Kinetic models for wave propagation in random media

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Outline

- 1. Waves in heterogeneous media
- 2. High Frequency regime and Geometrical optics
- 3. Wigner transforms
- 4. Radiative Transfer model in the weak coupling regime
- 5. More general Radiative Transfer models
- 6. Validity of Radiative Transfer Models
- 7. Applications to Detection and Imaging

Model for acoustic wave propagation

The linear system of acoustic wave equations for the pressure $p(t, \mathbf{x})$ and the velocity field $\mathbf{v}(t, \mathbf{x})$ takes the form of the following first-order hyperbolic system

$$\rho(\mathbf{x})\frac{\partial \mathbf{v}}{\partial t} + \nabla p = 0, \quad \kappa(\mathbf{x})\frac{\partial p}{\partial t} + \nabla \cdot \mathbf{v} = 0, \quad p(0, \mathbf{x}) = p_0(\mathbf{x}), \ \mathbf{v}(0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x}),$$

where $\rho(\mathbf{x}) = \rho_0$ (to simplify notation) is density and $\kappa(\mathbf{x})$ compressibility.
Energy conservation is characterized by

$$\mathcal{E}_B(t) = \frac{1}{2} \int_{\mathbb{R}^d} \left(\rho(\mathbf{x}) |\mathbf{v}|^2(t, \mathbf{x}) + \kappa(\mathbf{x}) p^2(t, \mathbf{x}) \right) d\mathbf{x} = \mathcal{E}_B(0).$$

We know that total energy is conserved. The role of a kinetic model is to describe its spatial distribution (at least asymptotically).

Another system model

Let us define $q(t, \mathbf{x}) = c^{-2}(\mathbf{x}) \frac{\partial p}{\partial t}(t, \mathbf{x})$. Then $\mathbf{u} = (p, q)$ solves the following (non-symmetric) 2×2 system

$$\frac{\partial \mathbf{u}}{\partial t} + A\mathbf{u} = 0, \qquad A = -\begin{pmatrix} 0 & c^2(\mathbf{x}) \\ \Delta & 0 \end{pmatrix},$$

with appropriate initial conditions. Note that

$$A = J \Lambda(\mathbf{x}), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Lambda(\mathbf{x}) = \begin{pmatrix} -\Delta & 0 \\ 0 & c^2(\mathbf{x}) \end{pmatrix}$$
 symmetric ,

and that energy conservation may be recast as

$$\mathcal{E}(t) = \frac{1}{2\rho_0} \int_{\mathbb{R}^d} \mathbf{u} \wedge \mathbf{u} d\mathbf{x} = \mathcal{E}(0).$$

Kinetic models associated to each acoustic equation must therefore agree and provide the same spatial energy distribution.

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High Frequency scaling

Consider the framework where the typical distance of propagation L of the waves is much larger than the typical wavelength λ in the system. We introduce the small adimensionalized parameter $\varepsilon = \frac{\lambda}{L} \ll 1$. We thus rescale space $\mathbf{x} \to \varepsilon^{-1} \mathbf{x}$ and since $l = c \times t$ rescale time accordingly $t \to \varepsilon^{-1} t$ to obtain the two model equations

$$\begin{aligned} \varepsilon^2 \frac{\partial^2 p_{\varepsilon}}{\partial t^2} &= c_{\varepsilon}^2(\mathbf{x}) \varepsilon^2 \Delta p_{\varepsilon}, \qquad p_{\varepsilon}(0, \mathbf{x}) = p_{0\varepsilon}(\varepsilon^{-1}\mathbf{x}) \\ \varepsilon \frac{\partial \mathbf{u}_{\varepsilon}}{\partial t} + A_{\varepsilon} \mathbf{u}_{\varepsilon} &= 0, \qquad A_{\varepsilon} = -\begin{pmatrix} 0 & c_{\varepsilon}^2(\mathbf{x}) \\ \varepsilon^2 \Delta & 0 \end{pmatrix}, \qquad \mathbf{u}_{\varepsilon}(0, \mathbf{x}) = \mathbf{u}_{0\varepsilon}(\varepsilon^{-1}\mathbf{x}). \end{aligned}$$

