Local Anisotropy In Globally Isotropic Granular Packings

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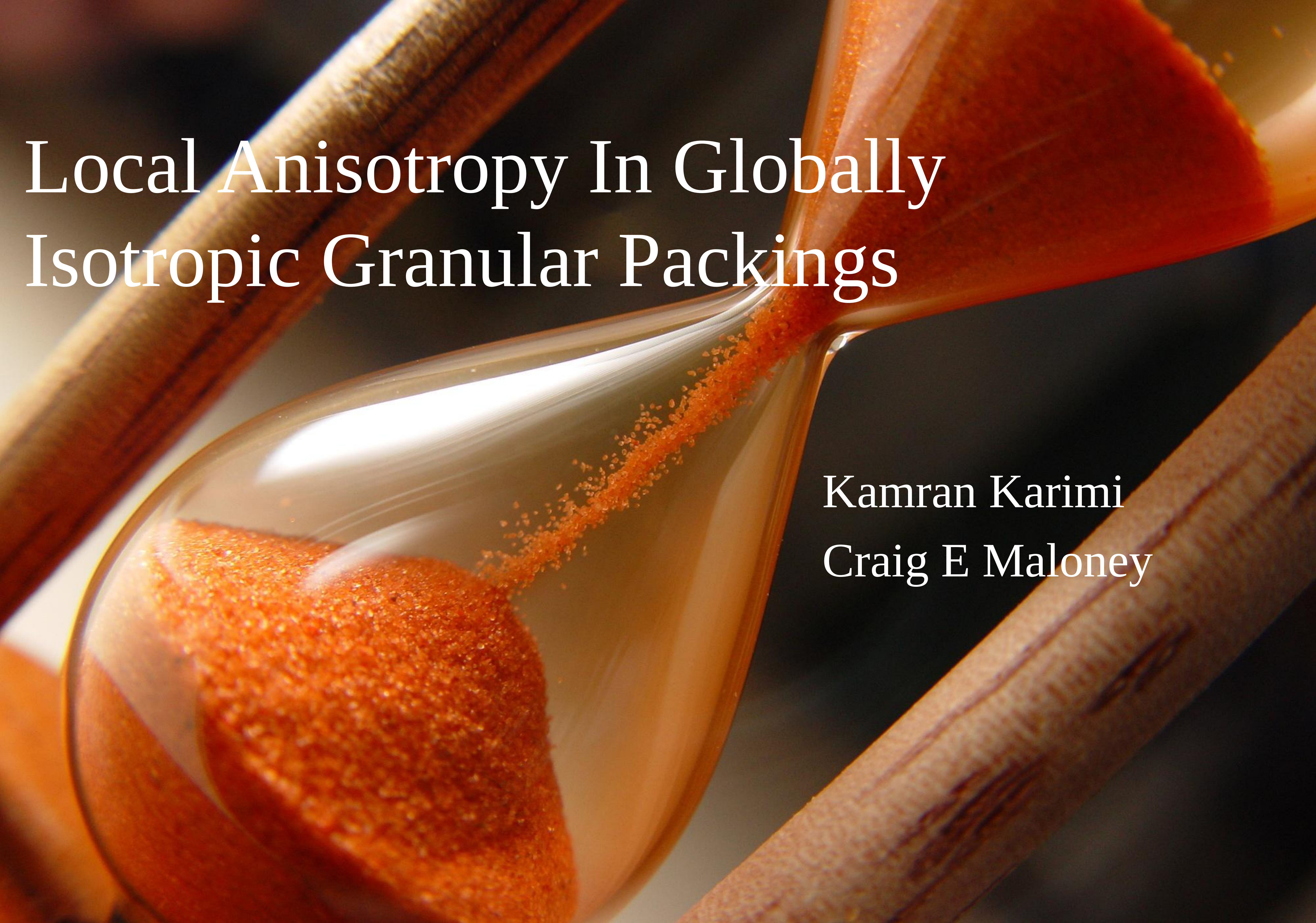
Craig E Maloney



Granular Materials

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Natural Phenomena:

Failure

Flows

Industry:

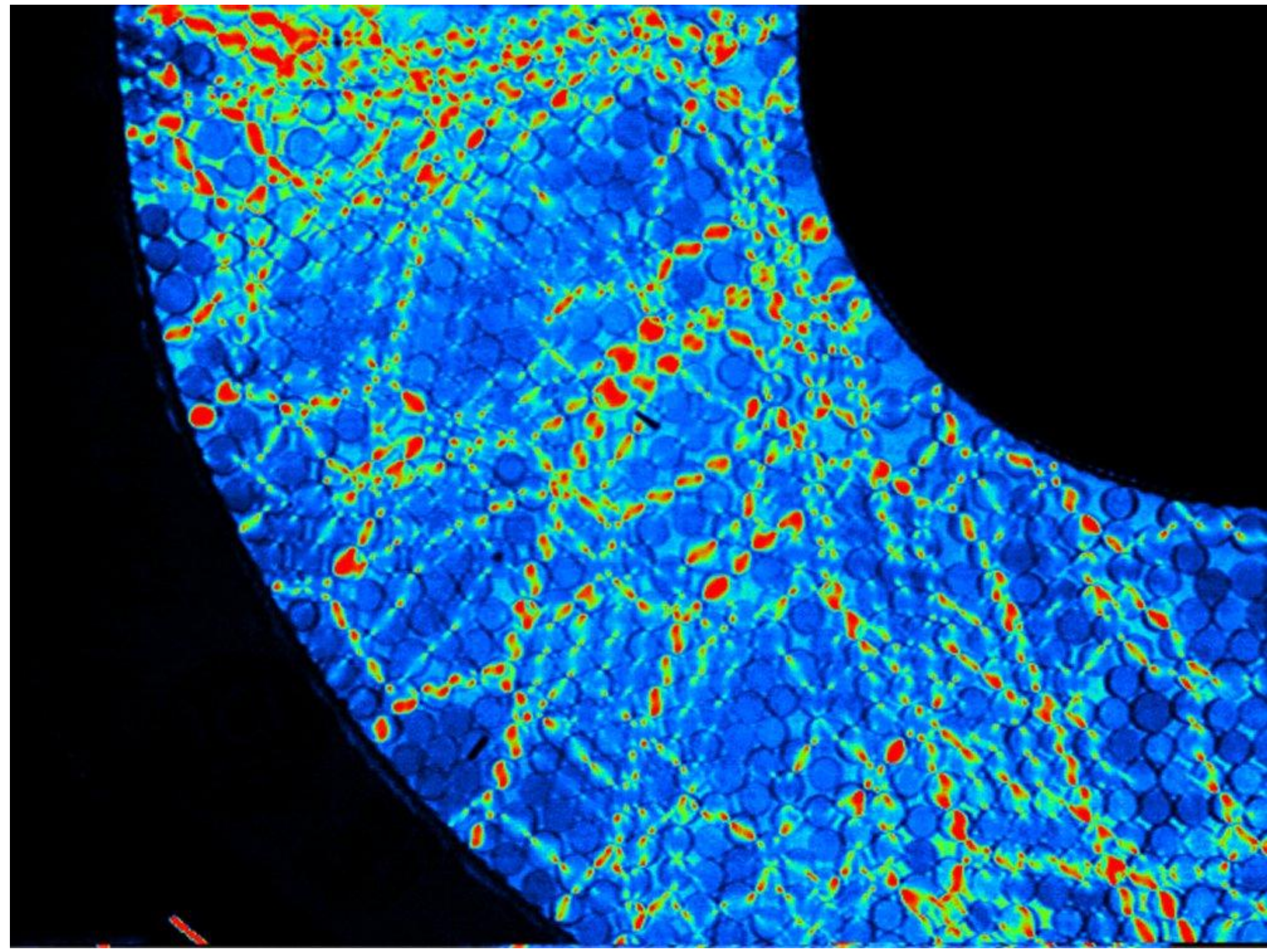
Tablets

Chemicals

Forces In Granular Materials

Walk On The Beach Without Sinking: *Forces* in Granular Materials

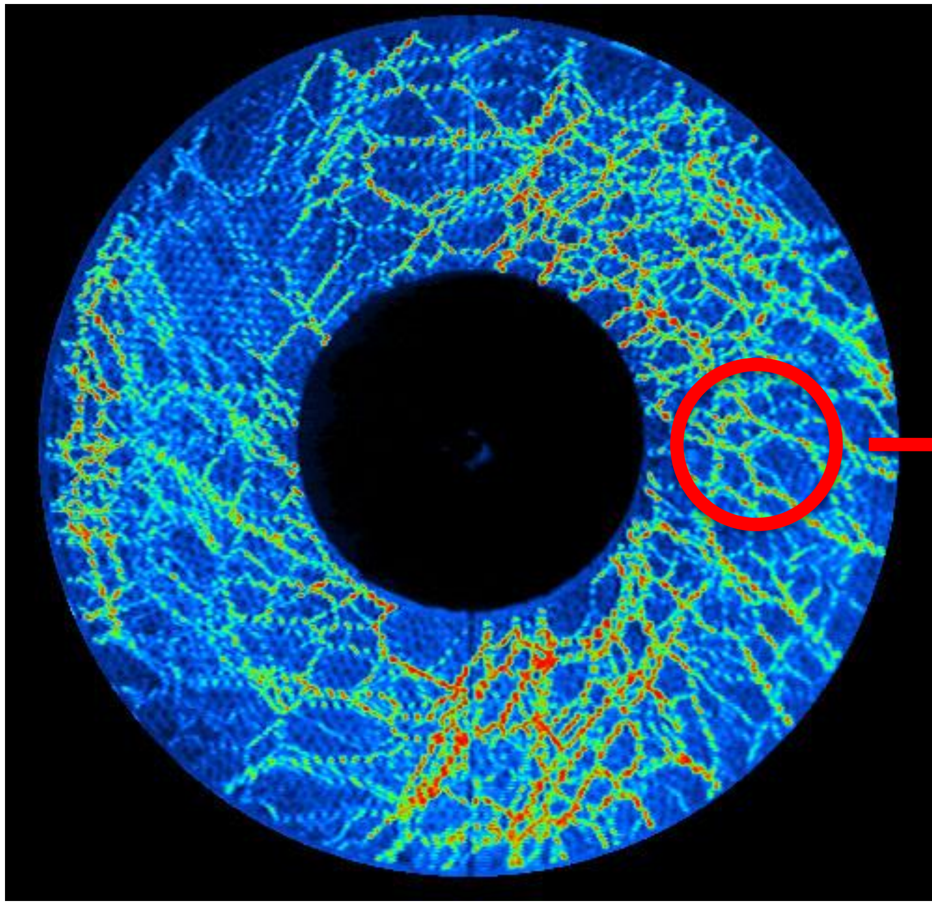
Veje, Howell, and Behringer, *Phys.Rev.E*, **59**, 739 (1999)

