High frequency behaviour of the Helmholtz equation : radiation condition at infinity, bounds, and a counter-example.

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I- Introduction

• Helmholtz equation with **source term**, and **slowly varying refraction index** (scale ε)

$$i\varepsilon\alpha_\varepsilon w^\varepsilon(x) + \Delta_x w^\varepsilon(x) + n^2(\varepsilon x)\, w^\varepsilon(x) = S(x).$$

• $\alpha_\varepsilon > 0$: small **positive** absorption coefficient,
e.g. $\alpha_\varepsilon = \varepsilon^{1000}$, or even $\alpha_\varepsilon = 0^+$.
Makes the Helmholtz equation invertible, or specifies a **radiation condition at infinity** (see below).

$S(x)$ is a given source term (regularity to be precised later)
$n(x)$ is the refraction index, $C^\infty(\mathbb{R}^d)$ in any case.
$0 < n_1 \leq n(x) \leq n_2 < \infty$ (may be relaxed)
The unknown is $w^\varepsilon \in \mathbb{C}$, and $d \geq 3$.

• Corresponds to harmonic solutions of the **damped** wave equation, where the time variable has disappeared ("infinite time").

- **Question 1 :** Uniform **bounds** on w^ε ?

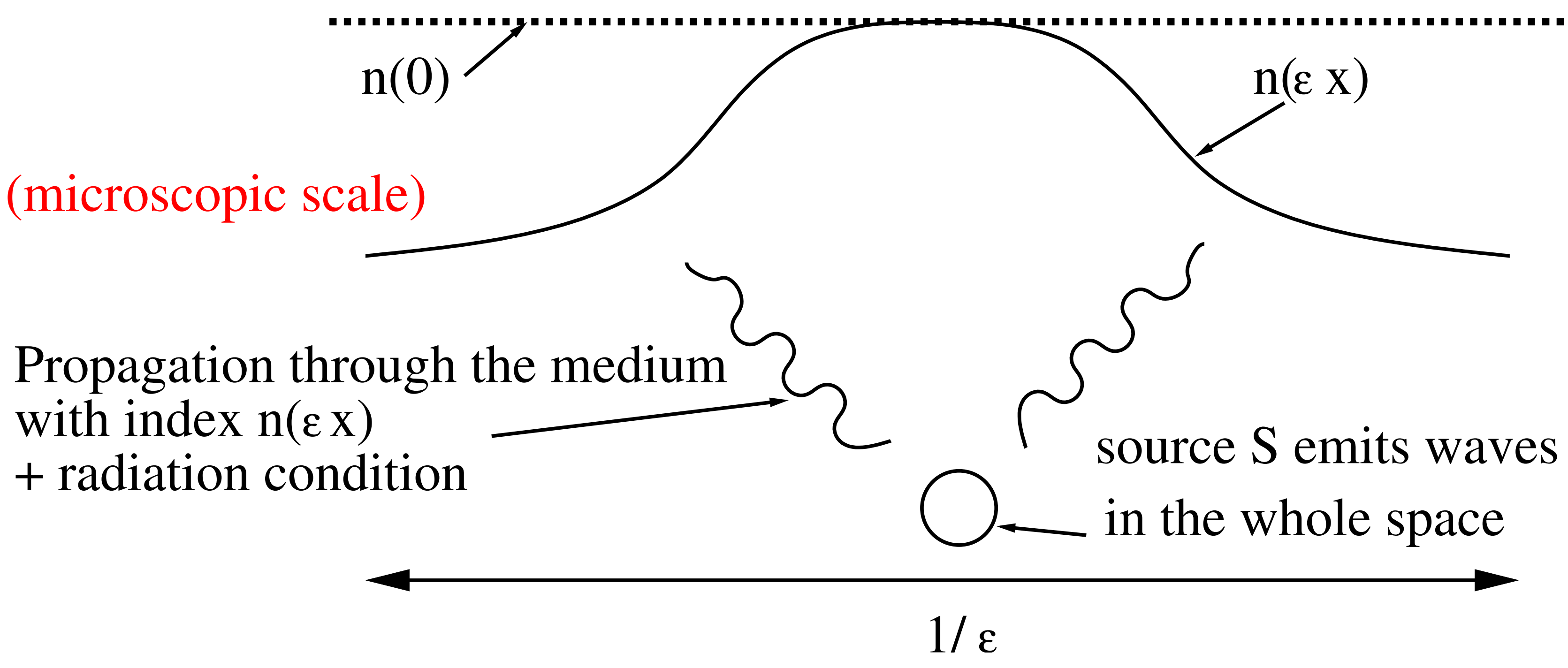
- **Question 2 :** Does w^ε, solution to