Energy conservation implies

$$\begin{split} \mathcal{E}_{H}(t) &= \frac{1}{2\rho_{0}} \int_{\mathbb{R}^{d}} \left(c_{\varepsilon}^{-2}(\mathbf{x}) \left(\varepsilon \frac{\partial p_{\varepsilon}}{\partial t} \right)^{2}(t, \mathbf{x}) + |\varepsilon \nabla p_{\varepsilon}|^{2}(t, \mathbf{x}) \right) d\mathbf{x} = \mathcal{E}_{H}(0), \\ \mathcal{E}(t) &= \frac{1}{2\rho_{0}} \int_{\mathbb{R}^{d}} \left(|\varepsilon \nabla p_{\varepsilon}|^{2}(t, \mathbf{x}) + c_{\varepsilon}^{2}(\mathbf{x})q_{\varepsilon}^{2}(t, \mathbf{x}) \right) d\mathbf{x} = \mathcal{E}(0). \end{split}$$

Geometrical optics

In the high frequency regime and for "low frequency" media, i.e. $c_{\varepsilon}(\mathbf{x}) = c(\mathbf{x})$ independent of ε , wave propagation can be approximated by looking at solutions of the form

$$p_{\varepsilon}(t,\mathbf{x}) = \left(p(t,\mathbf{x}) + \varepsilon p_{1\varepsilon}(t,\mathbf{x})\right) e^{i\frac{S(t,\mathbf{x})}{\varepsilon}}$$

Then $S(t, \mathbf{x})$ solves the eikonal equation

$$\left(\frac{\partial S}{\partial t}\right)^2 = c^2(\mathbf{x}) |\nabla_{\mathbf{x}} S|^2,$$

and $p(t, \mathbf{x})$ the transport equation

$$\frac{\partial S}{\partial t}\frac{\partial p}{\partial t} - c^{2}(\mathbf{x})\nabla_{\mathbf{x}}S \cdot \nabla_{\mathbf{x}}p + \left(\frac{\partial^{2}S}{\partial t^{2}} - c^{2}(\mathbf{x})\Delta_{\mathbf{x}}S\right)p_{0} = 0,$$
priate initial conditions so that $p_{c}(0, \mathbf{x}) = p(0, \mathbf{x})e^{i\frac{S(0, \mathbf{x})}{\varepsilon}}$

with appropriate initial conditions so that $p_{\varepsilon}(0, \mathbf{x}) = p(0, \mathbf{x})e \quad \varepsilon$.

Limitations of Geometrical optics

The eikonal equation admits a unique (physical) solution only for suffciently short times that are very small in highly heterogeneous media.

When such caustics occur, the geometrical optics decomposition needs to be generalized as a superposition of propagating fronts:

$$p_{\varepsilon}(t,\mathbf{x}) = \sum_{n=1}^{N} \left(p_0^n(t,\mathbf{x}) + \varepsilon p_{1\varepsilon}^n(t,\mathbf{x}) \right) e^{i \frac{S^n(t,\mathbf{x})}{\varepsilon}}$$

It is unclear how such decompositions can be used to model wave propagation in very heterogeneous media.

It is more natural to replace the physical description of high frequency waves (S and p_0 depend on time and space only) by a phase space description, which also accounts for the direction in which waves propagate.

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Theory of Wigner transforms (I)

[L.P. RMI-1993; G.M.M.P CPAM-1997]. Define the Wigner transform

$$W_{\varepsilon}[\psi,\phi](\mathbf{x},\mathbf{k}) = \int_{\mathbb{R}^d} e^{i\mathbf{y}\cdot\mathbf{k}}\psi(\mathbf{x}-\varepsilon\frac{\mathbf{y}}{2})\phi^*(\mathbf{x}+\varepsilon\frac{\mathbf{y}}{2})\frac{d\mathbf{y}}{(2\pi)^d}.$$

For ϕ and ψ in $L^2(\mathbb{R}^d)$, W_{ε} is bounded in $\mathcal{A}'(\mathbb{R}^{2d})$ defined as the dual of functions $\eta(\mathbf{x}, \mathbf{k})$ such that $\int_{\mathbb{R}^d} \sup_{\mathbf{x}} \|\hat{\eta}(\mathbf{x}, \mathbf{y})\| d\mathbf{y}$ is bounded. This subset of $\mathcal{S}'(\mathbb{R}^{2d})$ includes bounded measures on \mathbb{R}^{2d} . The Wigner transform has "bounded" $L^2(\mathbb{R}^{2d})$ -norm of order $\varepsilon^{-d/2}$.

For bounded sequences $\psi_{\varepsilon}, \phi_{\varepsilon}$ in $L^2(\mathbb{R}^d)$, we can extract convergent subsequences of $W_{\varepsilon}[\psi_{\varepsilon}, \phi_{\varepsilon}]$ in $\mathcal{A}'(\mathbb{R}^{2d})$. The limits of W^0 of $W_{\varepsilon}[\phi_{\varepsilon}, \phi_{\varepsilon}]$ are positive measures.