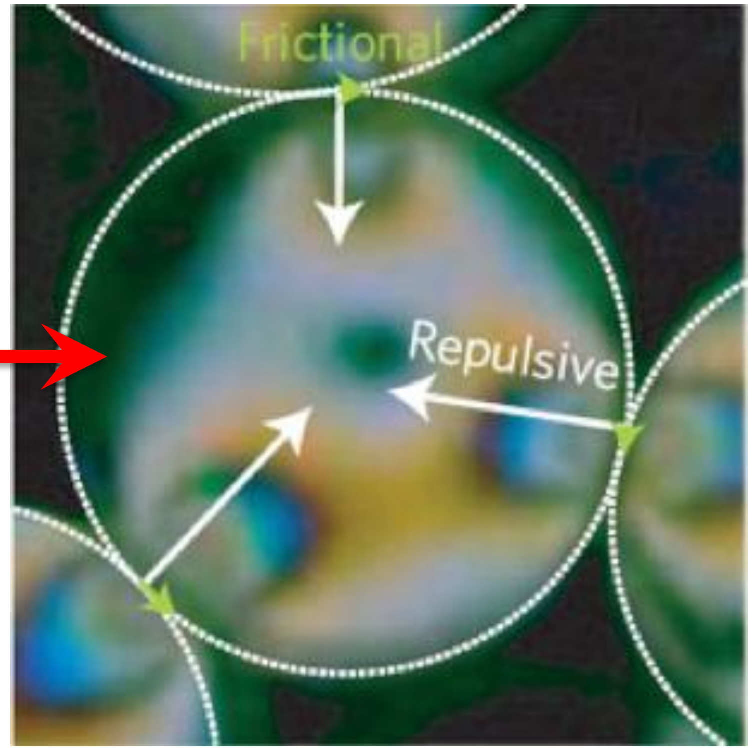


Transmission Of Loads :
Along "*Force Chains*"
Forces Are Highly *Heterogeneous*

Forces In Granular Materials

2d Shear Experiment

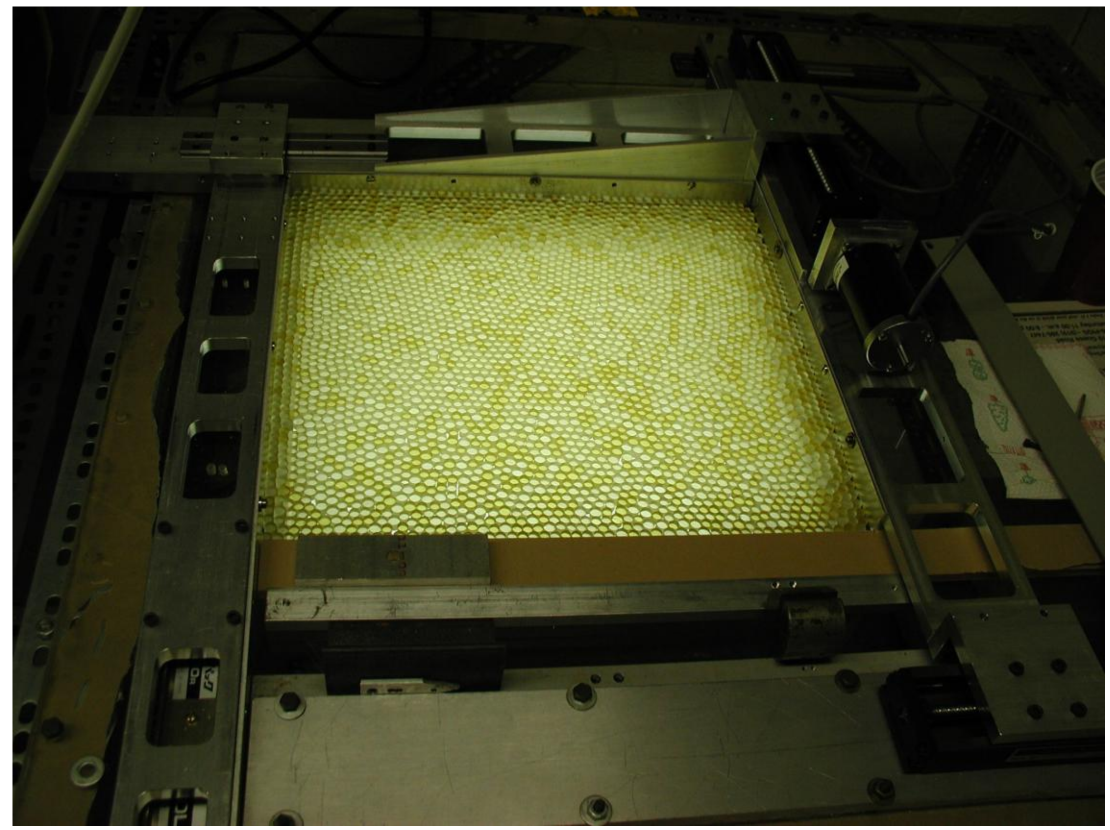




Howell et al (1999)

liu et al. (1995)
Mueth et al. (1998)
. (2003)
Majmudar et al. (2005)
Zhou et al. (2006)

Biaxial Tester And Photoelastic Particles

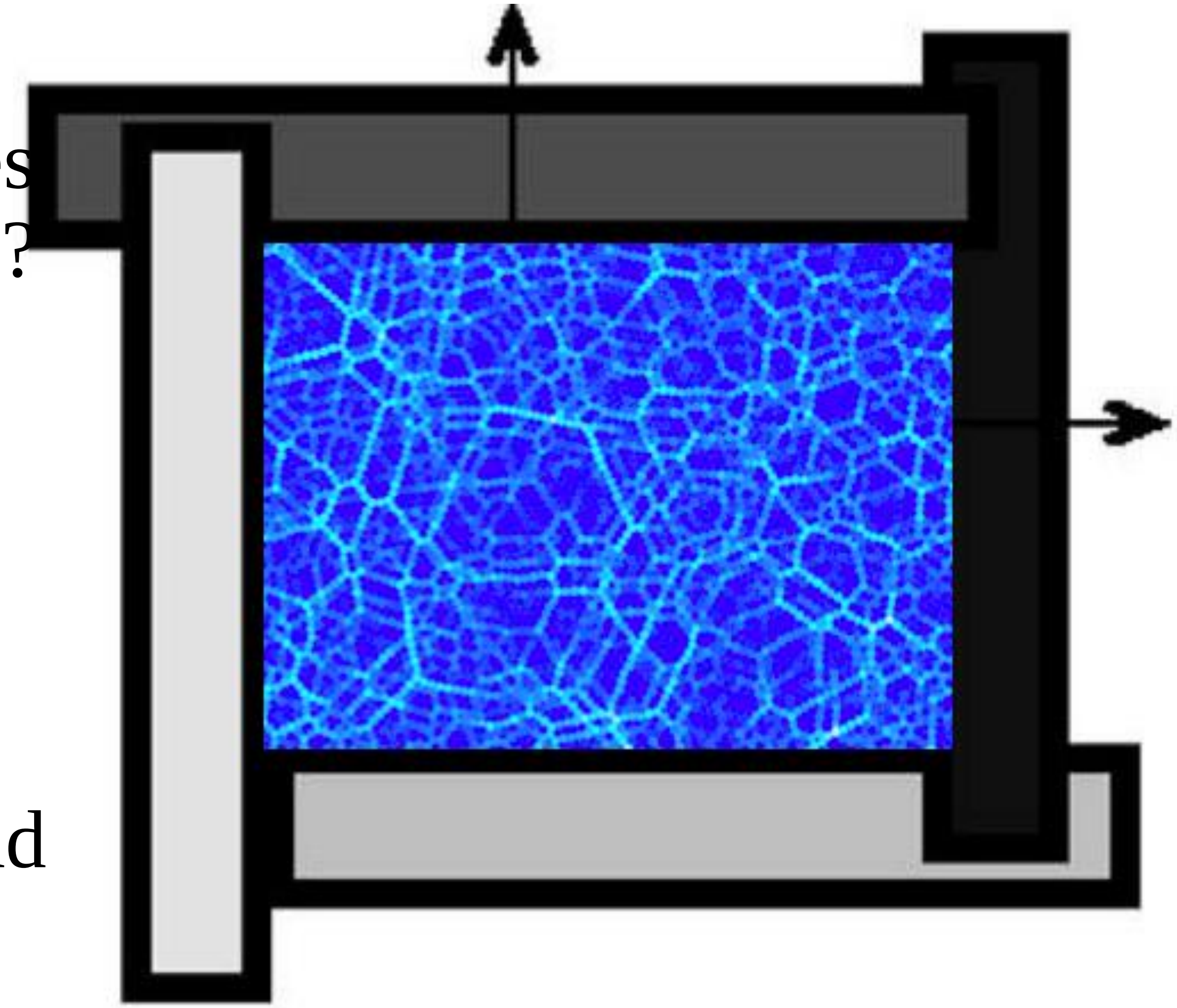


- Macroscopic Particles
- Forces Are Measurable

Main Questions

- **Disorder**
  - *Anisotropy* And *Heterogenity*
- **Question?**
  - How To Quantify These Properties In Geometrically Complex Forces?
  - Specifically: *Local Anisotropy* Despite *Isotropic Loading*
- **Question?**
  - Impact Of Forces In *Elasticity* And *Mechanical Response*
  - How Anisotropic Are *Elastic Moduli*?

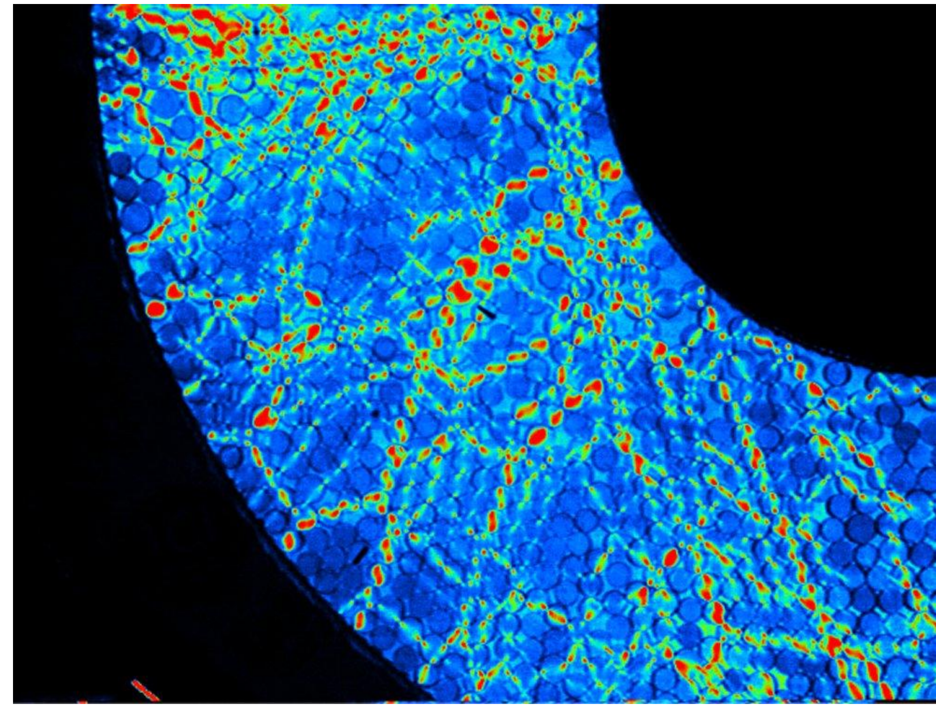
Apply Globally Isotropic Compression



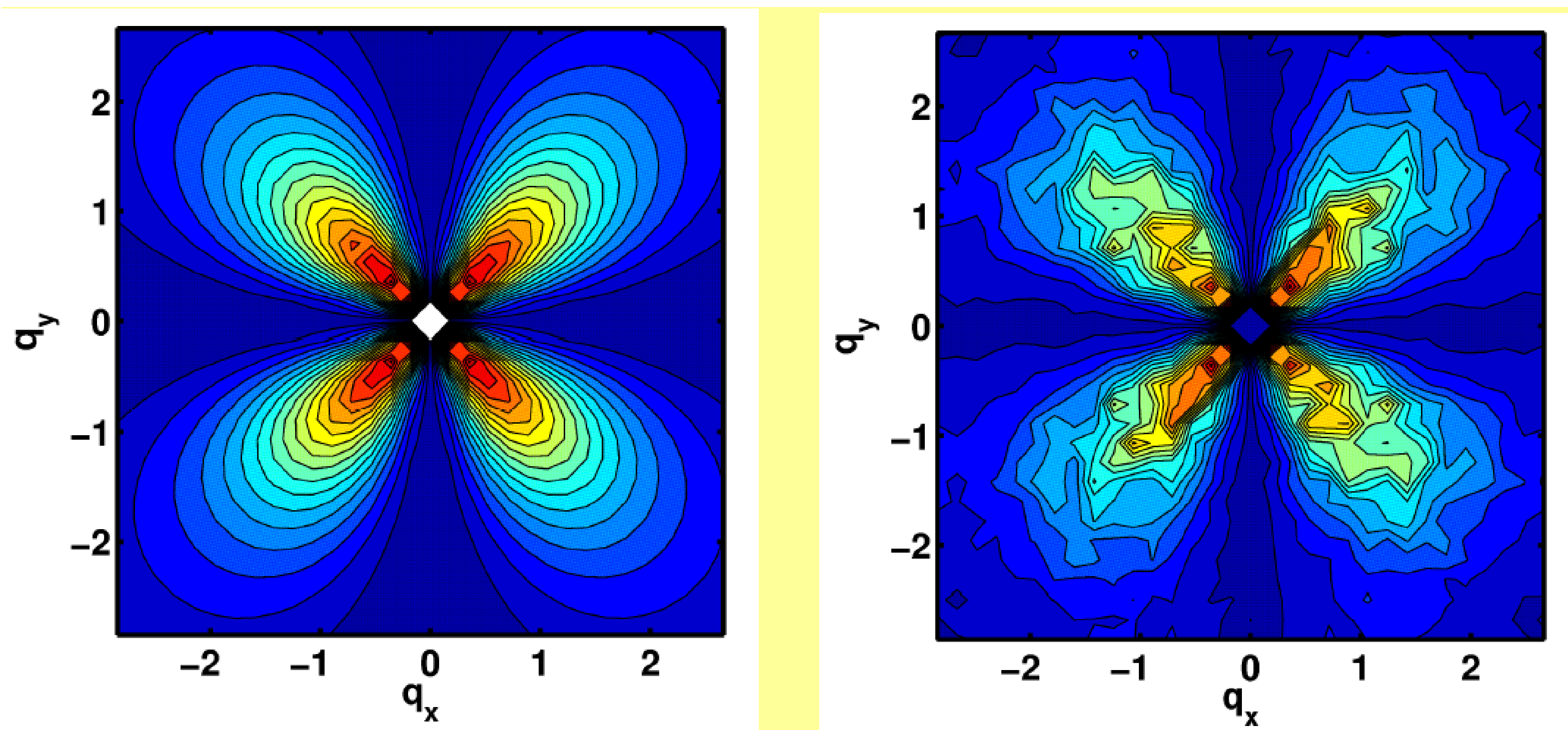
Previous Work: Study Forces

- **Question?**
  - How To Quantify These Geometrically Complex Forces?