$$i\varepsilon\alpha_\varepsilon w^\varepsilon(x) + \Delta_x w^\varepsilon(x) + n^2(\varepsilon x)\, w^\varepsilon(x) = S(x).$$

go to the **outgoing solution** w^{out}, solution to

$$i0^+ w^{\text{out}}(x) + \Delta_x w^{\text{out}}(x) + n^2(0)\, w^{\text{out}} = S(x) \;?$$

(Rmk : obviously OK if $n = \text{const} = n_\infty = n(0)$).



• **Motivation :** Through the L^2 unitary scaling $u^\varepsilon(x) := \varepsilon^{-d/2} w^\varepsilon(x/\varepsilon)$, the analysis of w^ε is related to the high-frequency Helmholtz equation

$$i\varepsilon\alpha_\varepsilon u^\varepsilon(x) + \varepsilon^2 \Delta_x u^\varepsilon(x) + n^2(x) u^\varepsilon(x) = \frac{1}{\varepsilon^{d/2}} S\left(\frac{x}{\varepsilon}\right).$$

The analysis of the propagation of the **semiclassical measure** of u^ε (related to $|u^\varepsilon|^2$), has been performed by J.D. Benamou, F.C., Th. Katsaounis, B. Perthame, **under the conjecture that the wave** w^ε (amplitude + phase) **goes to** w^{out}. Typically

$$\underbrace{0^+ f(x,\xi)}_{\text{radiation cdt}} + \underbrace{2\xi . \nabla_x f + \nabla_x n^2(x) \cdot \nabla_\xi f}_{\text{transport}} = |\widehat{S}(\xi)|^2 \underbrace{\delta(\xi^2 - n^2(x))}_{\text{dispersion relation}} \underbrace{\delta(x)}_{\text{source}}$$

$\rightarrow$ resonant interaction between $S(x/\varepsilon)$ and $\varepsilon^2 \Delta_x + n^2(x)$

See F.C., B. Perthame, O. Runborg and P. Zhang, X.P. Wang for more general oscillating/concentrating source terms. See E. Fouassier for the case of a discontinuous index $n(x)$ (reflection/transmission).

• **Difficulty in proving** $w^\varepsilon \to w^{\text{out}}$ **:** for constant coefficients $n(x) \equiv n_\infty$:

1 w^{out} solves $+\Delta_x w^{\text{out}}(x) + n_\infty^2\, w^{\text{out}}(x) = S\,(x)\,,$

with the Sommerfeld radiation condition : as $|x| \to \infty$, we impose

$$\frac{x}{|x|} \cdot \nabla_x w^{\text{out}}(x) + i n_\infty\, w^{\text{out}}(x) \to 0. \quad \Big(w^{\text{out}} \underset{|x|\to\infty}{\sim} \frac{\exp\left(-i n_\infty |x|\right)}{|x|} \text{ if } d = 3\Big).$$

2 $$\widehat{w^{\text{out}}}(\xi) = \frac{\widehat{S}(\xi)}{-\xi^2 + n_\infty^2 + i0^+}, \quad \text{where } \frac{1}{x + i0^+} = \text{pv}\left(\frac{1}{x}\right) + i\pi\delta(x), \text{ in } \mathcal{D}'(\mathbb{R}).$$

3 $$w^{\text{out}}(x) = i \int_0^{+\infty} \exp\left(it[\Delta_x + n_\infty^2]\right)\, S(x)\, dt,$$

is a quantity that is being propagated over positive times. Here,

$$\left[i\partial_t - \Delta_x - n_\infty^2\right]\, \left(e^{it[\Delta_x + n_\infty^2]}\, S(x)\right) = 0, \qquad e^{it[\Delta_x + n_\infty^2]}\, S(x)\Big|_{t=0} = S(x).$$