Theory of Wigner transforms (II)

Let ψ_{ε} be a (scalar) bounded family in $L^2(\mathbb{R}^d)$ which is ε -oscillatory and compact at infinity and such that the Wigner transform $W_{\varepsilon}[\psi_{\varepsilon}, \psi_{\varepsilon}]$ converges to the Wigner measure $W^0[\psi_{\varepsilon}]$. Then if $|\psi_{\varepsilon}|^2 \to \nu$ as measures on $\mathbb{R}^d_{\mathbf{x}}$, we have

$$\int_{\mathbb{R}^d} W^0[\psi_{\varepsilon}](\cdot, d\mathbf{k}) = \nu, \qquad \int_{\mathbb{R}^d} W^0[\psi_{\varepsilon}](d\mathbf{x}, d\mathbf{k}) = \overline{\lim_{\varepsilon \to 0}} \int_{\mathbb{R}^d} |\psi_{\varepsilon}|^2(\mathbf{x}) d\mathbf{x}.$$

The first equality shows that the Wigner measure may be interpreted as a probability density (energy density for classical waves) in the phase space. The second equality shows that provided that the field ψ_{ε} oscillates at the scale ε not too far from the origin, the limiting Wigner measure captures the whole probability density (energy density for classical waves).

Otherwise both equalities above are inequalities \leq .

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Equations for the Wigner transform

Consider two field equations and the Wigner transform:

 $\varepsilon \frac{\partial \mathbf{u}_{\varepsilon}^{\varphi}}{\partial t} + A_{\varepsilon}^{\varphi} \mathbf{u}_{\varepsilon}^{\varphi} = 0, \quad \varphi = 1, 2, \qquad W_{\varepsilon}(t, \mathbf{x}, \mathbf{k}) = W[\mathbf{u}_{\varepsilon}^{1}(t, \cdot), \mathbf{u}_{\varepsilon}^{2}(t, \cdot)](\mathbf{x}, \mathbf{k}).$ Here $\mathbf{u}_{\varepsilon}^{\varphi} = (p_{\varepsilon}^{\varphi}, (c_{\varepsilon}^{\varphi})^{-2}(\mathbf{x})\partial_{t}p_{\varepsilon}^{\varphi}).$ Then we verify that $\varepsilon \frac{\partial W_{\varepsilon}}{\partial t} + W[A_{\varepsilon}^{1}\mathbf{u}_{\varepsilon}^{1}, \mathbf{u}_{\varepsilon}^{2}] + W[\mathbf{u}_{\varepsilon}^{1}, A_{\varepsilon}^{2}\mathbf{u}_{\varepsilon}^{2}] = 0.$

Calculations of the type

$$W[P(\mathbf{x},\varepsilon\mathbf{D})\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}) = \int_{\mathbb{R}^{2d}} e^{-i\mathbf{y}\cdot\boldsymbol{\xi}} P(\mathbf{y},i\mathbf{k}+\frac{\varepsilon\mathbf{D}_{\mathbf{x}}}{2}) \left[e^{i\boldsymbol{\xi}\cdot\mathbf{x}} \underline{W[\mathbf{u},\mathbf{v}]}(\mathbf{x},\mathbf{k}-\frac{\varepsilon\boldsymbol{\xi}}{2})\right] \frac{d\boldsymbol{\xi}d\mathbf{x}}{(2\pi)^{d}}$$
$$W[V(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}) = \int_{\mathbb{R}^{2d}} e^{i\frac{\mathbf{x}\cdot\mathbf{p}}{\varepsilon}} e^{i\mathbf{x}\cdot\mathbf{q}} \widehat{V}(\mathbf{q},\mathbf{p}) \underline{W[\mathbf{u},\mathbf{v}]}(\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}-\frac{\varepsilon\mathbf{q}}{2}) \frac{d\mathbf{p}d\mathbf{q}}{(2\pi)^{2d}},$$

allow us to obtain an explicit equation for W_{ε} . The above formulas are amenable to asymptotic expansions in ε .