Behringer et al (1999)



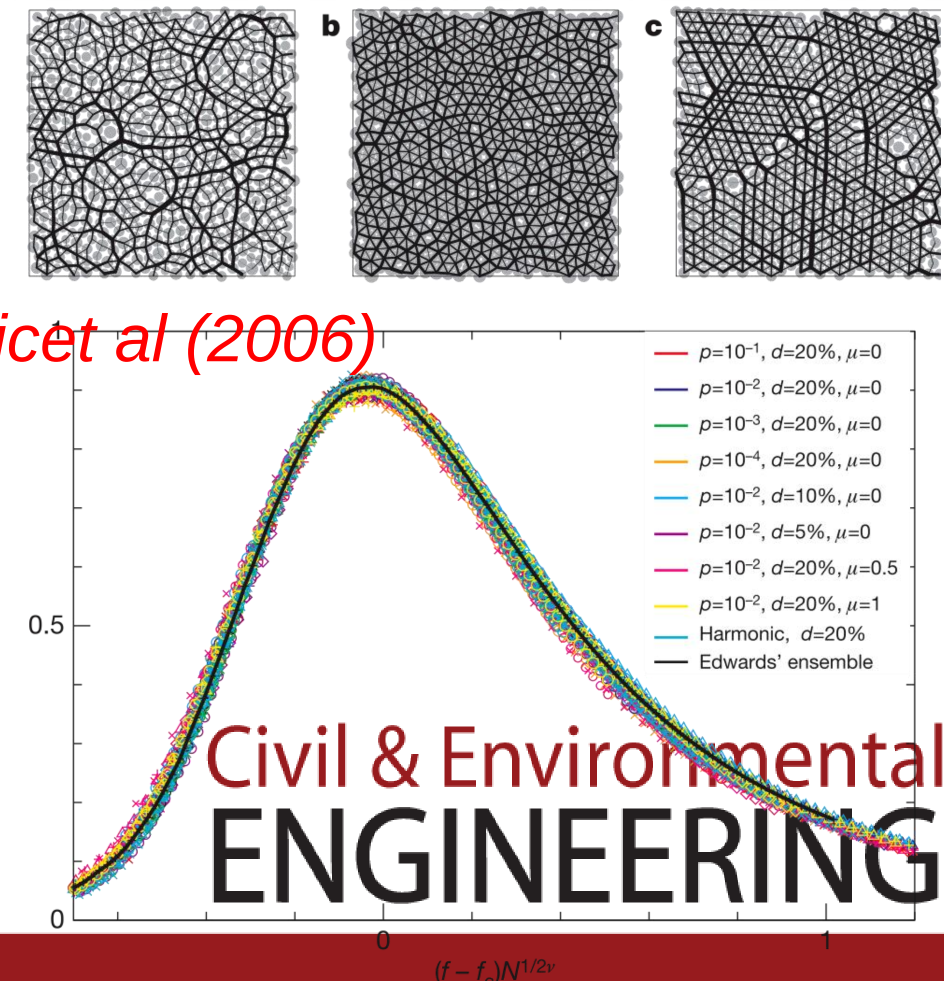
Henkes et al (2009)



Theoretical prediction

Simulation result

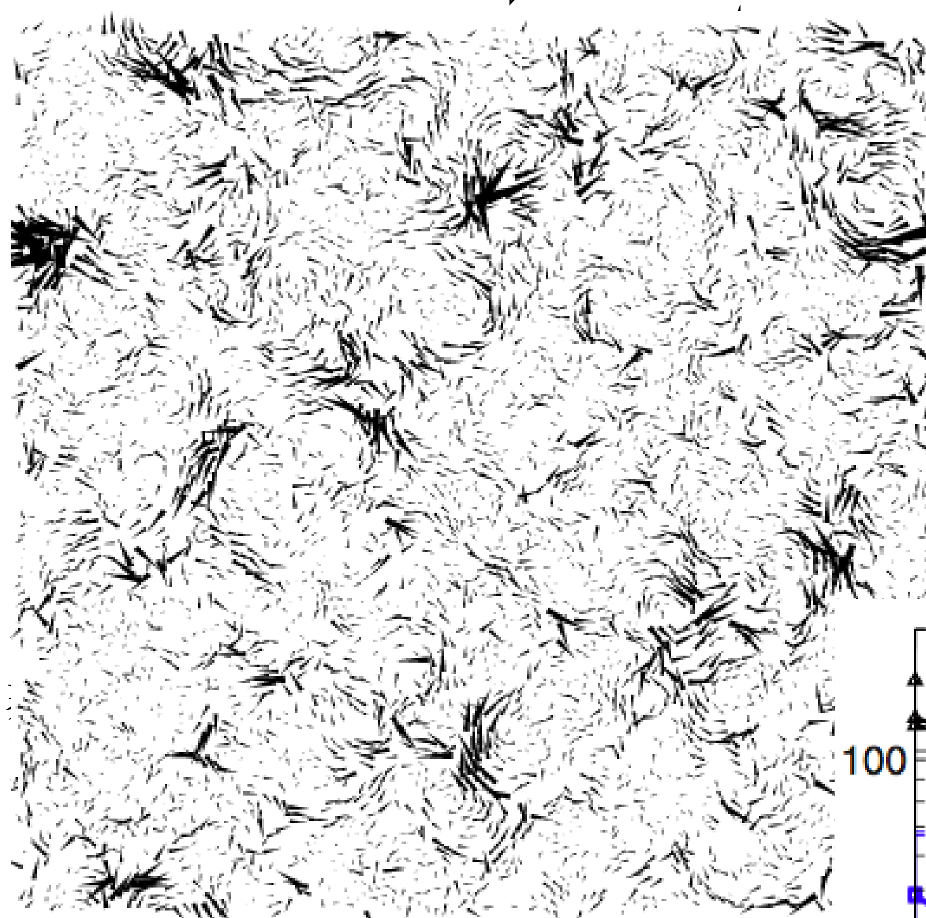
Ostojicet al (2006)



Previous Work: Mechanical Response

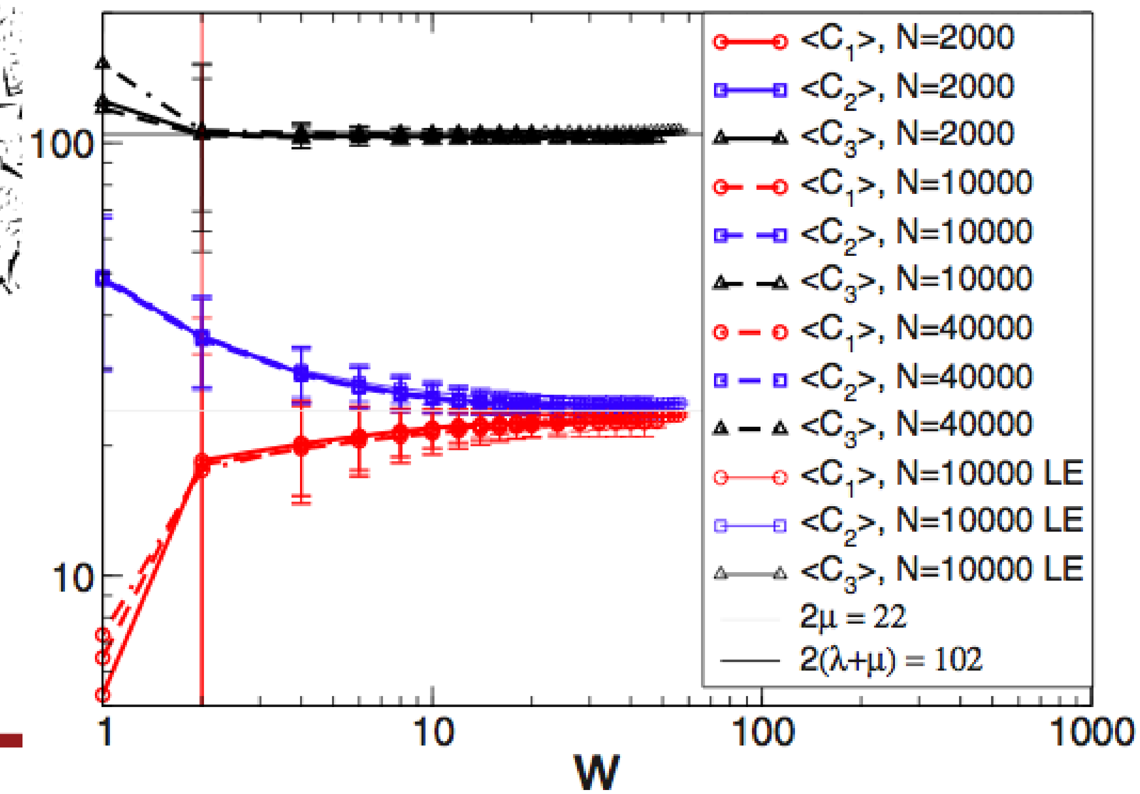
Homogeneous Displacements

Inhomogeneous Response, Non-Affine Displacements



- **Elastic Moduli**
  - 2D Jennard-Jones
  - Scale Dependent Corrections
  - Convergence To *Homogeneous* And *Isotropic* Elasticity

$$\left(\lambda_{eff},\mu_{eff}\right)\neq\left(\langle\lambda_a\rangle,\langle\mu_a\rangle\right)$$



Tanguy et al. (2002,2004,2005)
Lemaître and Maloney (2004,2006)
Williams et al. (1997)
Debrégeas et al. (2001)
Roux et al. (2002)
Kolb et coll. (2003)
Weeks et al. (2006)

Model And Protocol

- **Discrete Particle Model (LAMMPS)**
- **Generate Configurations**
  - ❑ Start With Random Positions
  - ❑ Quench At Fixed Φ
  - ❑ Viscous Damping: $\Delta t = 0.1\sqrt{\frac{\varepsilon}{mD^2}}$
  - ❑ Bi-Disperse Mixture:

    O'Hern et al (2003)