I- Introduction (6)

1, 2 or **3** $\rightarrow$ strong **nonlocal effects** induced by the "$i0^+$". It "brings back information" from infinity. If $n(0) \neq n_\infty = \lim_{|x|\rightarrow\infty} n(x)$, we have to explain why (say when $d = 3$ and $\alpha_\varepsilon = 0^+$)

$$w^{\text{out}} \underset{|x|\rightarrow\infty}{\sim} \exp\left(-in(0)|x|\right)/|x|$$

while

$$w^{\varepsilon} \underset{|x|\rightarrow\infty}{\sim} \exp\left(-in_\infty|x|\right)/|x|.$$

II- Bounds on w^ε and/or u^ε

• **When** <u>$n(x) =$ const</u>, standard **weighted** L^2 **bound** $(\forall \delta > 0)$

$$\left(\Delta_x + n_\infty^2 + i0^+\right)^{-1} \text{ maps } L^2(\,(1+|x|)^{+1+\delta} dx\,). \text{ to } L^2(\,(1+|x|)^{-1-\delta} dx\,).$$

Also a limiting, **optimal**, **homogeneous** version if $\delta = 0 \to B$-B^* **spaces** (Agmon-Hörmander).

• **When** <u>$n(x) \neq$ const</u>, similar **weighted** L^2 **bound** for the resolvents $\left(\Delta_x + n(x) + i0^+\right)^{-1}$, and $\left(\varepsilon^2 \Delta_x + n(x) + i0^+\right)^{-1}$, **provided** the index $n(x)$ **(i) is smooth**, **(ii) goes to a constant at infinity**, and **(iii) has no trapped rays** (Mourre, Burq, Wang, Gérard-Martinez, ...)

Optimal, homogeneous, B-B^* **estimate** (limiting case $\delta = 0$, independence on the frequency ε) obtained by Perthame-Vega, provided the rays of geometric optics go **monotonically to infinity** (stronger than (iii)).

• <u>Discontinuous $n(x)$</u> : Eidus, Bo Zhang, Fouassier.

II- Bounds (2)

Assumptions (valid throughout this talk)

• $n(x)$ is **constant at infinity** : $n(x) = n_\infty + O\left(1/|x|^\rho\right), \ (\rho > 0),$

as well as $\forall \beta \in \mathbf{N}^d,\ \partial_x^\beta n(x) = O\left(1/|x|^{\rho+|\beta|}\right).$

• The rays of geometric optics are **non trapping** : for any initial datum $(x, \xi) \in \mathbb{R}^{2d}$, the solution $(X(t), \Xi(t))$ to the Hamiltonian ODE

$$\partial_t X(t) = 2\,\Xi(t)\,, \qquad X(0) = x,$$
$$\partial_t \Xi(t) = +\nabla_x n^2\left(X(t)\right)\,, \quad \Xi(0) = \xi,$$

with zero enery : $H(x, \xi) := \xi^2 - n^2(x) = 0 \quad (= H(X(t), \Xi(t))$

satisfies $|X(t)| \to +\infty$ as $t \to \pm\infty$.

Example : $n(x) = \text{const} \neq 0 \Longrightarrow X(t) = x + t\xi,\ \xi \neq 0.$

Very roughly : "$u^\varepsilon(x) \approx \int_0^{+\infty} S(X(t))\,dt$" $\Rightarrow$ need for dispersion.

II- Bounds (3)

Under these assumption

Theorem 1 (F.C. - T. Jecko)
u^ε and w^ε are bounded in the **optimal, homogeneous,** B-B^* **spaces**.

III- Asymptotic propagation of the wave w^ε

Does $i\varepsilon\alpha_\varepsilon w^\varepsilon(x) + \Delta_x w^\varepsilon(x) + n^2(\varepsilon x) w^\varepsilon(x) = S(x)$
go to $i0^+ w^{\text{out}}(x) + \Delta_x w^{\text{out}}(x) + n^2(0) w^{\text{out}}(x) = S(x)$,
provided n is constant at infinity and non-trapping at zero energy ?