Weak-Coupling Regime

In the weak coupling regime, the <u>random fluctuations</u> of the media are modeled by

$$(c_{\varepsilon}^{\varphi})^{2}(\mathbf{x}) = c_{0}^{2} - \sqrt{\varepsilon}V^{\varphi}(\frac{\mathbf{x}}{\varepsilon}), \quad \varphi = 1, 2.$$

where c_0 is the background speed assumed to be constant to simplify. We consider two random media $V^{\varphi}(\varepsilon^{-1}\mathbf{x})$, $\varphi = 1,2$ and fields propagating in these media, i.e., solving

$$\varepsilon \frac{\partial \mathbf{u}_{\varepsilon}^{\varphi}}{\partial t} + A_{\varepsilon}^{\varphi} \mathbf{u}_{\varepsilon}^{\varphi} = 0, \quad A_{\varepsilon}^{\varphi} = -\begin{pmatrix} 0 & c_0^2 \\ p(\varepsilon \mathbf{D}) & 0 \end{pmatrix} + \sqrt{\varepsilon} V^{\varphi} (\frac{\mathbf{x}}{\varepsilon}) K, \ K = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

 $p(\mathbf{D}) = -\Delta$ for the wave equation. $V^{\varphi}(\mathbf{x})$ for $\varphi = 1, 2$ is a statistically homogeneous mean-zero random field with correlation function and power spectra:

$$c_0^4 R^{\varphi \psi}(\mathbf{x}) = \langle V^{\varphi}(\mathbf{y}) V^{\psi}(\mathbf{y} + \mathbf{x}) \rangle, \qquad 1 \le \varphi, \psi \le 2,$$

$$(2\pi)^d c_0^4 \widehat{R}^{\varphi \psi}(\mathbf{p}) \delta(\mathbf{p} + \mathbf{q}) = \langle \widehat{V}^{\varphi}(\mathbf{p}) \widehat{V}^{\psi}(\mathbf{q}) \rangle.$$

Equation for the Wigner Transform

Recalling that

$$W_{\varepsilon}(t,\mathbf{x},\mathbf{k}) = W[\mathbf{u}_{\varepsilon}^{1}(t,\cdot),\mathbf{u}_{\varepsilon}^{2}(t,\cdot)](\mathbf{x},\mathbf{k})$$

and that

$$\varepsilon \frac{\partial W_{\varepsilon}}{\partial t} + W[A_{\varepsilon}^{1}\mathbf{u}_{\varepsilon}^{1},\mathbf{u}_{\varepsilon}^{2}] + W[\mathbf{u}_{\varepsilon}^{1},A_{\varepsilon}^{2}\mathbf{u}_{\varepsilon}^{2}] = 0,$$

we obtain after (simple) pseudo-differential calculus that W_{ε} solves the following equation:

$$\varepsilon \frac{\partial W_{\varepsilon}}{\partial t} + P(i\mathbf{k} + \frac{\varepsilon \mathbf{D}}{2})W_{\varepsilon} + W_{\varepsilon}P^{*}(i\mathbf{k} - \frac{\varepsilon \mathbf{D}}{2}) + \sqrt{\varepsilon} \left(\mathcal{K}_{\varepsilon}^{1}KW_{\varepsilon} + \mathcal{K}_{\varepsilon}^{2*}W_{\varepsilon}K^{*}\right) = 0,$$

$$P(i\mathbf{k} + \frac{\varepsilon \mathbf{D}}{2}) = - \begin{pmatrix} 0 & c_{0}^{2} \\ p(i\mathbf{k} + \frac{\varepsilon \mathbf{D}}{2}) & 0 \end{pmatrix}, \quad \mathcal{K}_{\varepsilon}^{\varphi}W = \int_{\mathbb{R}^{d}} e^{i\frac{\mathbf{x}\cdot\mathbf{p}}{\varepsilon}}\hat{V}^{\varphi}(\mathbf{p})W(\mathbf{k} - \frac{\mathbf{p}}{2})\frac{d\mathbf{p}}{(2\pi)^{d}}.$$

Multiple scale expansion

Because of the presence of a highly-oscillatory phase $\exp(i(\mathbf{x}/\varepsilon) \cdot \mathbf{k})$ in the operator $\mathcal{K}_{\varepsilon}^{\varphi}$, direct asymptotic expansions on W_{ε} do not provide the correct limit. Instead we introduce the following two-scale version of W_{ε} :

$$W_{\varepsilon}(t,\mathbf{x},\mathbf{k}) = W_{\varepsilon}(t,\mathbf{x},\frac{\mathbf{x}}{\varepsilon},\mathbf{k}),$$

