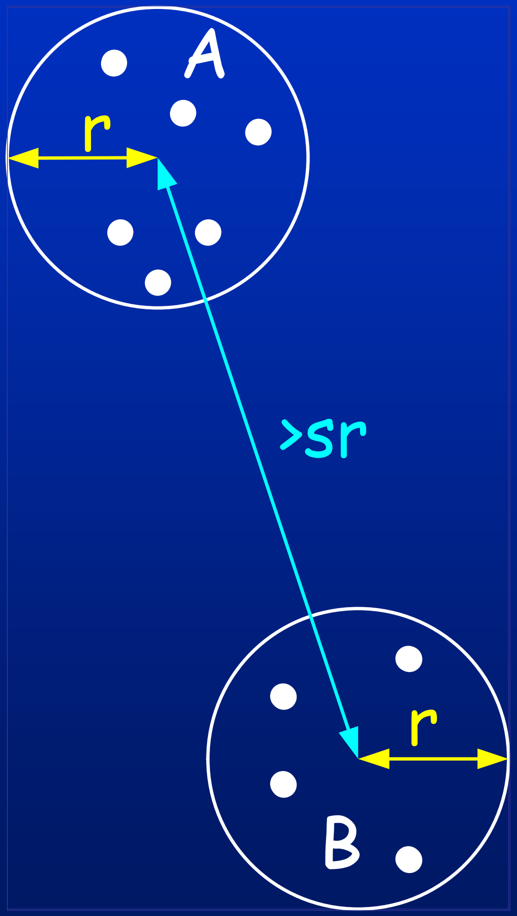
t=2



Basic Tools: WSPDs

Separation factor: $s > 2$.

Two sets A and B are **well-separated** if they can be enclosed in spheres of radius r, whose centers are at distance least sr.

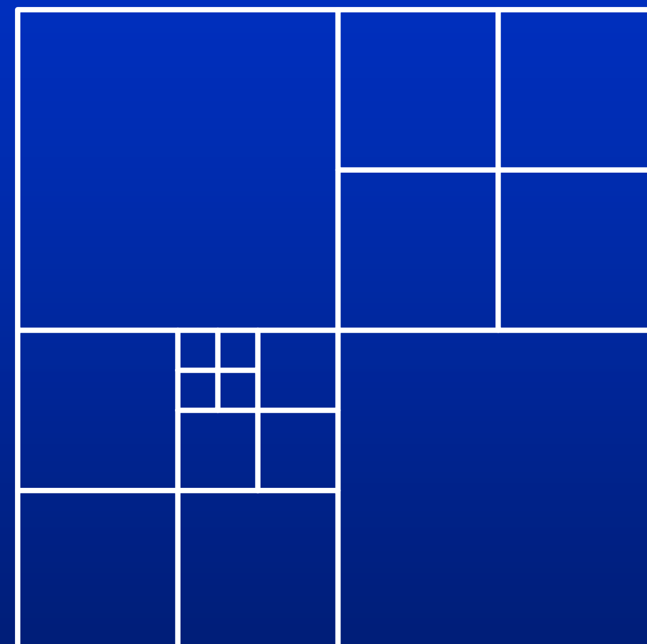


Basic Tools: WSPDs

Well-Separated Pair Decomposition (WSPD): Given a set of n points and separation factor s, it is possible to represent all $O(n^2)$ pairs as $O(s^d n)$ well-separated pairs. (Callahan, Kosaraju '95)

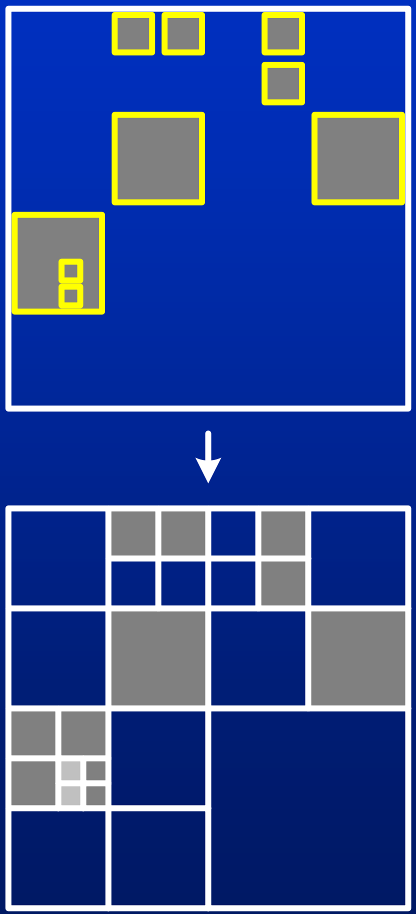
Basic Tools: BBD Trees

Quadtree Box: A box that can be obtained by repeatedly splitting the unit hypercube into 2^d identical boxes.