$N_A/N_B = 1,\ D_A/D_B = 1.4$

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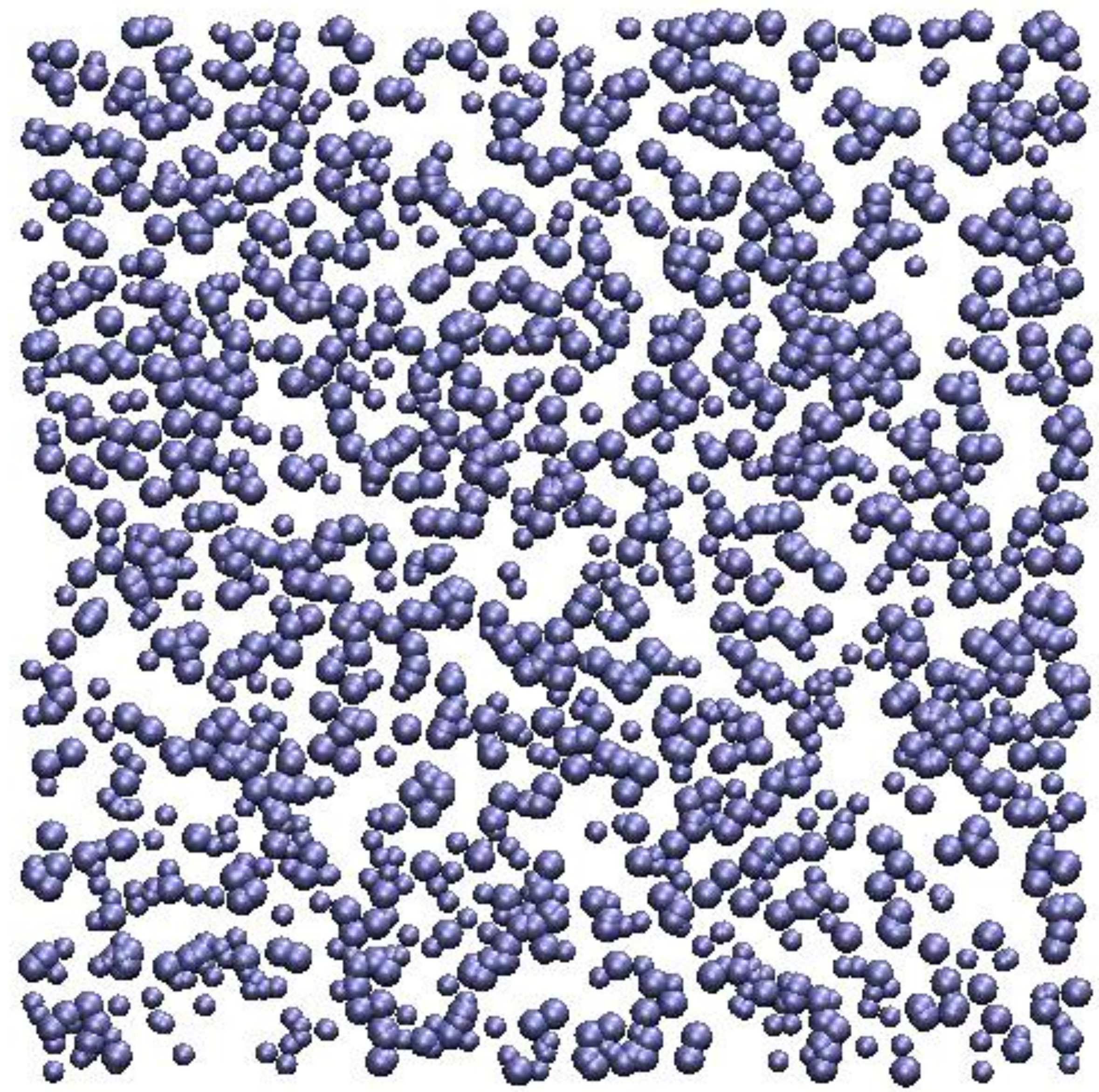
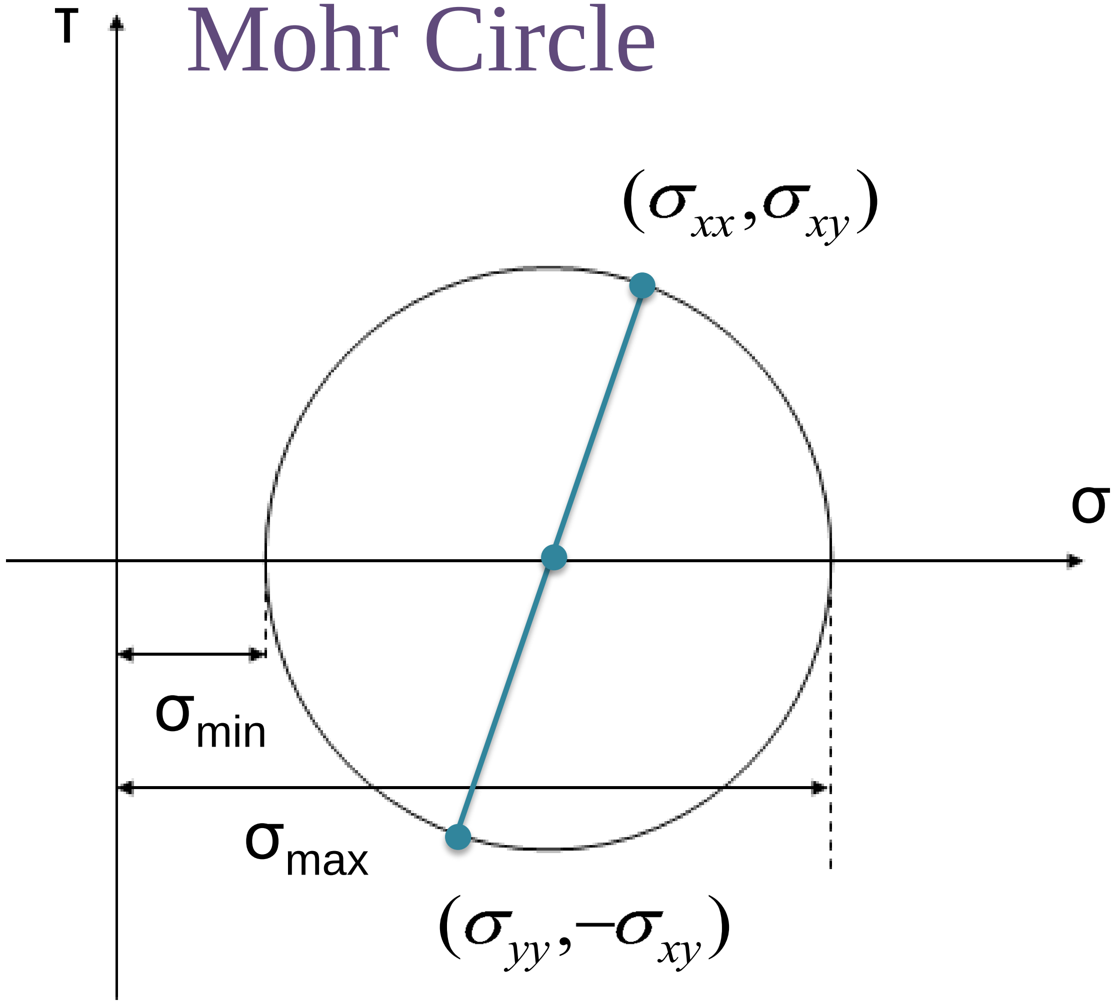


Periodic Boundaries

Stress

Stress Tensor: $\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix}$

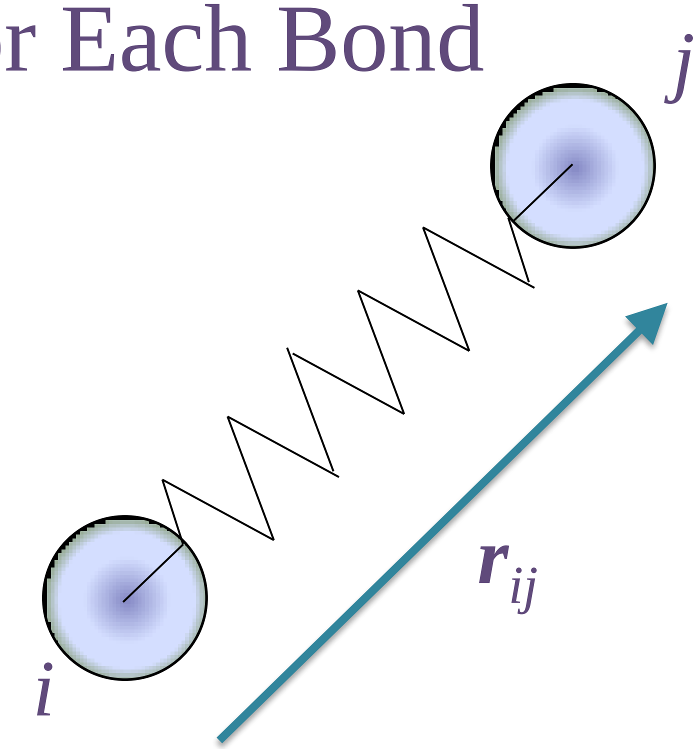
Mohr Circle



Microscopic Expression

$$\sigma_{\alpha\beta} = \frac{1}{V} \sum \sum_{i<j} f_{ij}^{\alpha} r_{ij}^{\beta}$$

For Each Bond



$$P = (\sigma_{max} + \sigma_{min})/2$$

$$\tau = (\sigma_{max} - \sigma_{min})/2$$

$$\varepsilon_s = \frac{\tau}{P}$$

$$\text{Globally}: \varepsilon_s << 1 \quad \text{isotropic}$$

$$\text{Locally}: \varepsilon_s \le 1 \quad \text{anisotropic}$$

Coarse-grained Stress

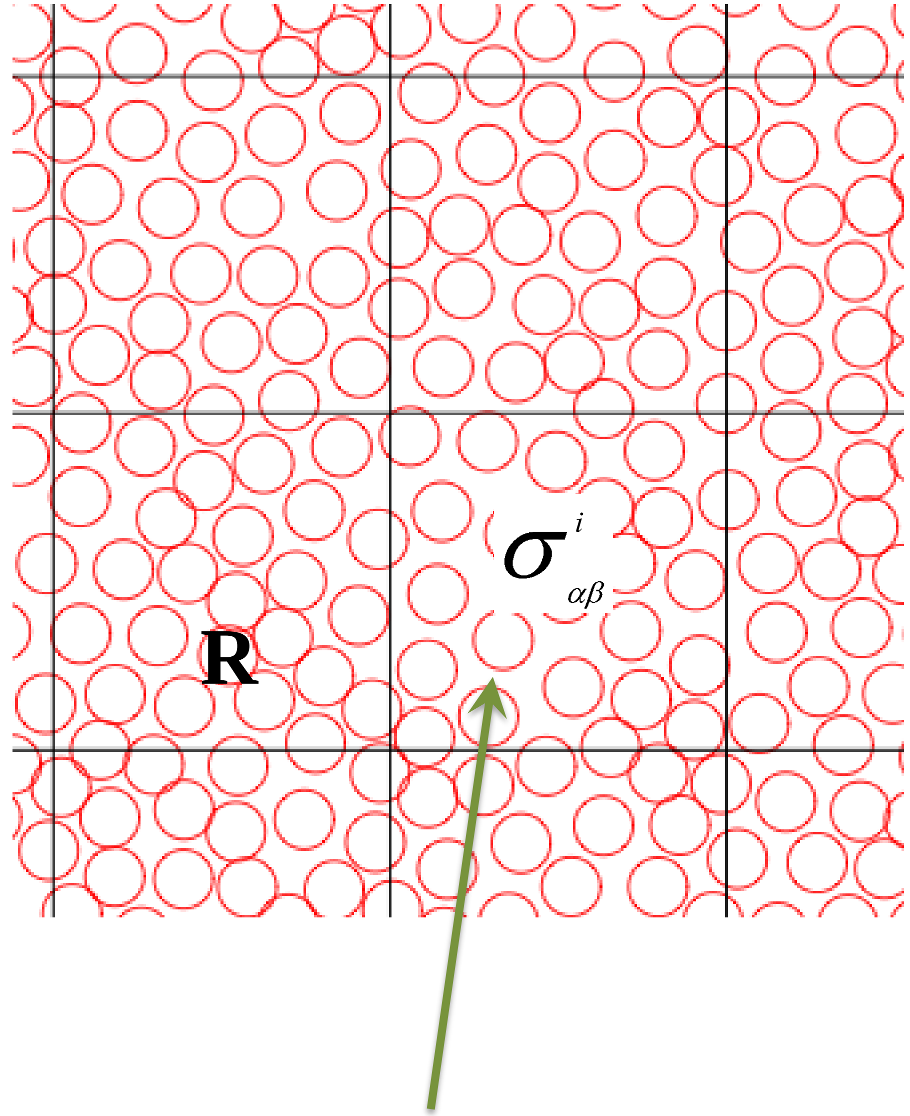
Coarse-grained Quantity

$$\varepsilon_S = \frac{\tau}{P}$$

$$\overline{\varepsilon}_{s(R)} = \frac{1}{N}\sum_{i=1}^{N}\varepsilon^{i}_{s(R)}$$