and using that $D \to D_x + \varepsilon^{-1} D_y$, find the equation

$$\varepsilon \frac{\partial W_{\varepsilon}}{\partial t} + P(i\mathbf{k} + \frac{\mathbf{D}_{\mathbf{y}}}{2} + \frac{\varepsilon \mathbf{D}_{\mathbf{x}}}{2})W_{\varepsilon} + W_{\varepsilon}P^{*}(i\mathbf{k} - \frac{\mathbf{D}_{\mathbf{y}}}{2} - \frac{\varepsilon \mathbf{D}_{\mathbf{x}}}{2}) + \sqrt{\varepsilon} \Big(\mathcal{K}^{1}KW_{\varepsilon} + \mathcal{K}^{2*}W_{\varepsilon}K^{*}\Big) = 0,$$
$$\mathcal{K}^{\varphi}W = \int_{\mathbb{R}^{d}} e^{i\mathbf{y}\cdot\mathbf{p}} \hat{V}^{\varphi}(\mathbf{p})W(\mathbf{k} - \frac{\mathbf{p}}{2})\frac{d\mathbf{p}}{(2\pi)^{d}}.$$

Radiative Transfer Equations

We find that $W_0 = a_+ \mathbf{b}_+ \mathbf{b}_+^* + a_- \mathbf{b}_- \mathbf{b}_-^*$, for \mathbf{b}_{\pm} and \mathbf{c}_{\pm} eigenvectors of the dispersion relation matrix, and a_+ solves the following RTE:

$$\frac{\partial a_{+}}{\partial t} - \nabla_{\mathbf{k}}\omega_{+} \cdot \nabla_{\mathbf{x}}a_{+} + (\Sigma(\mathbf{k}) + i\Pi(\mathbf{k}))a_{+}$$

=
$$\int_{\mathbb{R}^{d}} \sigma(\mathbf{k}, \mathbf{q})a_{+}(\mathbf{q})\delta(\omega_{+}(\mathbf{q}) - \omega_{+}(\mathbf{k}))d\mathbf{q}.$$

with

$$\begin{split} \Sigma(\mathbf{k}) &= \frac{\pi \omega_{\pm}^{2}(\mathbf{k})}{2(2\pi)^{d}} \int_{\mathbb{R}^{d}} \frac{\hat{R}^{11} + \hat{R}^{22}}{2} (\mathbf{k} - \mathbf{q}) \delta \left(\omega_{\pm}(\mathbf{q}) - \omega_{\pm}(\mathbf{k}) \right) d\mathbf{q}, \\ i \Pi(\mathbf{k}) &= \frac{1}{4(2\pi)^{d}} \text{ p.v.} \int_{\mathbb{R}^{d}} (\hat{R}^{11} - \hat{R}^{22}) (\mathbf{k} - \mathbf{q}) \sum_{i=\pm} \frac{\lambda_{\pm}(\mathbf{k}) \lambda_{i}(\mathbf{q})}{\lambda_{\pm}(\mathbf{k}) - \lambda_{i}(\mathbf{q})} d\mathbf{q}, \\ \sigma(\mathbf{k}, \mathbf{q}) &= \frac{\pi \omega_{\pm}^{2}(\mathbf{k})}{2(2\pi)^{d}} \hat{R}^{12} (\mathbf{k} - \mathbf{q}). \end{split}$$

Here $\omega_{+}(\mathbf{k}) = i\lambda_{+}(i\mathbf{k}) = -c_{0}q_{0}(i\mathbf{k}) = -c_{0}|\mathbf{k}|$ for the wave equation.

Rigorous derivations of radiative transfer

Theorem[Erdös-Yau-2000]. Consider the Schrödinger equation in the weak-coupling regime: $i\varepsilon \frac{\partial \psi_{\varepsilon}}{\partial t} + \frac{\varepsilon^2}{2} \Delta \psi_{\varepsilon} - \sqrt{\varepsilon} V(\frac{\mathbf{x}}{\varepsilon}) \psi_{\varepsilon} = 0$, with smooth WKB-type initial conditions in dimension $d \geq 2$, and where $V(\mathbf{x})$ is a mean-zero real Gaussian field with smooth power spectrum $\hat{R}(\mathbf{p})$. Then $\mathbb{E}\{W_{\varepsilon}(t, \mathbf{x}, \mathbf{k})\}$, the expectation of the Wigner transform of ψ_{ε} converges weakly in $\mathcal{S}'(\mathbb{R}^{2d})$ to the solution of the kinetic equation

$$\frac{\partial W}{\partial t} + \mathbf{k} \cdot \nabla_{\mathbf{x}} W = 2\pi \int_{\mathbb{R}^d} \widehat{R}(\mathbf{k} - \mathbf{q}) \Big(W(\mathbf{q}) - W(\mathbf{k}) \Big) \delta\Big(\frac{|\mathbf{k}|^2}{2} - \frac{|\mathbf{q}|^2}{2} \Big) d\mathbf{q}.$$