Basic Tools: BBD Trees

BBD Tree: Given a set of m quadtree boxes, we can build a BBD-tree of size O(m) and height O(log m) whose induced subdivision is a refinement of the box subdivision. (AMN+98)



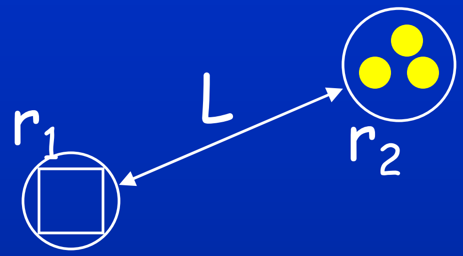
Separation: Intuition

The greater the separation from a set of points, the fewer representatives are needed to guarantee that one is an ε-NN.

Disjoint & Concentric Balls

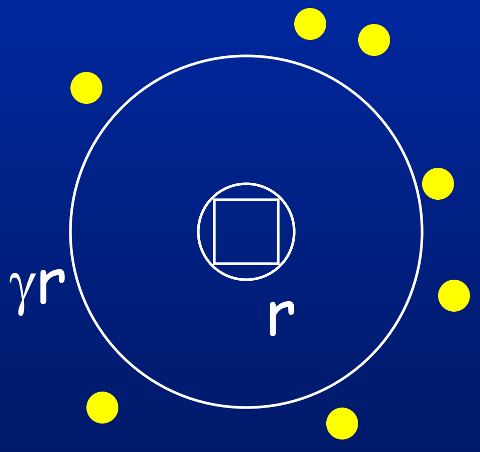
Disjoint Ball Lemma: Given disjoint balls of radii r_1 and r_2 separated by L, the number of representatives needed is

$$\left(r_1 r_2 / \left(\varepsilon L^2\right)\right)^{\frac{d-1}{2}}$$



Concentric Ball Lemma: Given concentric balls of radii r and γr, the number of representatives needed is

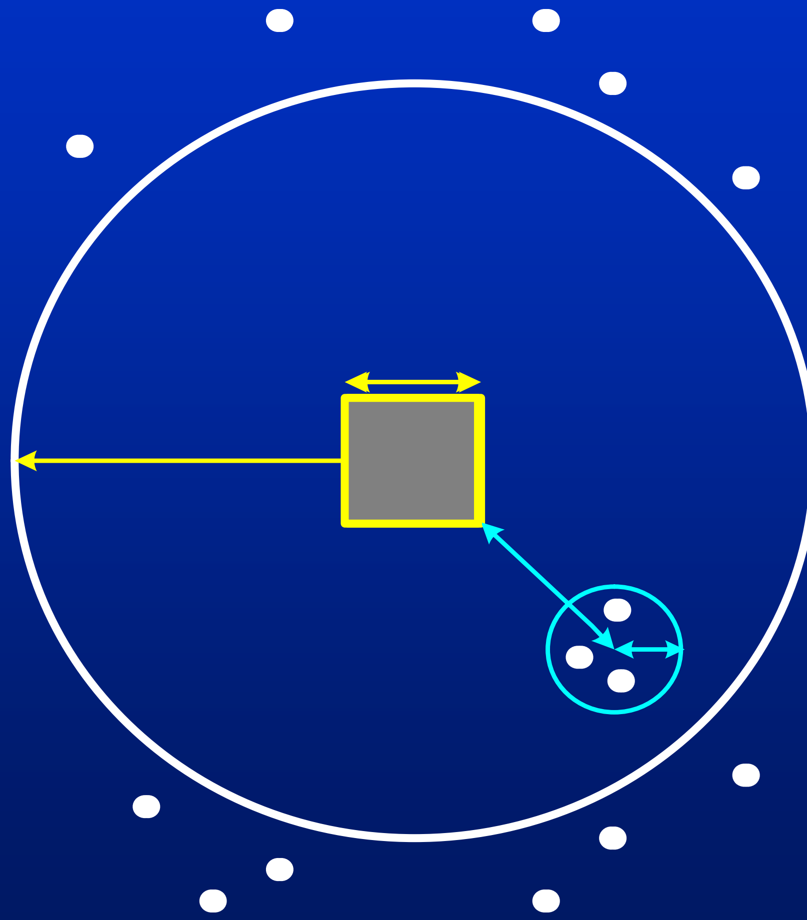
$$1/(\varepsilon\gamma)^{\frac{d-1}{2}}$$