N: Number Of Squares

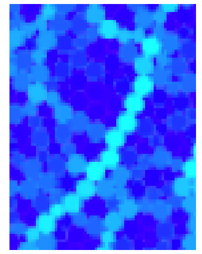
*ε_S Is **NOT** Linear!*

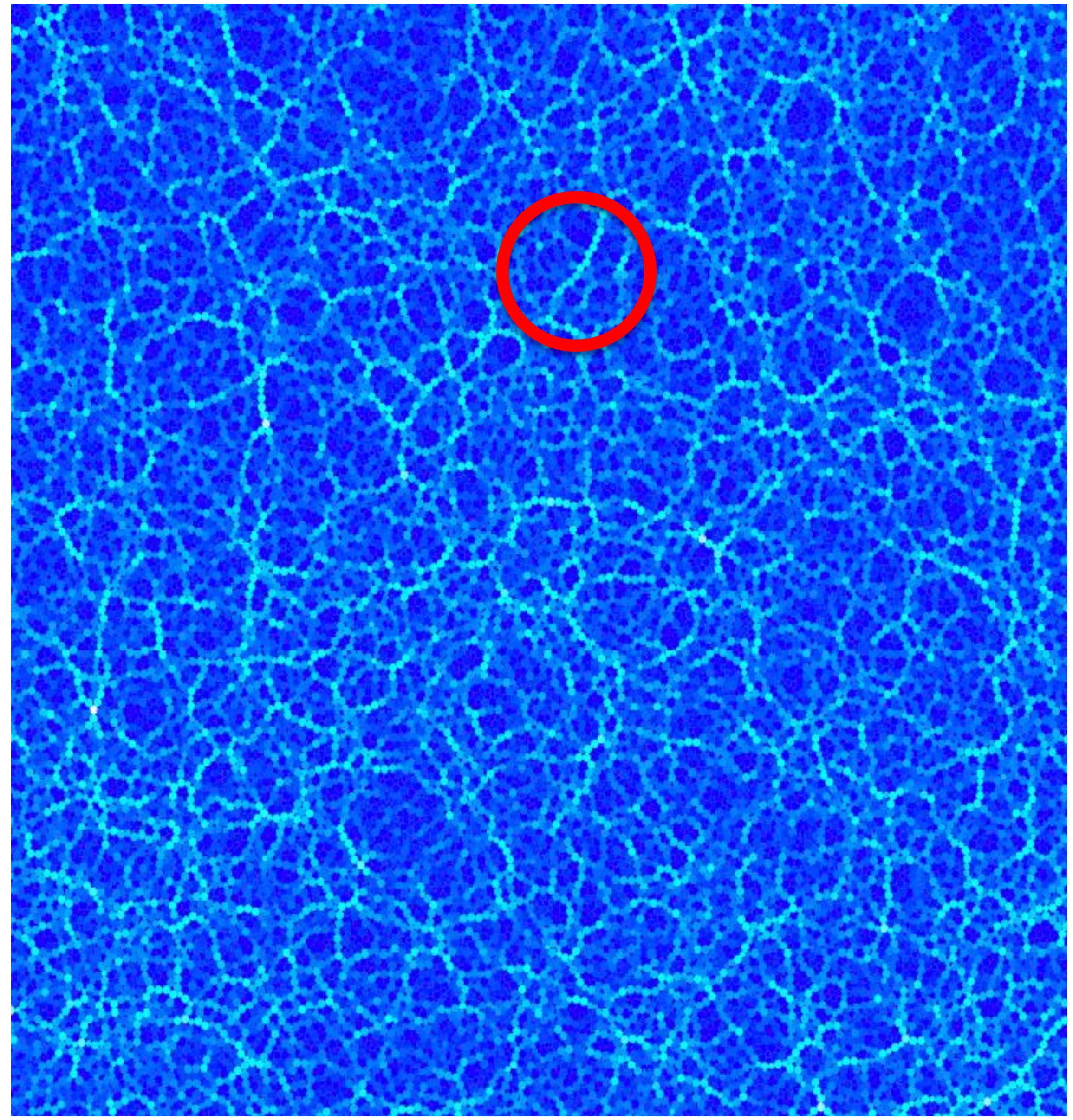


On Each Square
Region Of Length R

Coarse-grained Stress

Locally ε_S Is Large But
Globally $\varepsilon_S << 1$ *!*



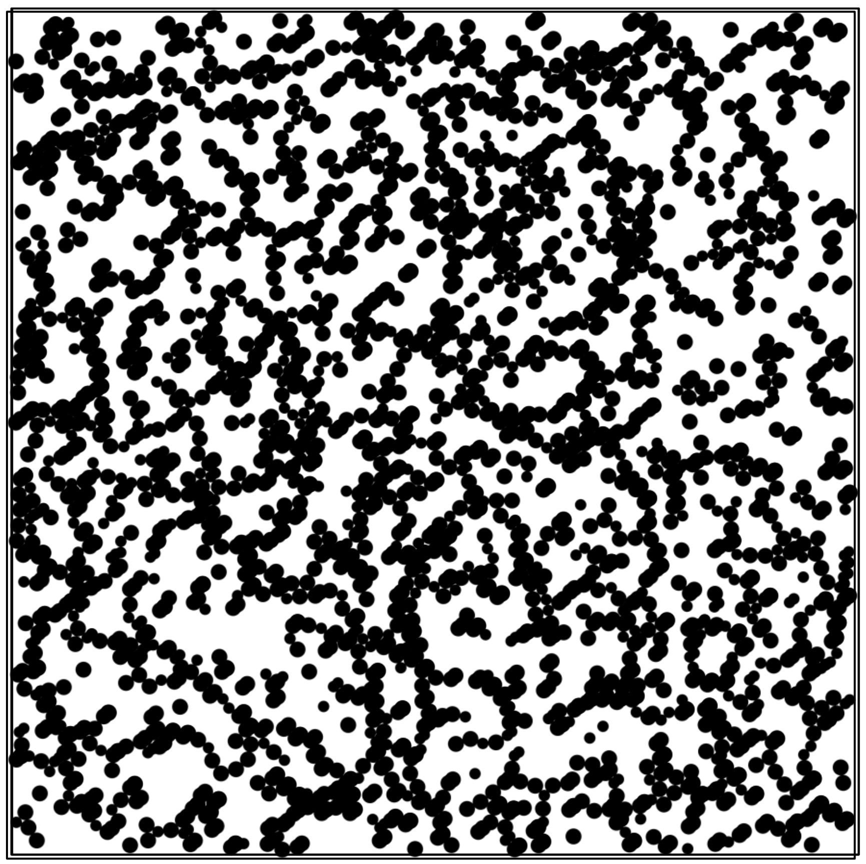


Local Anisotropy Despite
Global Isotropy

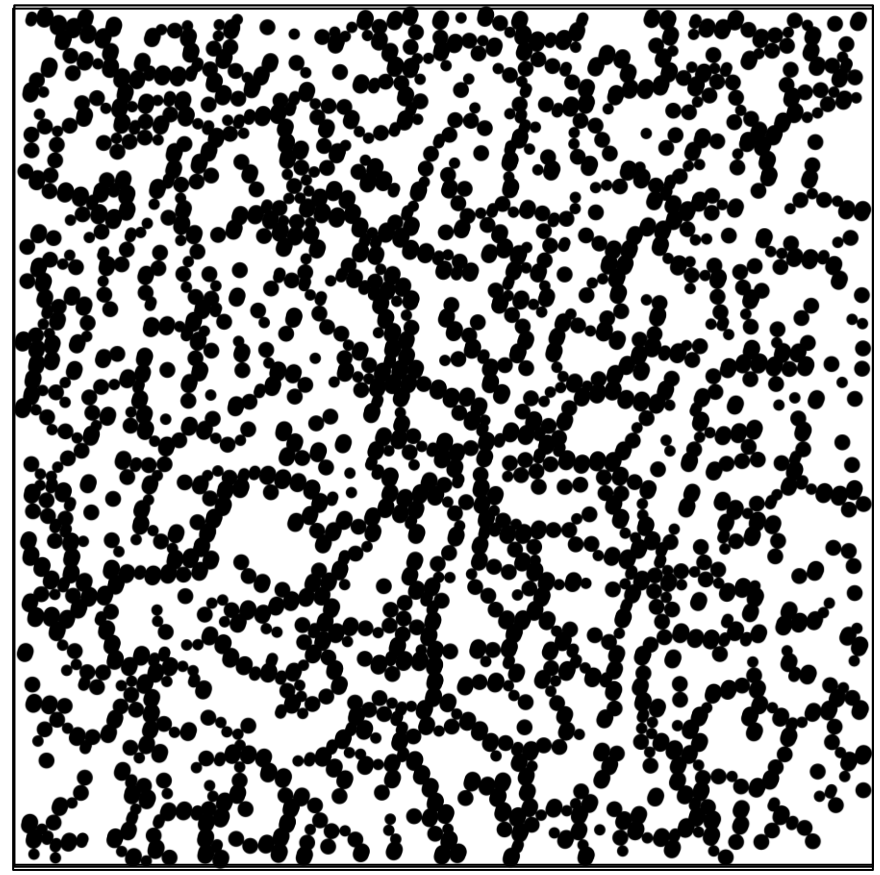
Pressure

Compact

φ=0.89



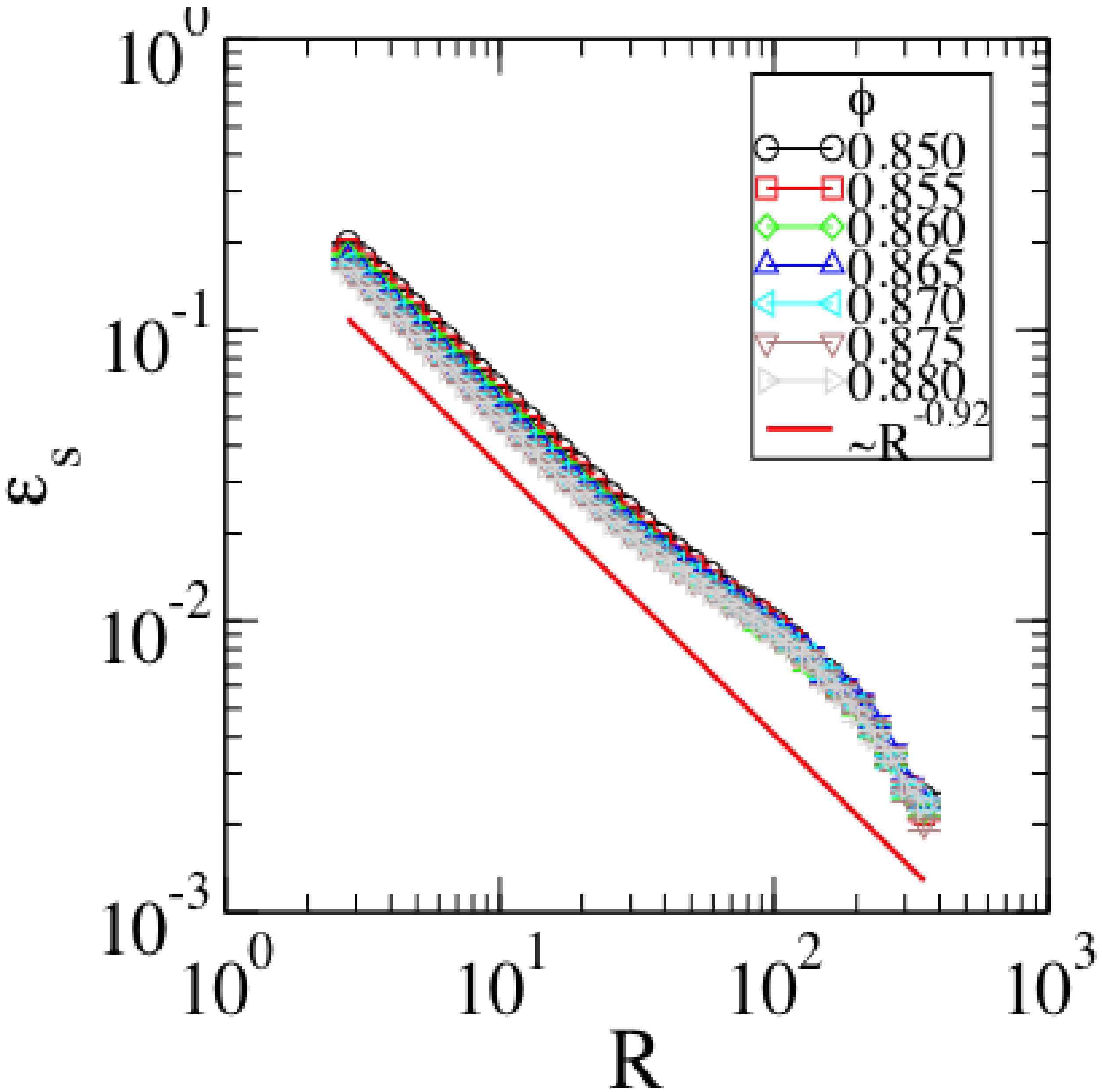
Chain-Like Clusters



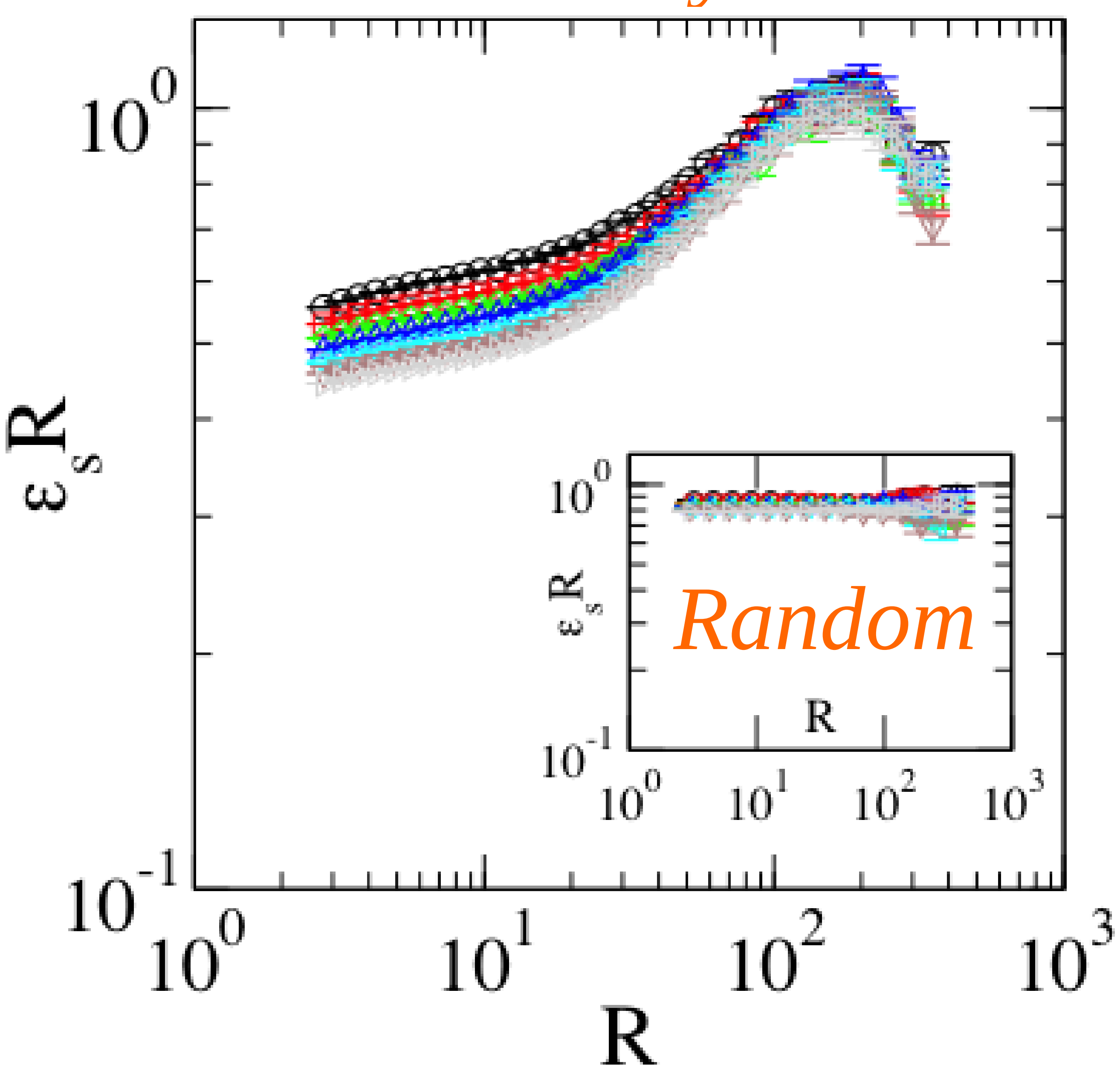
φ=0.86

Extended

- No *Spatial Correlations*: $\varepsilon_s \sim 1/R$
- More *Anisotropic* Closer to *Jamming*
- *Power-Law* Regime: $\varepsilon_s \sim R^{-0.92}$
- The *Characteristic Length* Grows With $\varphi - \varphi c \rightarrow 0$

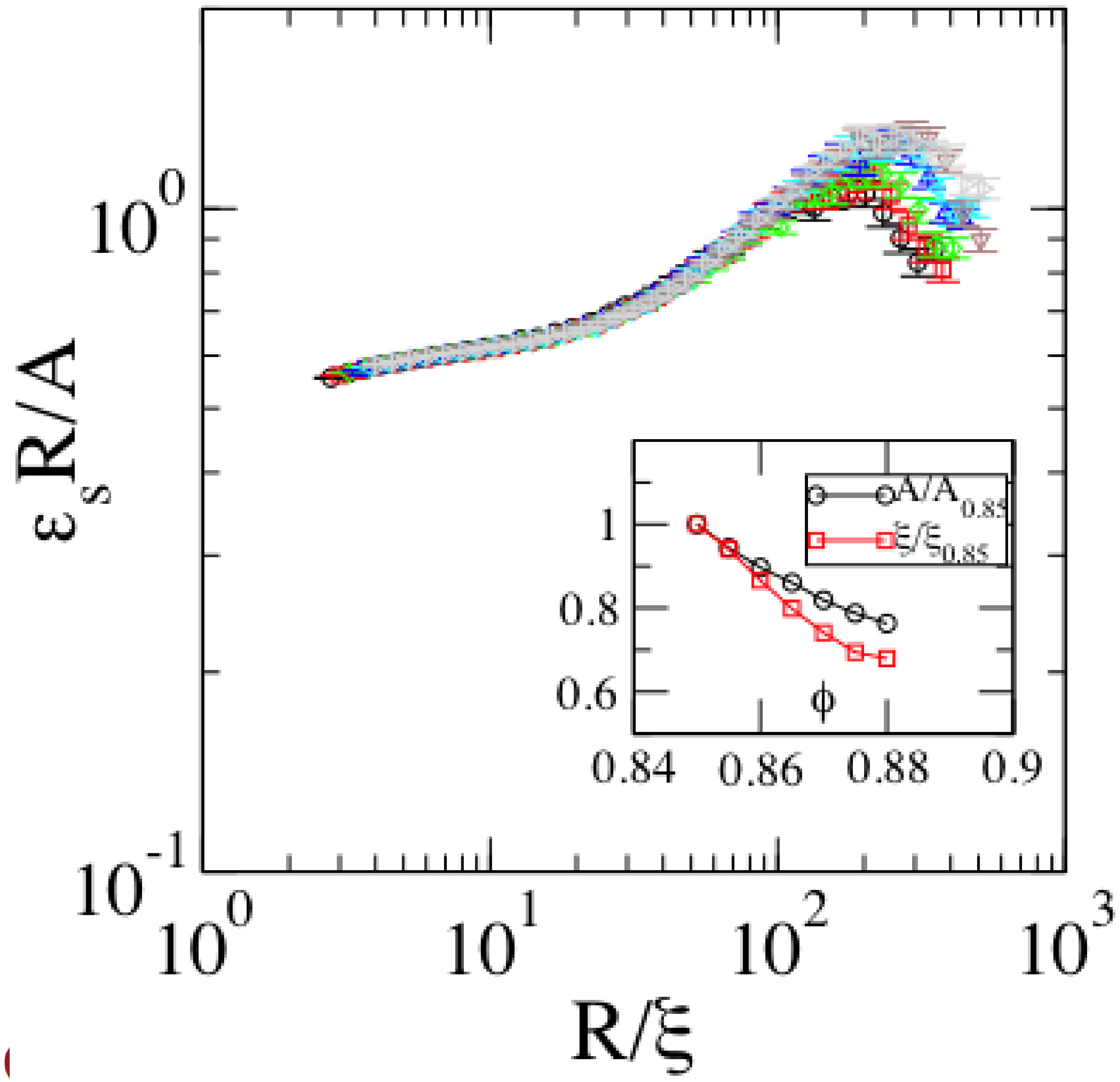


Scaled by R^{-1}



Average Stress

Rescale R by ξ and ε_s by A



Diverging Length Scale!

Elastic Moduli

2D Case (*Voigt Notation*):

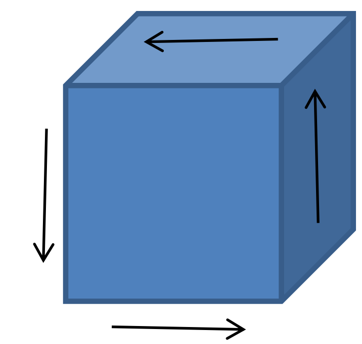
$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{yx} \end{bmatrix} = \begin{bmatrix} C_{xxxx} & C_{xxyy} & C_{xxxy} & C_{xxyx} \\ C_{yyxx} & C_{yyyy} & C_{yyxy} & C_{yyyx} \\ C_{xyxx} & C_{xyyy} & C_{xyxy} & C_{xyyx} \\ C_{yxxx} & C_{yxyy} & C_{yxxy} & C_{yxyx} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \\ \varepsilon_{yx} \end{bmatrix}$$

Independent Moduli From Symetries