Theorem 2 (F.C.) Let the rays $(X(t), \Xi(t))$ emitted from the origin $x = 0$ with zero energy $\xi^2 = n^2(x)$ satisfy the **non-focusing condition** below. Then,

$$w^\varepsilon \to w^{\text{out}} \text{ weakly } .$$

Theorem 3 (F.C.) If the above condition is violated, one may construct a situation for which $w^\varepsilon \not\to w^{\text{out}}$.

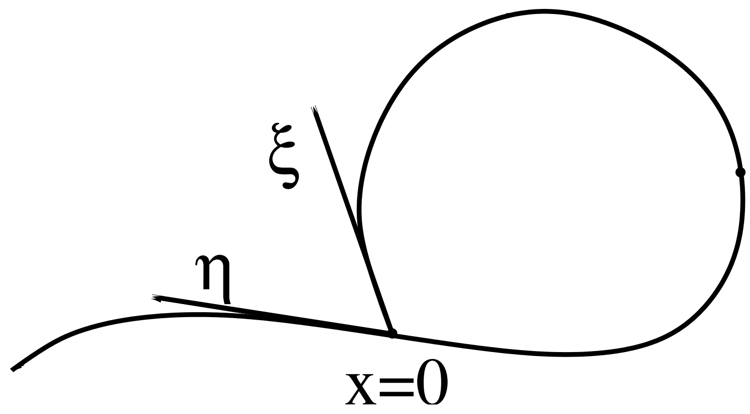
Rmk : a result similar to Thm 2 proved independently by Wang, Zhang.

Rmk : "weakly" refers here to $\lim \left\langle \left(i\varepsilon\alpha_\varepsilon + \Delta_x + n^2(\varepsilon x) \right)^{-1} S, \varphi \right\rangle$,

either when $S, \varphi \in \mathcal{S}(\mathbb{R}^d)$ (in any case),
or when $S, \varphi \in L^2((1 + |x|)^{1+\delta} dx)$,
or even in the limiting case ($\delta = 0$) when $S, \varphi \in B$
(this needs to reinforce the nontrapping condition a bit).

Non-focusing condition

The set $\left\{(\xi,\eta,t)\in\mathbb{R}^{2d}\times\mathbb{R}^*_+/\eta^2=n^2(0),\ X(t)=0,\ \Xi(t)=\eta\right\}$ *is a smooth submanifold of* $\mathbb{R}^{2d}\times\mathbb{R}^*_+$, *with a dimension $k<d-1$.*



In other words, the rays of geometric optics issued from the origin $x=0$ **do not refocus at the origin later**.
Generic condition. In any case $k\le d-1$.

Caricatural example : $n^2(x)=n^2(0)-x_1^2-\cdots-x_d^2\Rightarrow k=d-1$,
$n^2(x)=n^2(0)-\omega_1^2x_1^2-\cdots-\omega_d^2x_d^2$, $(\omega_1,\ldots,\omega_d)$ $\mathbb{Q}-$ independent $\Rightarrow k=0$.

Idea of proof : Time dependent approach. First observe, for any test function φ, $\langle w^\varepsilon, \varphi\rangle = \langle u^\varepsilon, \varphi_\varepsilon\rangle$ (defining $\varphi_\varepsilon(x) := \varepsilon^{-d/2}\varphi(x/\varepsilon)$). Second

$$\langle u^\varepsilon, \varphi_\varepsilon\rangle = \frac{i}{\varepsilon}\int_0^{+\infty} \mathrm{e}^{-\alpha_\varepsilon t} \left\langle \underbrace{\exp\left(i\frac{t}{\varepsilon}\left[\varepsilon^2\Delta_x + n^2(x)\right]\right) S_\varepsilon}_{=: \exp(it\, H_\varepsilon/\varepsilon) S_\varepsilon}, \varphi_\varepsilon \right\rangle dt\ ,$$

where $\quad \left[i\varepsilon\partial_t - \varepsilon^2\Delta_x - n^2(x)\right]\left(\mathrm{e}^{itH_\varepsilon/\varepsilon} S_\varepsilon\right) = 0, \quad \mathrm{e}^{itH_\varepsilon/\varepsilon} S_\varepsilon\Big|_{t=0} = S_\varepsilon.$