Separation: Goal

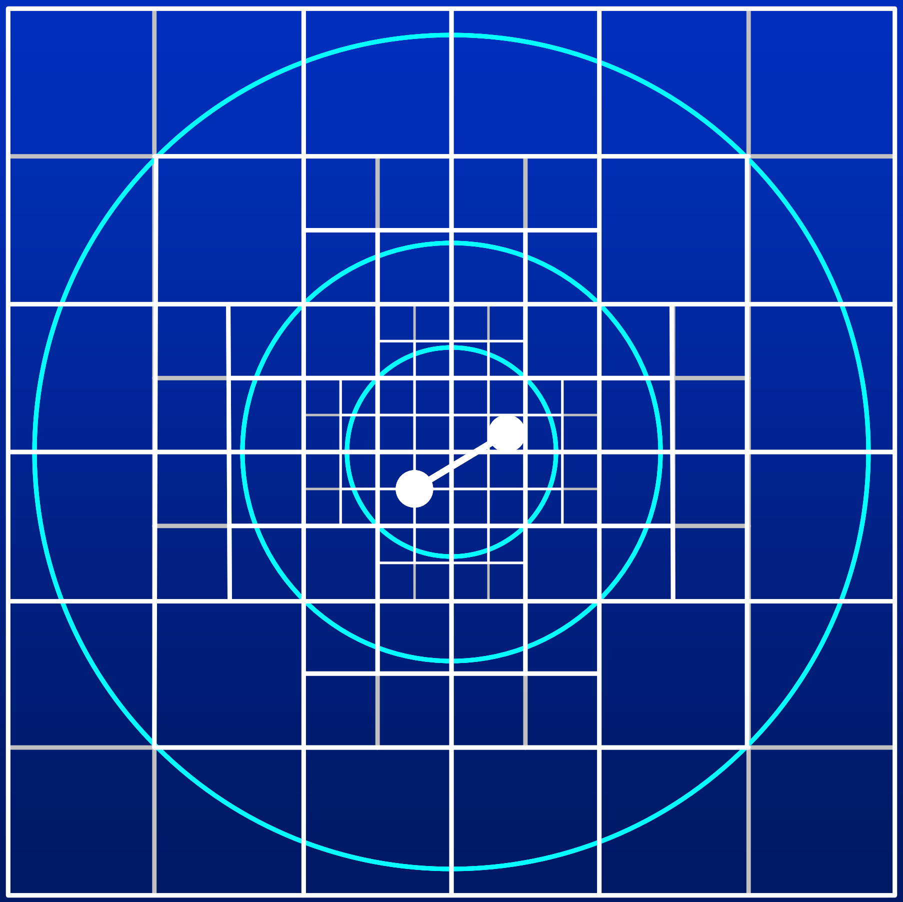
Low-γ: Assume γ=2.

Goal: Subdivide space into O(n) cells. For each cell of size s, all sites within distance 4s can be enclosed within a ball whose factor-2 expansion does not intersect the cell.



Construction

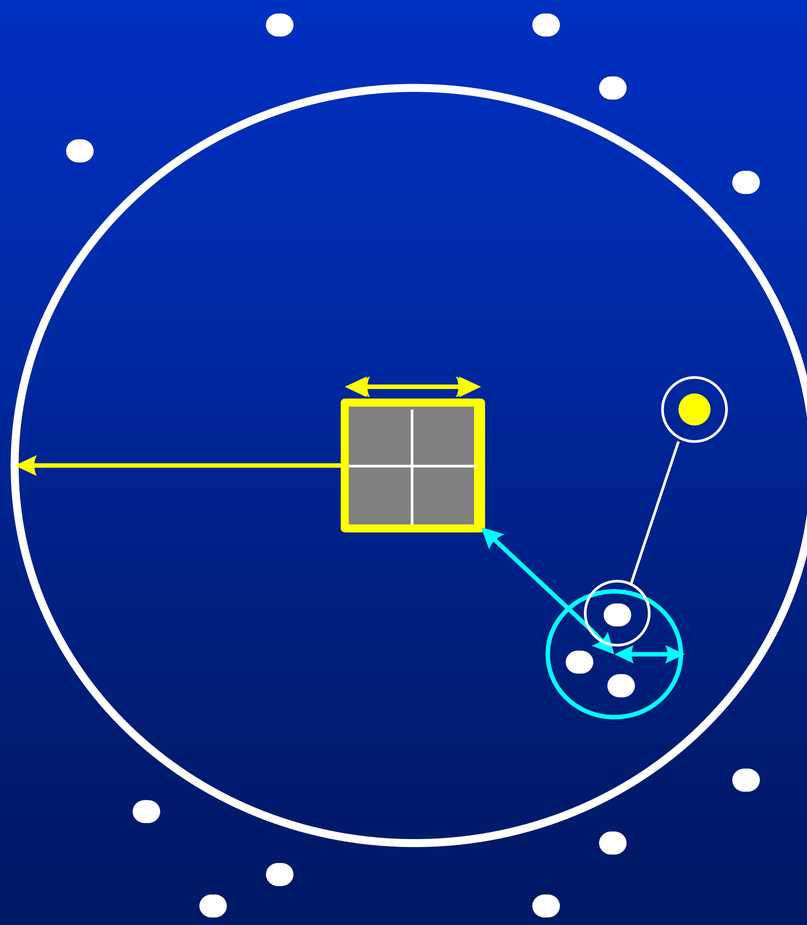
- Create a **WSPD** with separation 4.
- For each WSP, create a set of **quadtree boxes** whose sizes depend on the dist from this WSP.
- Build a **BBD tree** for these boxes.