$\boldsymbol{C}_{\alpha\beta\gamma\delta} = \boldsymbol{C}_{\gamma\delta\alpha\beta} \quad \rightarrow$ 10 Moduli

$\boldsymbol{\sigma}_{\alpha\beta} = \boldsymbol{\sigma}_{\beta\alpha} \quad \rightarrow$ 6 Moduli

Pure Rotation:



$$\varepsilon_4 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma \equiv 0$$

$\boldsymbol{C}_{\alpha\beta\gamma\delta} = \boldsymbol{C}_{\alpha\beta\delta\gamma} \quad \rightarrow$ 7 Moduli

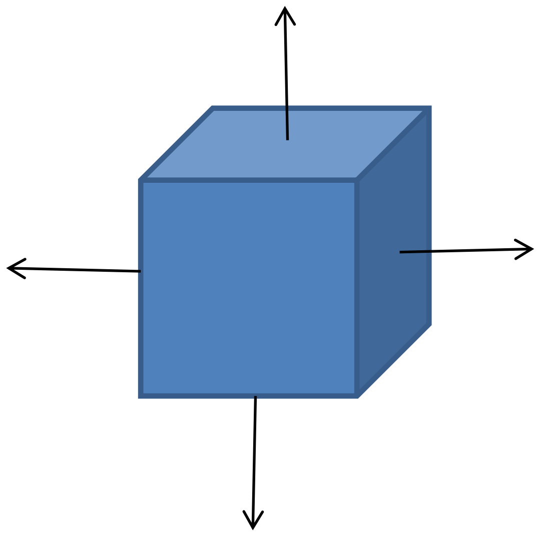
An *Isotropic* Material: *Lame-Navier*

Form Only 2 Constants

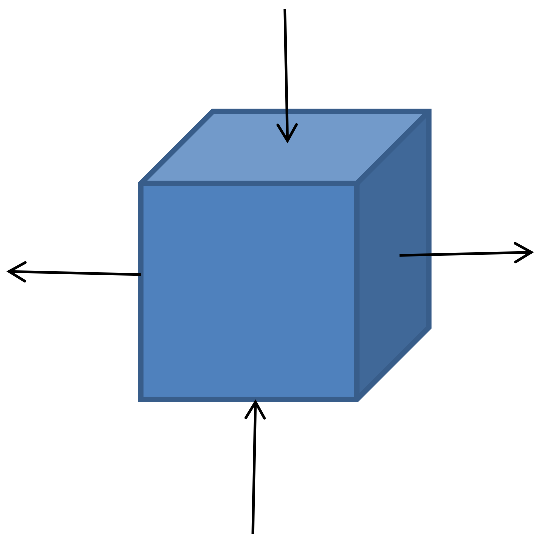
$$C^L = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 & 0 \\ \lambda & \lambda + 2\mu & 0 & 0 \\ 0 & 0 & \mu & \mu \\ 0 & 0 & \mu & \mu \end{bmatrix}$$

Elastic Moduli

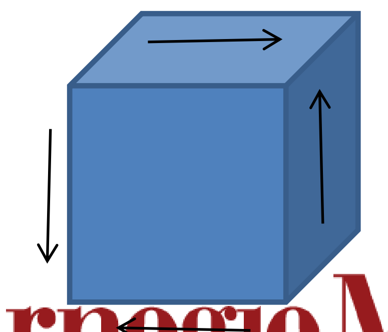
3 Nontrivial Eigenvalues/vectors For C^L:



Isotropic Expansion $\Lambda_1 = 2(\lambda + \mu), \quad \varepsilon_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$



Axial Pure Shear $\Lambda_2 = 2\mu, \quad \varepsilon_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$



Pure Shear $\Lambda_3 = 2\mu, \quad \varepsilon_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Form orthogonal basis

$$[\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4]$$

$$C = \begin{bmatrix} K & C_{12} & C_{13} & 0 \\ C_{21} & \mu_1 & C_{23} & 0 \\ C_{31} & C_{32} & \mu_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Or