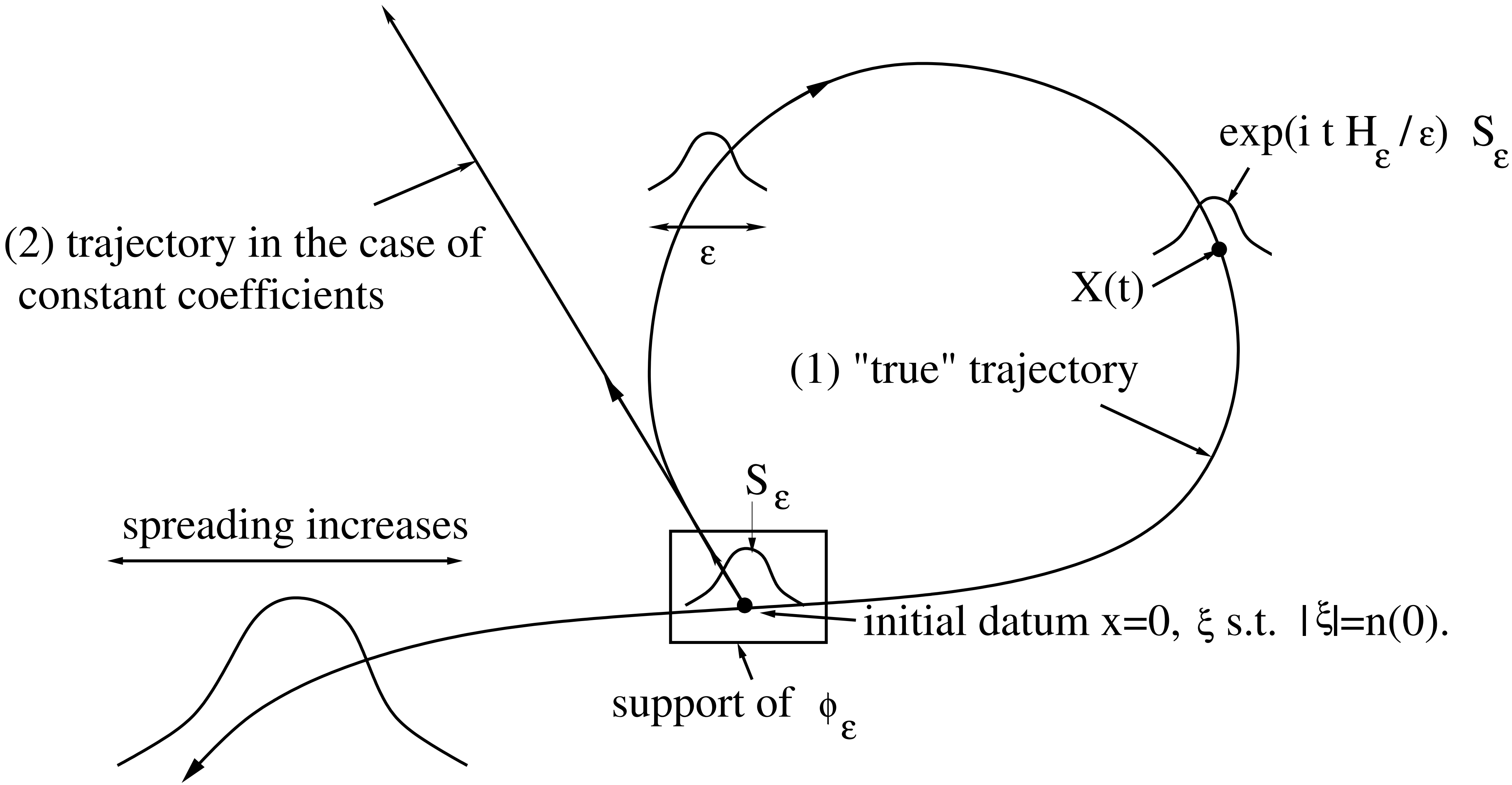
- We are thus left with the question :