Achieving Separation

Why does this work?

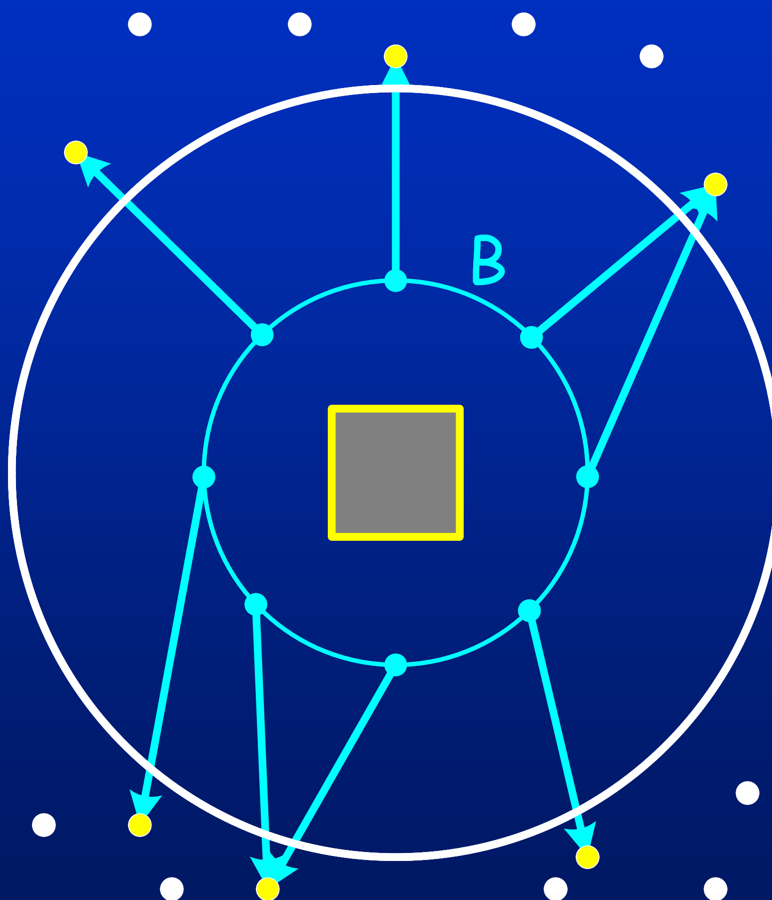
Suppose that the points within the 4s expansion are not contained within a separated ball. Then there would be a well-separated pair, which would force the cell to be split.



Selecting Representatives

Two-Step Approach:

- Construct a set of $1/\varepsilon^{(d-1)/2}$ points uniform on an intermediate sphere B.
- Reps are the nearest neighbors of these points.