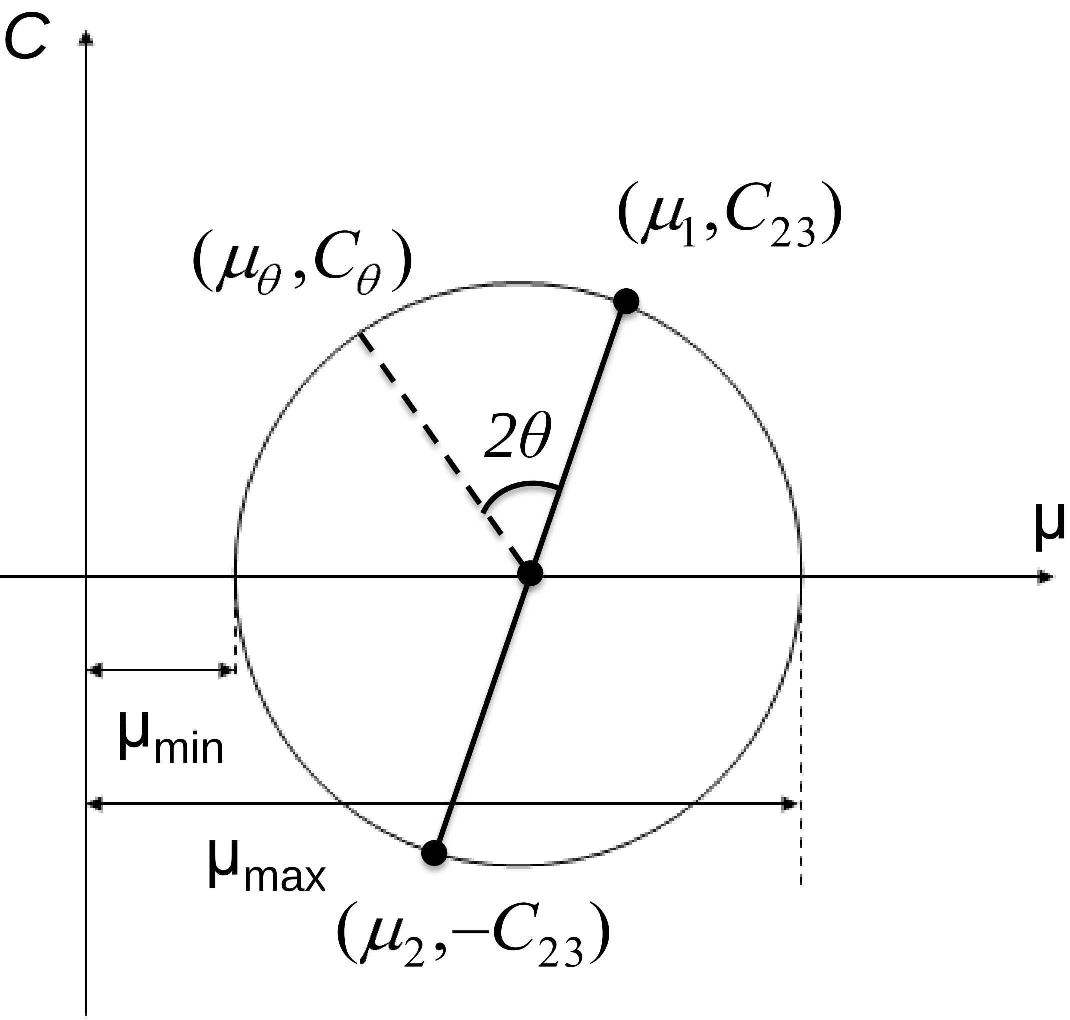
$$\begin{bmatrix} K & C_{12} & C_{13} \\ C_{21} & \mu_1 & C_{23} \\ C_{31} & C_{32} & \mu_2 \end{bmatrix}$$

Anisotropy in Elastic Moduli

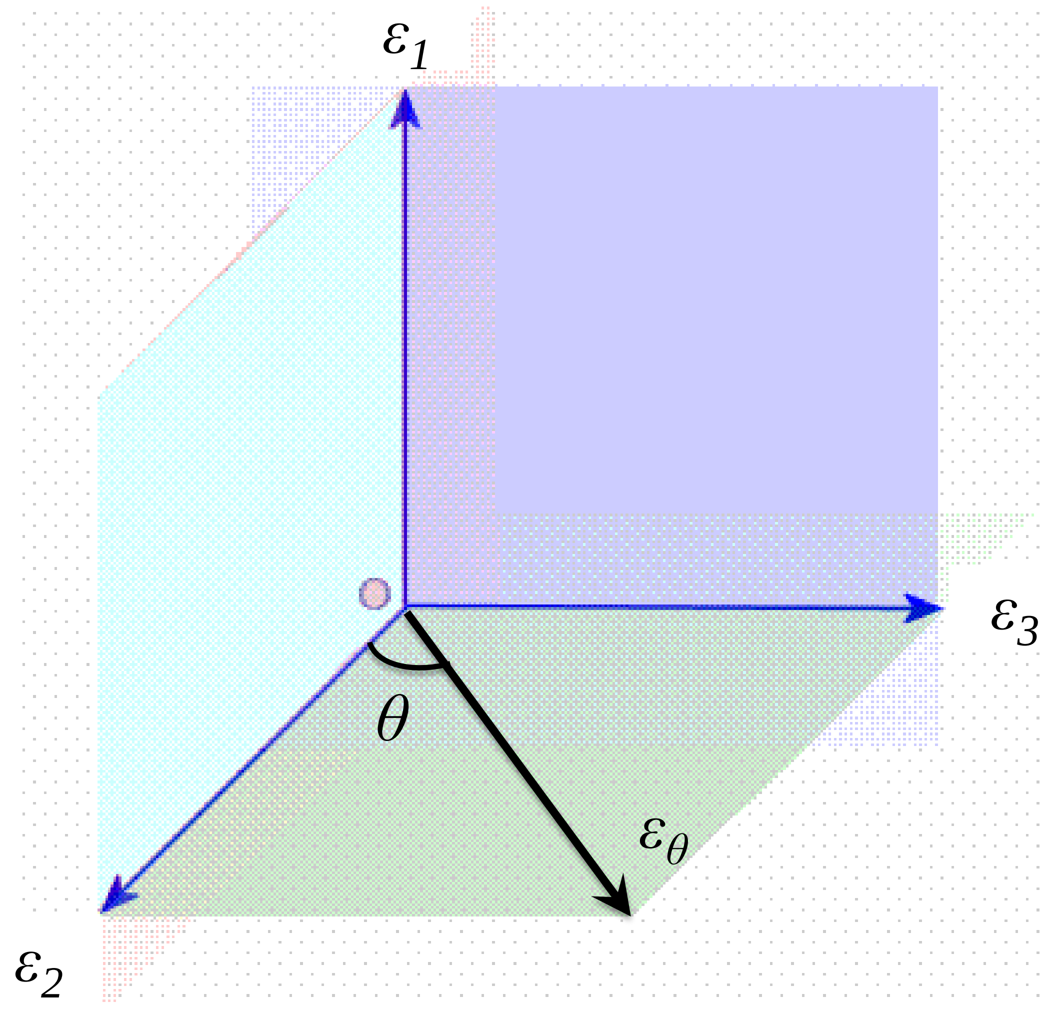
$$C_{(\varepsilon_1,\, \varepsilon_2,\, \varepsilon_3)} = \begin{bmatrix} K & C_{12} & C_{13} \\ C_{21} & \mu_1 & C_{23} \\ C_{31} & C_{32} & \mu_2 \end{bmatrix}$$

K: Bulk Modulus

μ_1, μ_2: Shear Moduli



$$\boxed{\begin{aligned} \bar{\mu} &= (\mu_{max} + \mu_{min})/2 \\ \Delta\mu &= (\mu_{max} - \mu_{min})/2 \\ \varepsilon_m &= \frac{\Delta\mu}{\bar{\mu}} \end{aligned}}$$



$$\mu_\theta = \varepsilon_\theta \,.\, C \,.\, \varepsilon_\theta$$

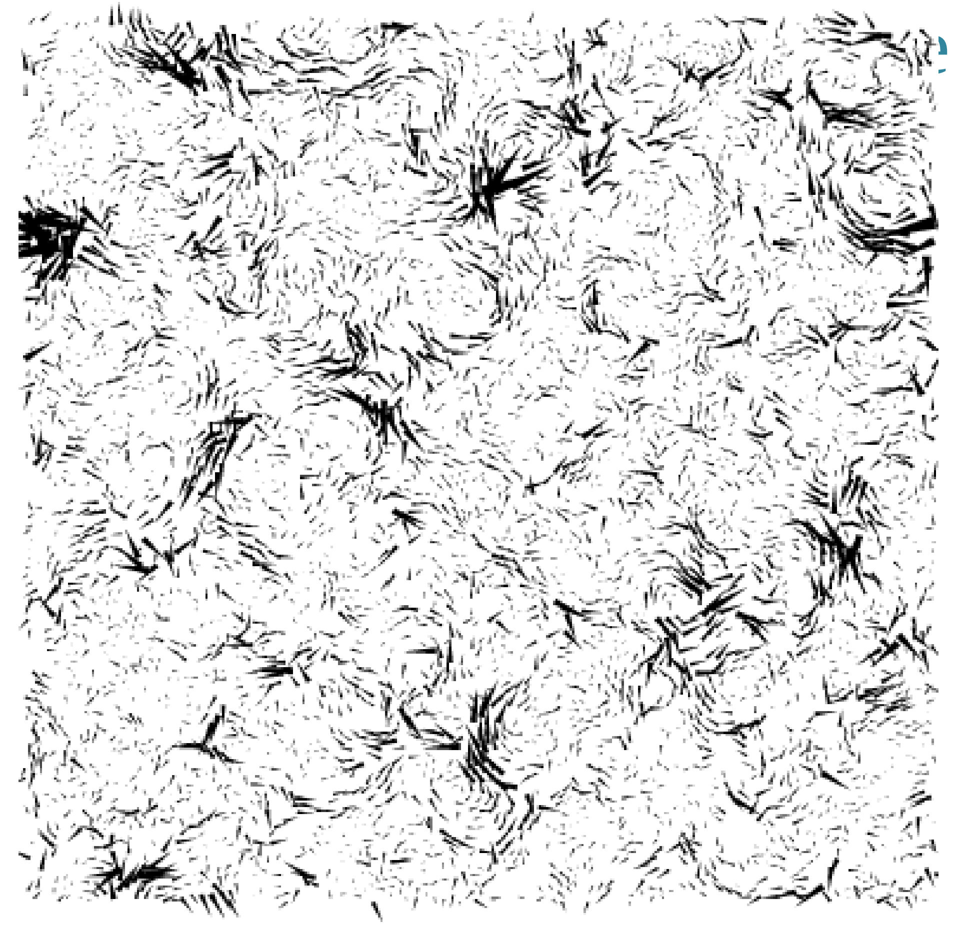
$$\boxed{\begin{aligned} &\text{Globally}: \varepsilon_m << 1 \ \text{Isotropic} \\ &\text{Locally}: \varepsilon_m \le 1 \quad \text{Anisotropic} \end{aligned}} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}^T \begin{bmatrix} \mu_1 & C_{23} \\ C_{32} & \mu_2 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

Microscopic Elastic Moduli

Born-Huang Approximation*:*

$$C^{Born}_{\alpha\beta\gamma\delta} = \frac{1}{V}\sum_{ij}(r_{ij}c_{ij} - t_{ij})r_{ij}n^{\alpha}_{ij}n^{\beta}_{ij}n^{\gamma}_{ij}n^{\delta}_{ij}$$

$$c_{ij} = \frac{\partial^2 V}{\partial r_{ij}^2} \quad \text{and} \quad t_{ij} = \frac{\partial V}{\partial r_{ij}}$$



For The *Disrdered* Case

$$C_{\alpha\beta\gamma\delta} = C^{Born}_{\alpha\beta\gamma\delta} - \frac{1}{V}\boldsymbol{\Xi}_{\alpha\beta} . \mathbf{H}^{-1} . \boldsymbol{\Xi}_{\gamma\delta}$$

Assemble *Local Affine Force* and *Hessian* to get full matrices

$$\Xi^{\alpha}_{i,\gamma\delta} = -\sum_{j}(r_{ij}c_{ij} - t_{ij})n^{\alpha}_{ij}n^{\gamma}_{ij}n^{\delta}_{ij} \qquad H^{i}_{\alpha\beta} = \sum_{j}(c_{ij} - \frac{t_{ij}}{r_{ij}})n^{\alpha}_{ij}n^{\beta}_{ij}$$