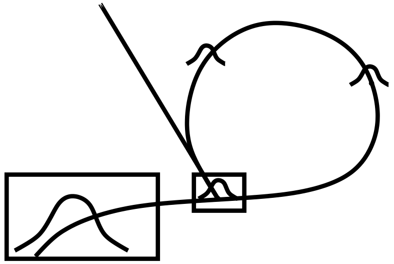
$$(1) \quad \langle w^\varepsilon, \varphi\rangle = \frac{i}{\varepsilon}\int_0^{+\infty} \mathrm{e}^{-\alpha_\varepsilon t} \left\langle \exp\left(i\frac{t}{\varepsilon}\left[\varepsilon^2\Delta_x + n^2(x)\right]\right) S_\varepsilon, \varphi_\varepsilon \right\rangle dt\ ,$$

$$\underset{\varepsilon\to 0}{\longrightarrow} (2) \quad \left\langle w^{\text{out}}, \varphi\right\rangle = \frac{i}{\varepsilon}\int_0^{+\infty} \left\langle \exp\left(i\frac{t}{\varepsilon}\left[\varepsilon^2\Delta_x + n^2(0)\right]\right) S_\varepsilon, \varphi_\varepsilon \right\rangle dt \ \ ?$$

- **Nota :** Set $H_\varepsilon = \varepsilon^2\Delta_x + n^2(x)$. Integration by parts in time shows that for **large values of** t ($\leftrightarrow$ main difficulty), the part of S_ε corresponding to **non-zero energies**, i.e. $(1-\chi)(H_\varepsilon)\, S_\varepsilon$, **does essentially not contribute** $\to$ One may replace everywhere S_ε by $\chi(H_\varepsilon) S_\varepsilon$.

III- Asymptotic propagation of w^ε (4)





Large times :

• A result by X.P. Wang gives, with $H_\varepsilon = \varepsilon^2\Delta_x + n^2(x)$,

$$\left\| \langle x\rangle^{-s} \exp\left(i\frac{t}{\varepsilon} H_\varepsilon\right) \chi(H_\varepsilon)\langle x\rangle^{-s} \right\|_{\mathcal{L}(L^2)} \leq C_{s,\delta}\, t^{-s+\delta}, \qquad \forall s,\delta > 0.$$

Hence $\left| \left\langle \exp\left(i\frac{t}{\varepsilon} H_\varepsilon\right) \chi(H_\varepsilon) S_\varepsilon, \varphi_\varepsilon \right\rangle \right| \leq C_s\, t^{-s}, \qquad \forall s > 0.$

The contribution of times that are at least polynomially large, $t \geq \varepsilon^{-\kappa}$ ($\kappa > 0$), is thus vanishing :

$$i\varepsilon^{-1} \int_{\varepsilon^{-\kappa}}^{+\infty} \langle \exp(itH_\varepsilon/\varepsilon)\chi(H_\varepsilon)S_\varepsilon, \varphi_\varepsilon\rangle \sim \varepsilon^{s\kappa-1} \to 0.$$

• For times $1 \ll T_1 \leq t \leq \varepsilon^{-\kappa}$, we need an Egorov Theorem that holds up to polynomially large times (i.e. "$\exp(itH_\varepsilon/\varepsilon)S_\varepsilon \approx S_\varepsilon(X(t))$+remainder"). Obtained through an adaptation of [Bouzouina-Robert].

III- Asymptotic propagation of w^ε (6) - large time contribution

- Roughly, using the notation $\Phi_t(x,\xi) = (X(t), \Xi(t))$, we have

$$\exp\left(i\frac{t}{\varepsilon}H_\varepsilon\right)\chi(H_\varepsilon)S_\varepsilon \approx \exp\left(i\frac{t}{\varepsilon}H_\varepsilon\right)\mathrm{Op}^w_\varepsilon\left(\chi\left[x=0,\xi^2=n^2(x)\right]\right)S_\varepsilon$$
$$\approx \mathrm{Op}^w_\varepsilon\left(\chi\left[\Phi_{-t}(x=0,\xi^2=n^2(x))\right]\right)\exp\left(i\frac{t}{\varepsilon}H_\varepsilon\right)S_\varepsilon + \mathrm{R}(t,\varepsilon),$$

and [BR] asserts, say : $\quad \mathrm{R}(t,\varepsilon) \leq \varepsilon^N \sup_{1\leq k\leq N}\|\partial^k_{x,\xi}\Phi_t(x,\xi)\|_{L^\infty} \quad (N \gg 1).$