The proof is based on diagrammatic expansions in the Duhamel formula $\psi_{\varepsilon}(t) = e^{-iH_{\varepsilon}t}\psi_{\varepsilon}(0)$. The law of the limiting measure is not characterized. A similar result was recently obtained for (a discrete version of) the wave equation by Jani Lukkarinen and Herbert Spohn (Kinetic Limit for Wave Propagation in a Random Medium; math-ph/0505075).

Stability in the Random Liouville case

Consider the connected regime of random Liouville. **Theorem.** Let u_{ε} be a propagating mode. Then:

$$\mathbb{E}\{u_{\varepsilon}(t,\mathbf{x},\mathbf{k})\} \rightarrow F(t,\mathbf{x},\mathbf{k}) \text{ weakly as } \delta(\varepsilon) \rightarrow 0,$$

where F satisfies the following Fokker-Planck equation

$$\frac{\partial F}{\partial t} + c_0 \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} F - \mathcal{L}F = 0,$$

$$\mathcal{L}F(\mathbf{k}) = \sum_{p,q=1}^d |\mathbf{k}|^2 D_{p,q}(\hat{\mathbf{k}}) \partial_{k_p,k_q}^2 F(\mathbf{k}) + \sum_{p=1}^d |\mathbf{k}| E_p(\hat{\mathbf{k}}) \partial_{k_p} F(\mathbf{k}).$$

Moreover, we obtain the stability result

$$\mathbb{E}\left\{\int \left|\left\langle u_{\varepsilon}(T,\mathbf{x}_{0},\mathbf{k})-F(T,\mathbf{x}_{0},\mathbf{k}),\lambda(\mathbf{k})\right\rangle\right|^{2}d\mathbf{x}_{0}\right\}\to0\quad\text{ as }\quad\delta(\varepsilon)\to0.$$

So u_{ε} converges in probability to the deterministic solution F.

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Spatio-temporal Wigner transform

To handle more general differential or pseudo-differential opeators in the time variable, we introduce the *spatio-temporal* Wigner transform

$$W[\mathbf{u}, \mathbf{v}](t, \omega, \mathbf{x}, \mathbf{k}) = \int_{\mathbb{R}^{d+1}} e^{i\mathbf{k}\cdot\mathbf{y}+i\tau\omega} \mathbf{u}(t-\frac{\varepsilon\tau}{2}, \mathbf{x}-\frac{\varepsilon\mathbf{y}}{2}) \mathbf{v}^*(t+\frac{\varepsilon\tau}{2}, \mathbf{x}+\frac{\varepsilon\mathbf{y}}{2}) \frac{d\mathbf{y}d\tau}{(2\pi)^{d+1}}.$$

Let us illustrate the use of the spatio-temporal Wigner transform by considering the following constant coefficient equation

$$R(\varepsilon D_t)\mathbf{u}_{\varepsilon}(t,\mathbf{x}) + P(\varepsilon \mathbf{D}_{\mathbf{x}})\mathbf{u}_{\varepsilon}(t,\mathbf{x}) = 0.$$

For $R(i\omega) = i\omega$, we are back to first-order equations in time. Then clearly,

$$W[R(\varepsilon D_t)\mathbf{u}_{\varepsilon},\mathbf{u}_{\varepsilon}] + W[P(\varepsilon \mathbf{D}_{\mathbf{x}})\mathbf{u}_{\varepsilon},\mathbf{u}_{\varepsilon}] = \mathbf{0}.$$

The same calculus as earlier gives for $W_{\varepsilon} = W[\mathbf{u}_{\varepsilon}, \mathbf{u}_{\varepsilon}]$ the equations

$$\left(R(i\omega + \frac{\varepsilon D_t}{2}) + P(i\mathbf{k} + \frac{\varepsilon \mathbf{D}_{\mathbf{x}}}{2}) \right) W_{\varepsilon}(t, \omega, \mathbf{x}, \mathbf{k}) = 0,$$

$$W_{\varepsilon}(t, \omega, \mathbf{x}, \mathbf{k}) \left(R^*(i\omega - \frac{\varepsilon D_t}{2}) + P^*(i\mathbf{k} - \frac{\varepsilon \mathbf{D}_{\mathbf{x}}}{2}) \right) = 0.$$