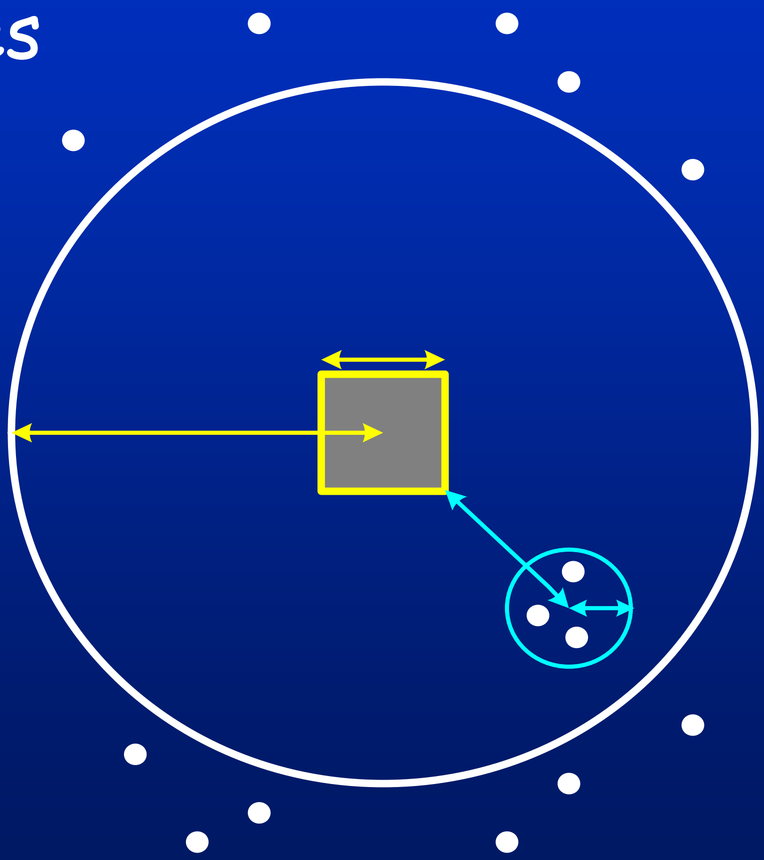


Space Reduction: Sampling

Recall that representatives come from two sources:

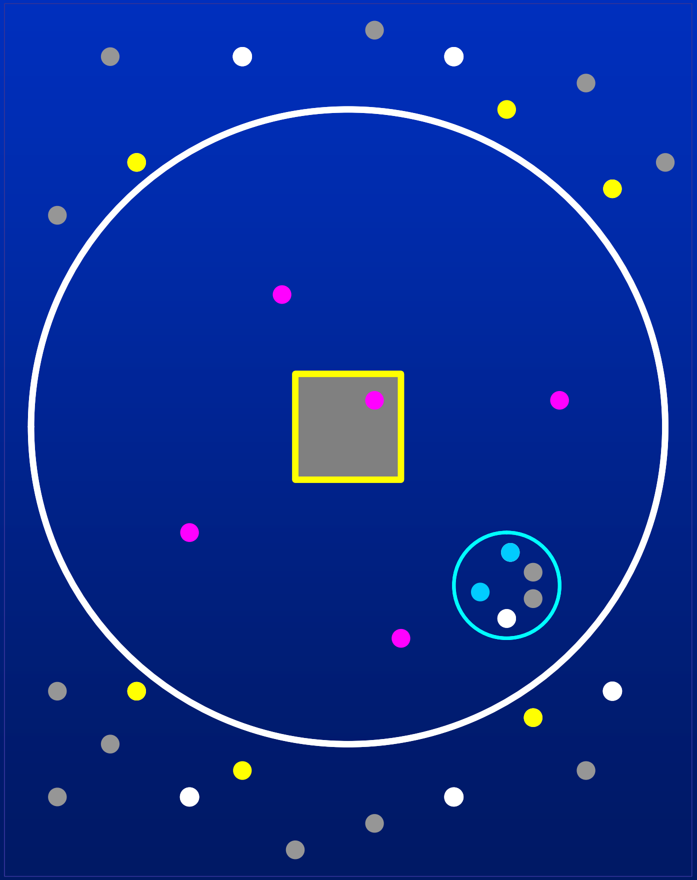
- From outside large ball
- From inner cluster
- No points exist in the remaining "no-man's land"

Idea: Allow more points into no-man's land, and make them all reps.