- Non-trapping implies $X(t)$ is far from $x=0$, provided $t \geq T_1 \gg 1$. Hence,

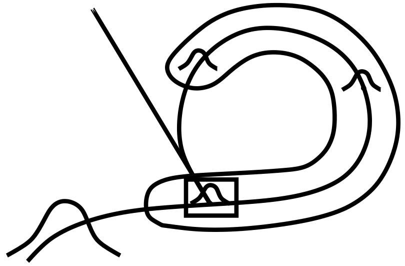
$$i\varepsilon^{-1}\int_{T_1}^{\varepsilon^{-\kappa}}\left\langle \mathrm{e}^{itH_\varepsilon/\varepsilon}\chi(H_\varepsilon)S_\varepsilon,\varphi_\varepsilon\right\rangle \approx 0 + i\varepsilon^{-1}\int_{T_1}^{\varepsilon^{-\kappa}}\mathrm{R}(t,\varepsilon). \text{ (orthogonal supports)}$$

- Long-range potential + symplecticness of Φ_t implies

$$\sup_{1\leq k\leq N}\|\partial^k_{x,\xi}\Phi_t(x,\xi)\|_{L^\infty} \leq t^{N^2} \qquad \text{(polynomial, rather than } \mathrm{e}^t).$$

- As a consequence

$$i\varepsilon^{-1}\int_{T_1}^{\varepsilon^{-\kappa}}\langle\exp\left(i\frac{t}{\varepsilon}H_\varepsilon\right)\chi(H_\varepsilon)S_\varepsilon,\varphi_\varepsilon\rangle \leq \varepsilon^{N-1}\int_{T_1}^{\varepsilon^{-\kappa}}t^{N^2} \approx \varepsilon^{\kappa N^2+N-2} \longrightarrow 0.$$



Moderate times : $\theta \leq t \leq T_1$ $(\theta \ll 1 \ll T_1)$

- Idea : $e^{itH_\varepsilon/\varepsilon} S_\varepsilon$ is almost explicitely known if S_ε is a wave packet, i.e. a "gaussian centered at $x = q$, $\xi = p$". More precisely,

$$\Phi^\varepsilon_{q,p} = \varepsilon^{-d/4} \exp(-(x-q)^2/\varepsilon) \exp(i(x-q/2)\cdot p/\varepsilon)$$

$$e^{itH_\varepsilon/\varepsilon}\Phi^\varepsilon_{q,p}(x) \sim \varepsilon^{-d/4} \exp\left(-\Gamma(t)(x-Q(t))^2/\varepsilon\right) \exp\left(i(x-Q(t)/2)\cdot P(t)/\varepsilon\right)$$
$$\exp\left(i\, S(t)/\varepsilon\right) \times (1 + O(\varepsilon^{1/2})),$$

where $\Gamma(t)$, $S(t)$ are classical quantities [Hepp, Combescure-Robert, Hagedorn-Joye, Ralston, Robinson, ...]

- Hence

$$\varepsilon^{-1}\int_\theta^{T_1} dt \left\langle e^{itH_\varepsilon/\varepsilon} S_\varepsilon, \varphi_\varepsilon\right\rangle = \varepsilon^{-1}\int_\theta^{T_1} dt dq dp \left\langle S_\varepsilon, e^{-itH_\varepsilon/\varepsilon}\Phi^\varepsilon_{q,p}(x)\right\rangle \left\langle \Phi^\varepsilon_{q,p}(x), \varphi_\varepsilon\right\rangle$$
$$= \ldots = \varepsilon^{-(d+2)/2}\int_\theta^{T_1} dt d\xi d\eta \text{ Amplitude}(t,\xi,\eta) \exp(i \text{ Phase}(t,\xi,\eta)/\varepsilon).$$

• Using the "explicit" value of the complex phase, the stationnary phase theorem gives the singular set

$$\{(t,\xi,\eta) \text{ s.t. } X(t)=0, \Xi(t)=\eta, \eta^2=n^2(0)\}.$$

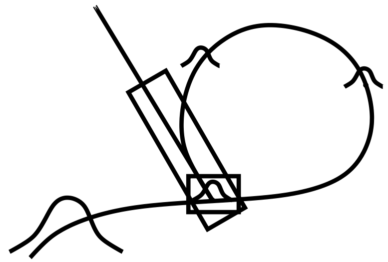
Caution : one needs to integrate by parts in **time** as well $\Rightarrow$ small times $0 \leq t \leq \theta$ are excluded.