Application to discrete wave equations

Consider the wave equation with dispersive effects:

$$R(\varepsilon D_t)\mathbf{u}_{\varepsilon}^{\varphi} + A_{\varepsilon}^{\varphi}\mathbf{u}_{\varepsilon}^{\varphi} = 0, \qquad \varphi = 1, 2,$$

where $\overline{R}(i\omega) = -R(i\omega)$. For instance $i\Delta^{-1}\sin(\omega\Delta)$ corresponds to secondorder time discretization. Then the energy density (or correlation function) associated to the above field equation is still modeled by a kinetic model. The radiative transfer equation for the propagating mode a_+ is

$$\frac{\partial a_{+}}{\partial t} - \nabla_{\mathbf{k}} \omega_{+} \cdot \nabla_{\mathbf{x}} a_{+} + (\tilde{\boldsymbol{\Sigma}}(\mathbf{k}) + i \tilde{\boldsymbol{\Pi}}(\mathbf{k})) a_{+} = \int_{\mathbb{R}^{d}} \tilde{\sigma}(\mathbf{k}, \mathbf{q}) a_{+}(\mathbf{q}) \delta\left(\omega_{+}(\mathbf{q}) - \omega_{+}(\mathbf{k})\right) d\mathbf{q},$$

where the above coefficients are related those with $R(i\omega) = i\omega$ by

$$\tilde{\Sigma}(\mathbf{k}) = \frac{\Sigma(\mathbf{k})}{|R'(i\omega_{+}(\mathbf{k}))|^{2}}, \quad \tilde{\sigma}(\mathbf{k},\mathbf{q}) = \frac{\sigma(\mathbf{k},\mathbf{q})}{|R'(i\omega_{+}(\mathbf{k}))|^{2}}, \quad \tilde{\Pi}(\mathbf{k}) = \frac{\Pi(\mathbf{k})}{R'(i\omega_{+}(\mathbf{k}))}.$$

This quantifies the effects of e.g. numerical discretizations on the kinetic parameters. Scalar equations can also be treated similarly.

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Numerical validation of radiative transfer

Wave propagation in heterogeneous media may sometimes be difficult to control in real experiments. Numerical simulations offer an interesting complement to physical experiments.

In order to be relevant the simulations need to consider spatial domains that are much larger than the typical wavelength in the system. This requires us to use multi-processor architectures and parallelized codes.

We have developed such a computational tool to solve acoustic waves (easily extendible to micro-waves) in the time domain.

Details of the wave (microscopic) code.

The codes solves a discrete version (centered second-order discretization in space and time) of the following acoustic wave system of equation

$$\frac{\partial \mathbf{v}}{\partial t} + \rho^{-1}(\mathbf{x})\nabla p = 0,$$
$$\frac{\partial p}{\partial t} + \kappa^{-1}(\mathbf{x})\nabla \cdot \mathbf{v} = 0.$$

The domain is surrounded by a perfectly matched layer (PML) method so that outgoing waves are not reflected at the domain boundary. The (random) physical coefficients $\rho(\mathbf{x})$ and $\kappa(\mathbf{x})$ are carefully chosen to verify prescribed statistical properties.

The FDFT (Finite difference forward in time) method has been parallelized by using the software PETSc developed at Argonne. Forward calculations for T = 1500 (typical times necessary to validate the diffusive model; for $\lambda = 1$ and average sound speed $c_0 = 1$) require 3-4 days of calculations.

Details of the macroscopic codes.

In both the direct and time reversal measurements, the data are the macroscopic energy densities

$$\mathcal{E}(t,\mathbf{x}) = \frac{1}{2} \left(\rho(\mathbf{x}) |\mathbf{v}|^2(t,\mathbf{x}) + \kappa(\mathbf{x}) p^2(t,\mathbf{x}) \right).$$

We consider two macroscopic models for \mathcal{E} : a radiative transfer equation and a diffusion equation. The radiative transfer equation is solved by a Monte Carlo method (requiring in excess of 50*M* particles to achieve a reasonable accuracy even with good variance reduction technique conditioning particles on hitting the inclusion). The diffusion approximation to transport is solved by the finite element method.