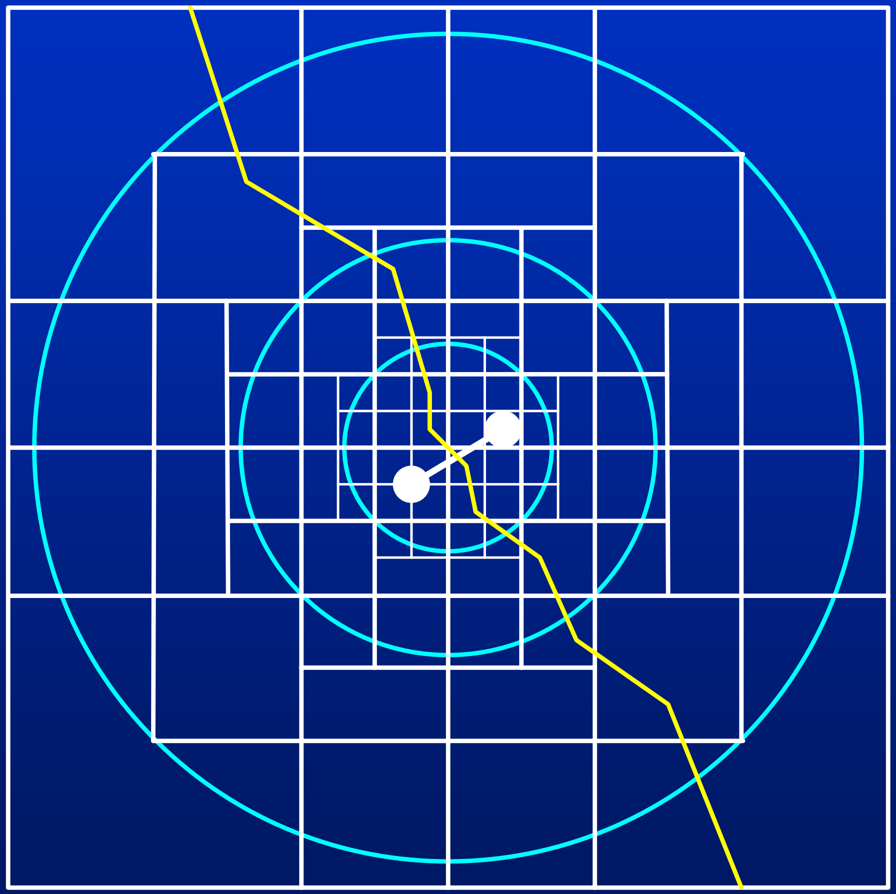
Space Reduction: Sampling

- **Intuition:** Use a sample S' of $n\varepsilon^{(d-1)/2}$ points in the basic AVD construction. We expect $O(1/\varepsilon^{(d-1)/2})$ points of S to lie in no-man's land.
- **Representatives:** From outer, inner cluster, and no-man's land.



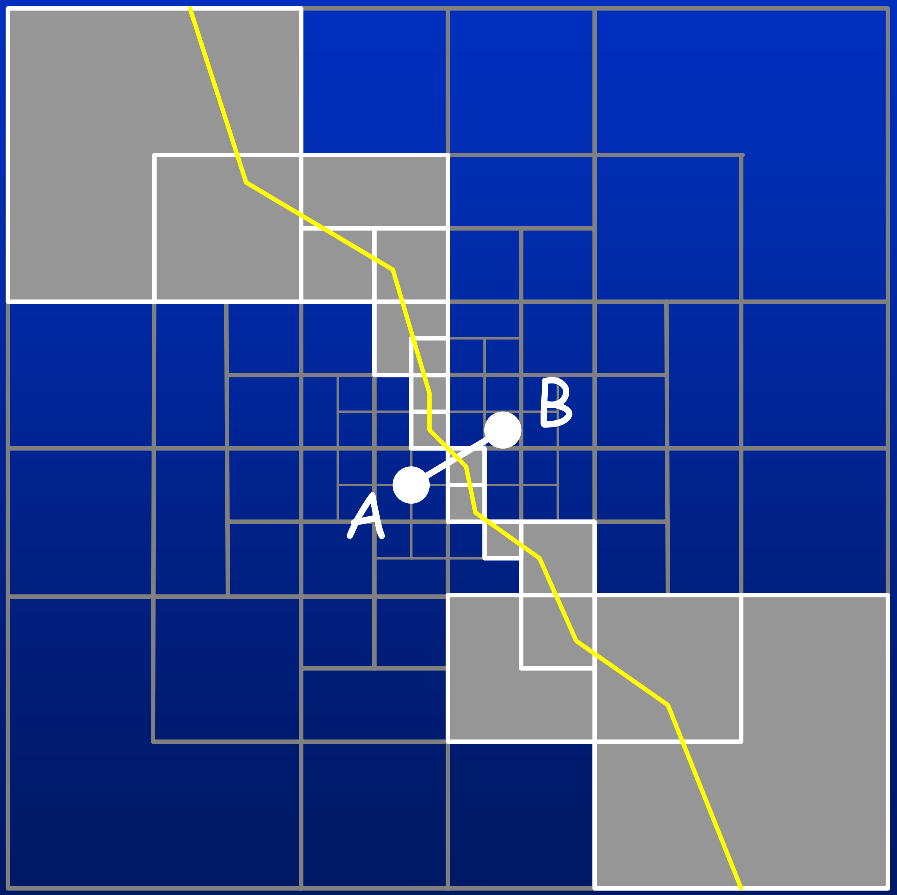
Bisector-Sensitivity

- Recall that the basic AVD construction creates quadtree boxes uniformly around each WSP.
- **Idea:** Concentrate boxes along bisector.



Bisector-Sensitivity

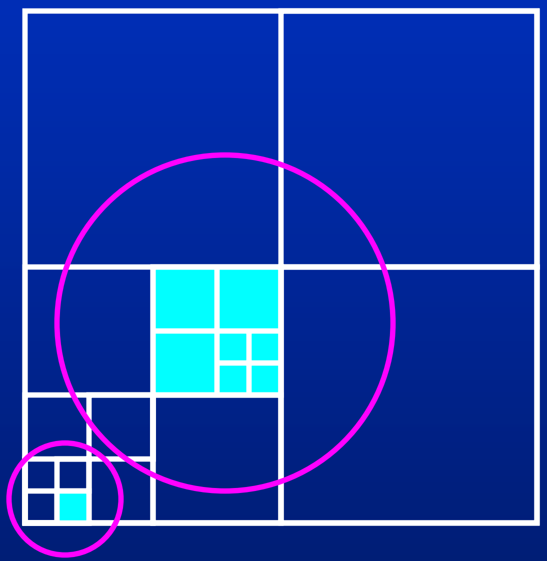
Bisector Sensitive Construction: For each WSP (A,B), create quadtree boxes as before, but only for those that intersect the A-B bisector.