• As a consequence,

$$\varepsilon^{-1}\int_\theta^{T_1} dt \left\langle e^{itH_\varepsilon/\varepsilon} S_\varepsilon, \varphi_\varepsilon \right\rangle$$

$$\sim \varepsilon^{(d-1-k)/2} \int_{\text{singular set}} dt d\xi d\eta \text{ Amplitude}(t,\xi,\eta) \exp(i \text{ Phase}(t,\xi,\eta)/\varepsilon)$$

$$\longrightarrow 0, \qquad \text{provided } k < d-1 \text{ (geometric assumption)}.$$



Small times $0 \leq t \leq \theta \ll 1$

Here the constant coefficients trajectory, and the "true" trajectory, are tangent. One goes back to the microscopic scale, and uses Taylor expansions in the above phase (Brenner, Dos Santos Ferreira, ...) :

$$\varepsilon^{-1} \int_0^\theta dt \left\langle e^{itH_\varepsilon/\varepsilon} S_\varepsilon, \varphi_\varepsilon \right\rangle = \int_0^{\theta/\varepsilon} dt \left\langle e^{it[\Delta_x + n^2(\varepsilon x)]} S, \varphi \right\rangle$$

$$\sim \int_0^{\theta/\varepsilon} dt d\xi \ \widetilde{\text{Amplitude}}(\varepsilon t, \xi) \ \exp(i \ \widetilde{\text{Phase}}(\varepsilon t, \xi))$$

$$= \int_0^{\theta/\varepsilon} dt d\xi \ \widetilde{\text{Amplitude}}(\varepsilon t, \xi) \ \exp\left(it \ \frac{\widetilde{\text{Phase}}(\varepsilon t, \xi)}{\varepsilon t} \right)$$

with $\widetilde{\text{Phase}}(\varepsilon t, \xi)/\varepsilon t = \xi^2 - n^2(0) + O([\varepsilon t]) = \xi^2 - n^2(0) + O(\theta).$

Hence,

$$\varepsilon^{-1}\int_0^\theta dt\left\langle \mathrm{e}^{itH_\varepsilon/\varepsilon}S_\varepsilon,\varphi_\varepsilon\right\rangle \sim \int_0^{\theta/\varepsilon} dt d\xi\ \widetilde{\mathrm{Amplitude}}(\varepsilon t,\xi)\ \exp\left(\ it\ [\xi^2-n^2(0)]\right)$$

and, using the dispersion for the free Schrödinger flow,

$$\left|\int d\xi\ \widetilde{\mathrm{Amplitude}}(\varepsilon t,\xi)\ \exp\left(\ it\ [\xi^2-n^2(0)]\right)\right| \leq C\ t^{-d/2}$$

as well as the fact that $d \geq 3$, we recover

$$\begin{aligned}\varepsilon^{-1}\int_0^\theta dt\left\langle \mathrm{e}^{itH_\varepsilon/\varepsilon}S_\varepsilon,\varphi_\varepsilon\right\rangle &\sim \int_0^{+\infty} dt d\xi\ \widetilde{\mathrm{Amplitude}}(0,\xi)\ \exp\left(\ it\ [\xi^2-n^2(0)]\right)\\ &= \int_0^{+\infty} dt \int d\xi \exp(it[\xi-n^2(0)])\ \widehat{S}(\xi)\ \widehat{\varphi}(\xi)\\ &= \int d\xi\, \widehat{w}^{\mathrm{out}}(\xi)\ \widehat{\varphi}(\xi) = \langle w^{\mathrm{out}},\varphi\rangle.\end{aligned}$$

III- Asymptotic propagation of w^ε (11) - the counterexample

Difficulty : building a refraction index that **creates refocusing and is non-trapping** (!).