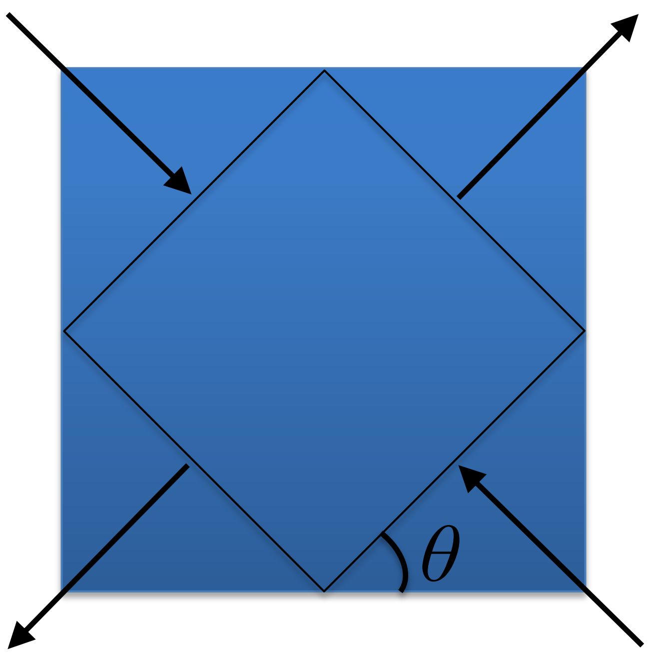
Elastic Moduli

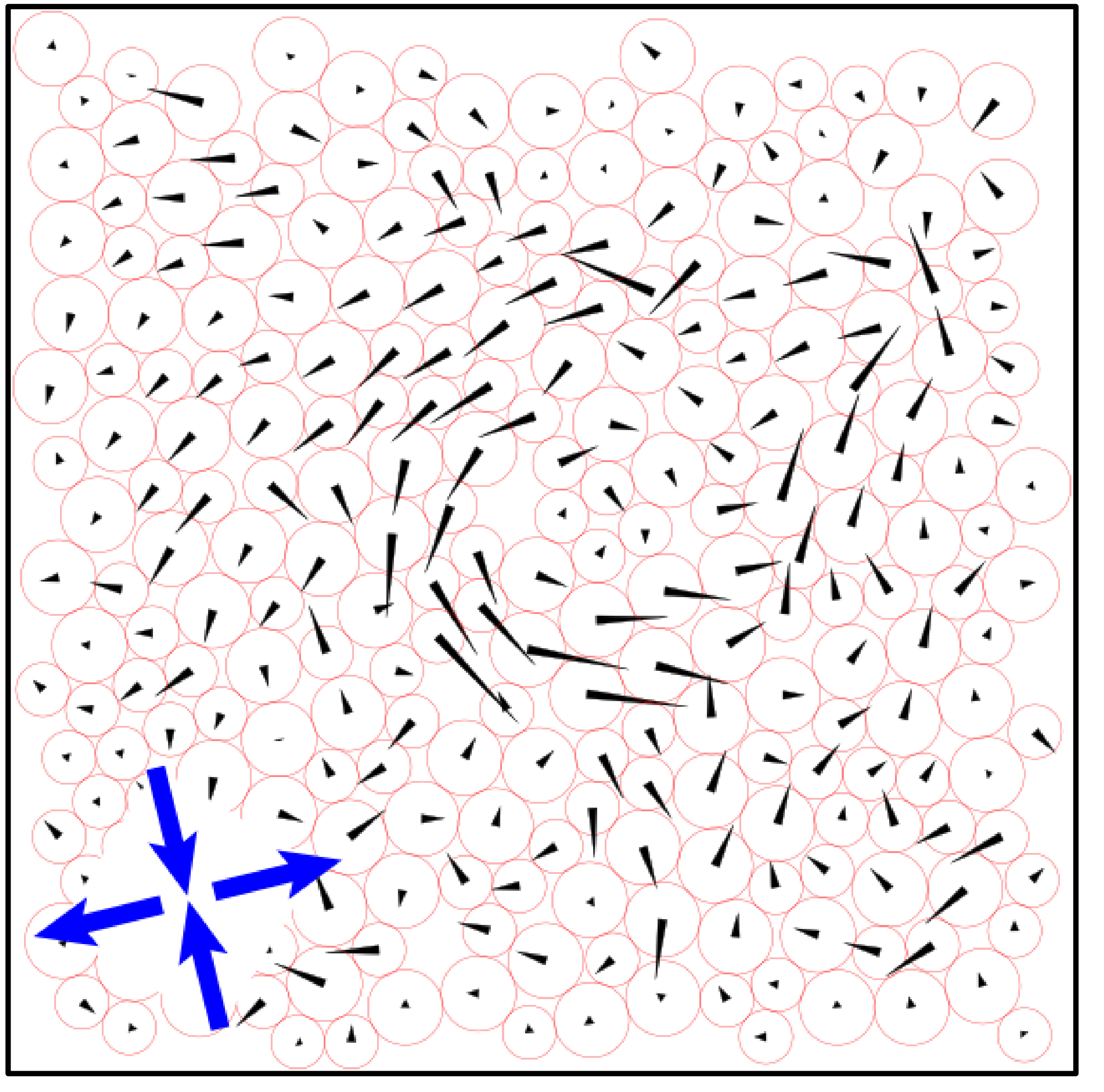
$$\mu = \varepsilon_{\alpha\beta}^{PureShear} \,.\, C_{\alpha\beta\gamma\delta} \,.\, \varepsilon_{\gamma\delta}^{PureShear}$$

$$\bar{\mu} = (\mu_{\max} + \mu_{\min})/2$$

$$\Delta\mu = (\mu_{\max} - \mu_{\min})/2$$

$$\varepsilon_m = \frac{\Delta\mu}{\bar{\mu}}$$





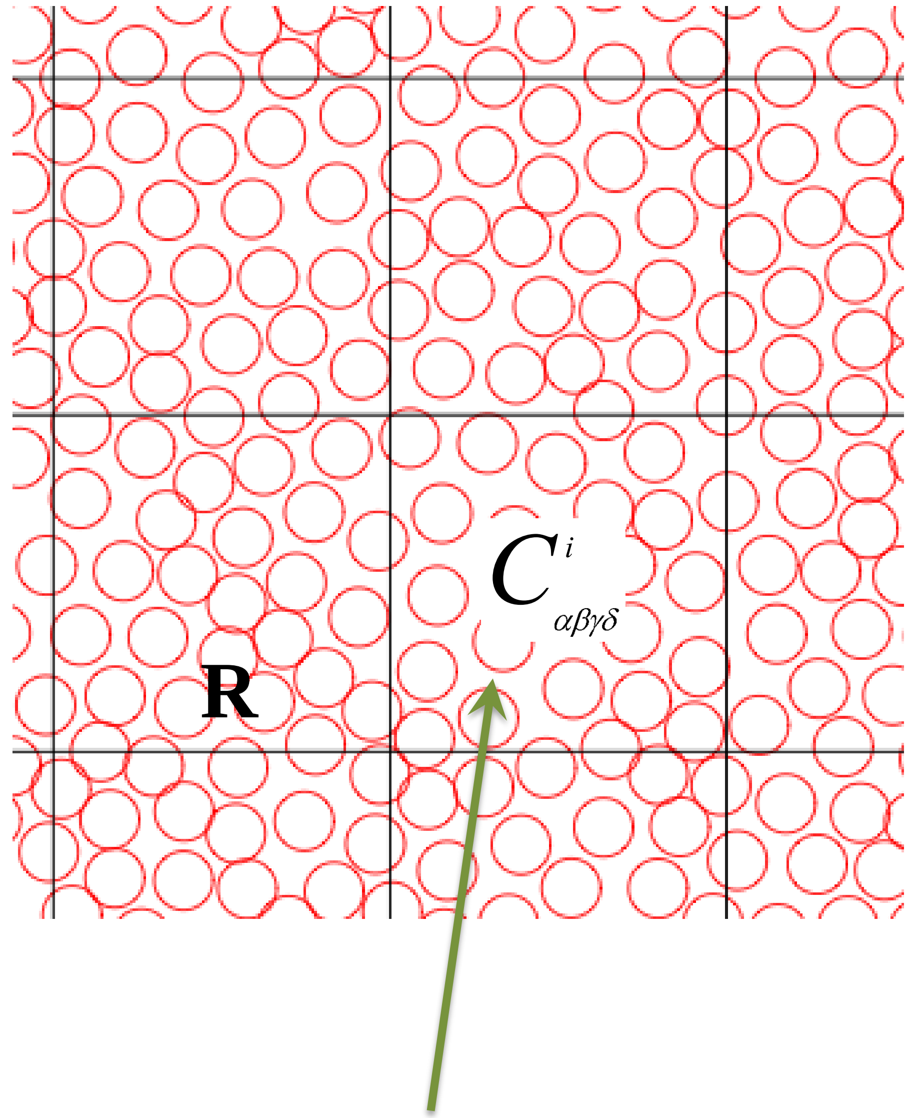
Coarse-graining

Coarse-grained Quantities

$$\bar{\varepsilon}_m = \frac{1}{N}\sum_{i=1}^{N}\varepsilon_m^i$$

N: Number Of Squares

*$C_{\alpha\beta\gamma\delta}$ and ε_m Are **NOT** Linear!*



On Each Square
Region Of Length R

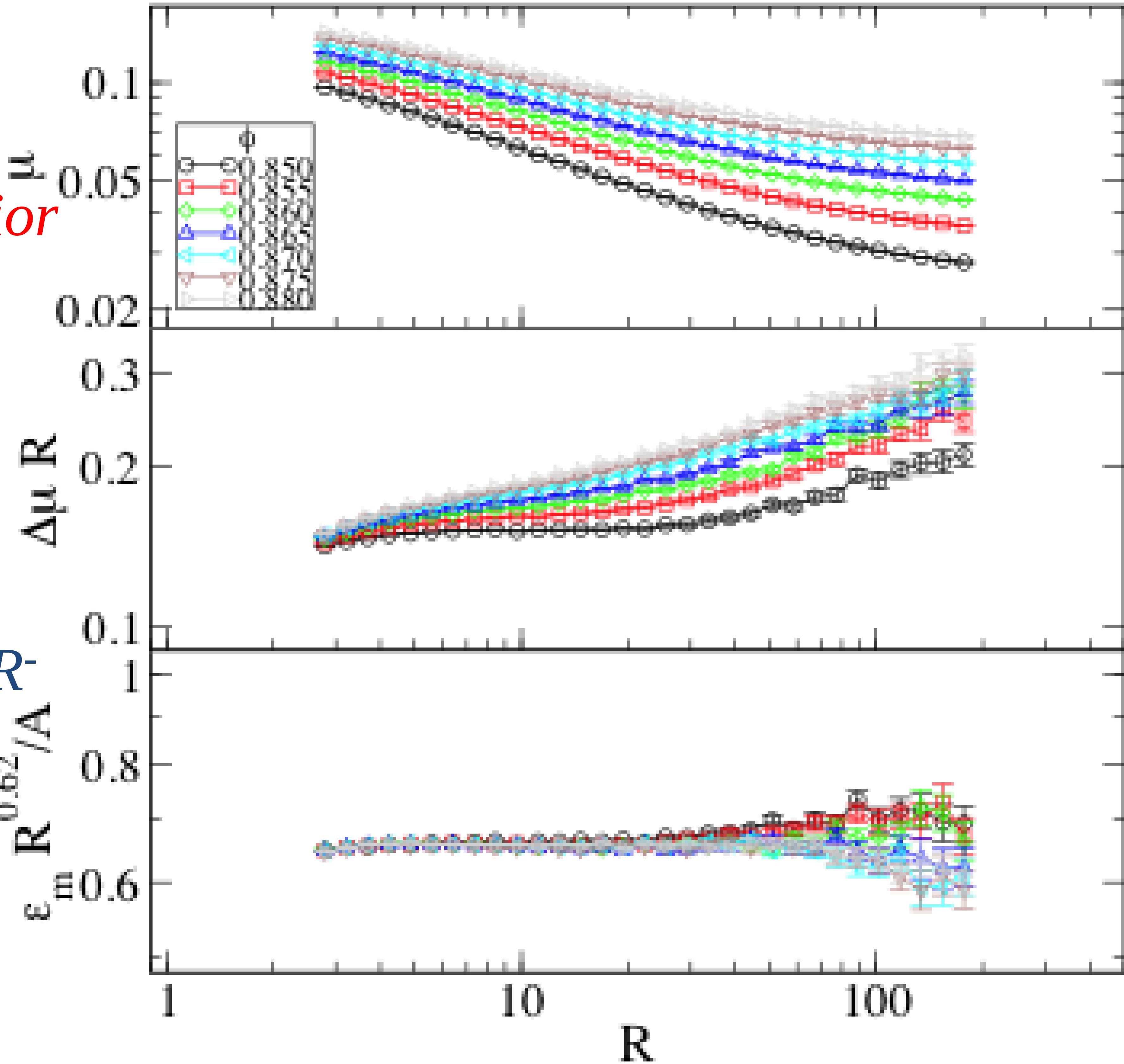
Average Moduli

- Small *R*: $\mu \sim \mu_{\mathrm{Born}}$
  Large *R*: $\mu \rightarrow \mu_g$
  No Simple *Scaling Behavior*

- Large *R*: $\delta\mu \rightarrow 0$
  Lame-like Elasticity

- *Power-Law* Regime: $\varepsilon_m \sim R^{-0.62}$

  NO Characteristic Length!



Conclusion

- **In Isotropically Prepared, Frictionless Granular Packings**
  - *Lengthscale Is exhibited By The Anisotropy In The Stress*
  - *Anisotropy In The Shear Modulus Shows No Characteristic Lengthscale*

Thank You!