AVDs and Range Searching

Adaptation: AVDs can be adapted to perform range searching. Rather than using just the leaf nodes, internal nodes are used as well for answering queries, where the query size is roughly γ times the size of the associated cell.

Auxiliary information: Each internal node stores information about surrounding region in order to answer queries.

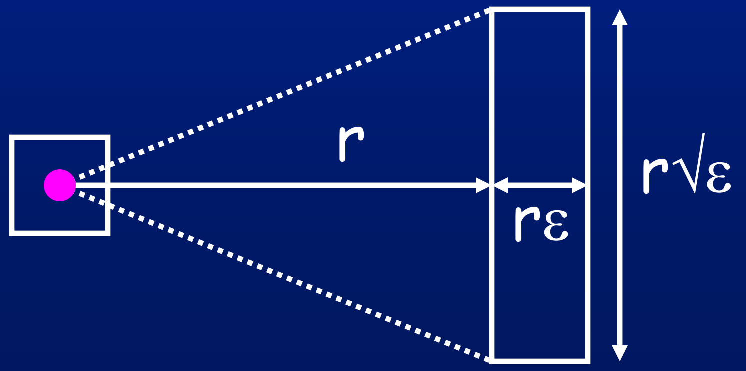


Polar kd-trees

With spherical ranges there are two sources of approximation error:

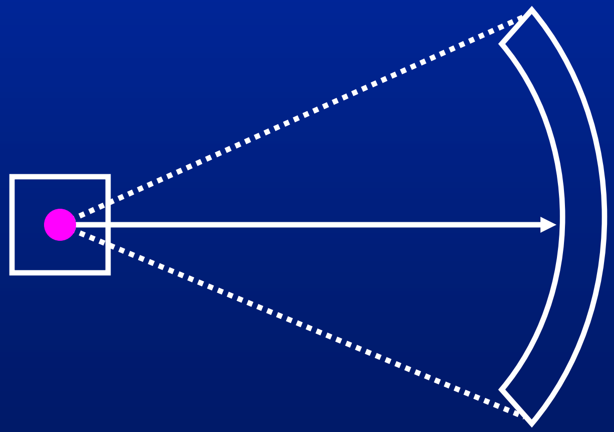
- radial distance from center
- angular error

It is possible to tolerate greater angular error than radial error.



Polar kd-trees

Polar kd-tree: Build a collection of hierarchical spatial subdivisions, based on a polar representation of points relative to some local center.



Conclusions

ε-AVD: A spatial subdivision in which ε-NN queries reduce to point location.

Space Efficiency: Through deterministic sampling and bisector sensitivity.

- $O(\log n + 1/\varepsilon^{(d-1)/2})$ time
- $O(n)$ space

AVDs for Other Problems: AVDs for other objects? AVD-like structures for interpolation?

Approximating Voronoi Cells: Some initial results by Arya and Vigneron.

The End

Thank you!

Implementation?

The WSPD construction is not very practical:

- Large constants.
- Bottom-up construction (must know all the AVDs to construct any part).
- Further study is warranted.

Partial construction: Given the size of the AVD, it is useful to build/rebuild portions of the structure. Need for top-down construction.

Top-Down Construction

Input: Point set S. Error factor ε, and number of representatives t.

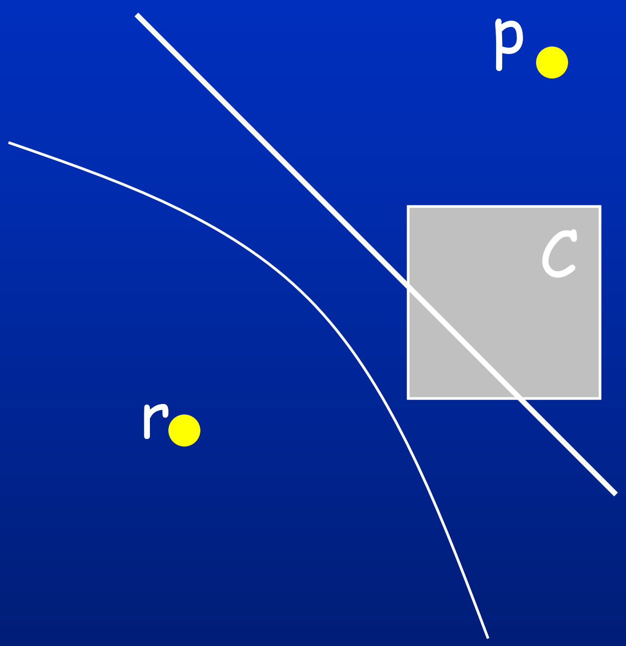
Basis: Start with bounding hypercube as cell and all points of S as candidate nearest neighbors.