A typical configuration for the wave solver



The domain size is roughly $20,000 \times 10,000 = 200M$ nodes

Wave-Transport-diffusion comparison



Experiment with isotropic scattering ($\hat{R} \equiv 1$ for this frequency; the source term is a localized Bessel function). The best transport fit is obtained for $\Sigma_{num}^{-1} = 88.5$ versus $\Sigma_{th}^{-1} = 83.00$. The best fit for the diffusion coefficient and the extrapolation length are $D_{num} = 43.2$ and $L_{ex} = 0.80$ versus $D_{\rm th} = (2\Sigma)^{-1} = 41.5$ and $L_{th} = 0.81$.

Averaged energy densities on detector as a function of time.

Effect of void inclusion



Right, radius of 50. Transport and diffusion generated by best energy fit. The diffusion fit is valid only for very long times, whereas transport performs extremely well.

Effect of increased randomness



and diffusion generated by best energy fit. The diffusion fit is now much more accurate.

Effect of perfectly reflecting inclusion



Correction generated by a perfectly reflecting inclusion (specular reflection for transport and Neumann conditions for diffusion). Left, radius of 30. Right, radius of 40. Transport and diffusion generated by best energy fit. Still very good agreement between wave and transport simulations.

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Modeling the inclusion

The detection and imaging of buried inclusions (which are large compared to the wavelength) is done as follows. We model the inclusion as a variation in the kinetic parameters of the radiative transfer equation that models the wave energy density.

The objective is to reconstruct these kinetic parameters from wave energy measurements at the boundary of a domain. This is severely ill-posed problem (in the sense that the reconstruction amplifies noise drastically). Because the inclusion is assumed to be of small volume (at the macro-scopic scale), further assumptions are possible. We consider asymptotics in the volume of the inclusion, which take the form

$$\delta a^{0}(t,\mathbf{x},\mathbf{k}) = -|B| \int_{0}^{t} G(t-s,\mathbf{x},\mathbf{x}_{b},\mathbf{k}) (Qa^{0})(s,\mathbf{x}_{b},\mathbf{k}) ds + 1.0.t.,$$

where a^0 is the unperturbed solution, G the transport Green's function, Q the scattering operator and $|B| \sim R^d$ the inclusion's volume.

Reconstruction of the inclusion

Detection and imaging based on the above asymptotic expansions allow us obtain the inclusion's location and volume:



$\sigma_{\rm n}/a_{\rm 0}$	error on R (%)	error on x_b	error on y_b
0.25%	12	9.0	3.5
0.5%	25	15	5.0
1%	33	30	10

Very accurate data are required to locate and estimate the inclusion.

Imaging based on correlations of fields

In the diffusive regime, the perturbation caused by a void inclusion is given approximately by

$$\delta u^{D}(t,\mathbf{x}) = d\pi D_{0} \mathbf{R}^{d} \int_{0}^{t} \nabla_{\mathbf{x}} u_{0}(t-s,\mathbf{x}_{b}) \cdot \nabla_{\mathbf{x}_{b}} G(s,\mathbf{x},\mathbf{x}_{b}) ds.$$

Here d is dimension and $G(s, \mathbf{x}, \mathbf{x}_b)$ the background Green's function.

When we have access to the measured wave field **both** in the presence and in the absence of the inclusion, we can consider the correlation of the two fields. In the diffusive regime, the corresponding perturbation is given by

$$\delta u(t, \mathbf{x}) = -4\pi R \int_{0}^{t} u_{0}(t - s, \mathbf{x}_{b})G(s, \mathbf{x}, \mathbf{x}_{b})ds + o(R), \qquad d = 3$$

$$\delta u(t, \mathbf{x}) = \frac{2\pi}{\ln R} \int_{0}^{t} u_{0}(t - s, \mathbf{x}_{b})G(s, \mathbf{x}, \mathbf{x}_{b})ds + o(\frac{1}{|\ln R|}), \qquad d = 2.$$

Since $O(R) \gg O(R^3)$ in d = 3 and $O(|\ln R|^{-1}) \gg O(R^2)$ in d = 2, it is much easier to detect and image in the presence of differential information.

Conclusions

We have a theory to express the high frequency limit of the correlation of two fields using a Wigner transform. The fields may also propagate in different environments.

In certain cases, we can rigorously characterize the high frequency limit of the Wigner transform and obtain its statistical stability. This has been done for the parabolic approximation and the Itô Schrödinger approximation, and in the random Liouville regime.

Radiative transfer was shown to be quite accurate numerically to model wave propagation in (certain) random media.

Wave propagation is (sometimes) sufficiently stable so that inverse problems based on transport equation can successfully be solved to detect and image buried inclusions. *Differential data* allow us to detect and image much *smaller* objects.