Recursive step: Given a quadtree cell C, and a collection of candidate nearest neighbors U.

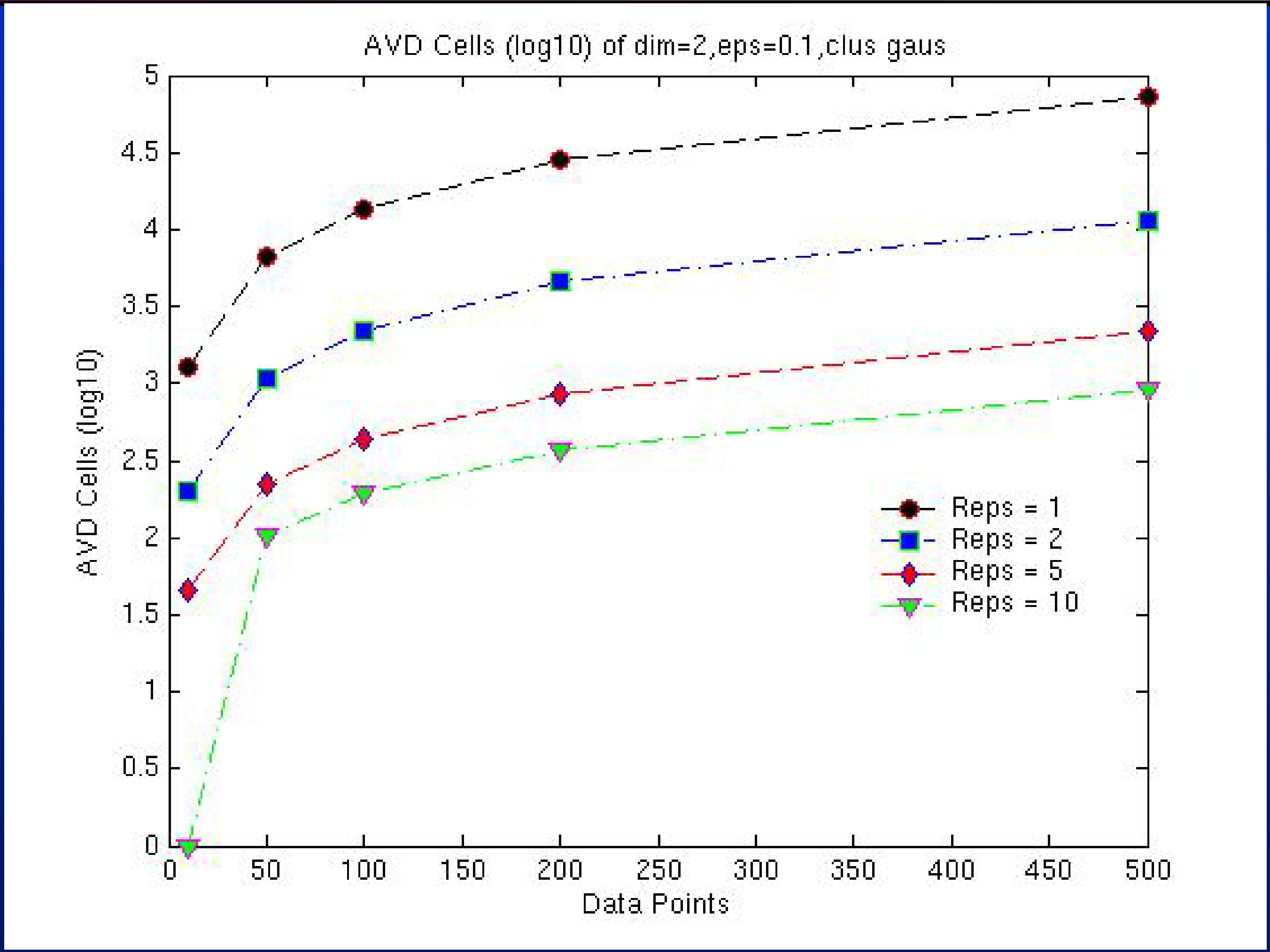
- Prune from U all points that cannot be an ε-NN to any point of C.
- If $|U| \leq t$, then done. Otherwise, split C and recurse.

How to Prune?

- Let p in U be the closest candidate to the center of the cell. Let r be some other candidate.
- If for all x in C, if $\text{dist}(x,p) \le (1+\varepsilon)\text{dist}(x,r)$ then prune r.
- This can be solved numerically.



Results: Cells vs n,t



Results: cells vs ε