Gaussian Beams

Asymptotic Solutions concentrated on single ray paths

(+) Valid approximations on **any** finite segment of a ray path – single rays do not lead to caustics.

(=) Computational requirements barely more than computing the ray itself.

(−) Most asymptotic solutions of interest are built from many rays – need superpositions.

Notation

Symbols: For a linear differential operator $P(x, D)$

$$P(x,D)(\int e^{ix\cdot\xi}\hat{f}(\xi)d\xi) = \int e^{ix\cdot\xi}p(x,\xi)\hat{f}(\xi)d\xi$$

defines $p(x, \xi)$, the symbol of $P(x, D)$. Grouping the terms in $p(x, \xi)$ by their orders of homogeneity in ξ one gets

$$p(x,\xi) = p_m(x,\xi) + p_{m-1}(x,\xi) + \cdots + p_0(x,\xi),$$

where m is the order of $P(x, D)$. $p_m(x, \xi)$ is the principal symbol. For example, the wave equation operator $\frac{\partial^2}{\partial t^2} - c^2(x, y, z)\Delta$, setting $t = x_0$ and $(x, y, z) = (x_1, x_2, x_3) = x'$, has symbol

$$p(x,\xi) = -\xi_0^2 + c^2(x')|\xi'|^2 = p_2(x,\xi).$$

$$\partial_x = (\frac{\partial}{\partial x_1}, .., \frac{\partial}{\partial x_n}),\ F_x = \partial_x F$$

– likewise for derivatives in ξ

The *ANSATZ*

(for solutions to $P(x, D)u = 0$ concentrated near $x = x(s)$)

$$u(x, k) = e^{ik\phi(x)}(a_0(x) + \frac{1}{k}a_1(x) + \cdots + \frac{1}{k^N}a_N(x))$$
$$= e^{ik\phi(x)}a(x, k)$$

Just like geometric optics – except that $\phi(x(s))$ is real, and $\mathrm{Im}\{\phi(x)\} > 0$ for $x \neq x(s)$.

However, that changes a lot.

Note that $|x|^r e^{-ck|x|^2} = O(k^{-r/2})$ on $\mathbb{R}^n$. The Hessian of Im ϕ will be positive definite on $x(s)$. So to make

$$P(x, D)u = O(k^{-M})$$

it will suffice to make

$$e^{-ik\phi}P(x, D)u = O(\sum_s k^s |x - x(s)|^{-s+2M})$$

So we will be setting derivatives of $\exp(-k\phi)P(x, D)u$ equal to zero on $x(s)$.

We assume that $P(x, \xi)$ has a real **principal** symbol $p(x, \xi)$.

$$P(x, D)u = k^m p(x, \phi_x(x))u + O(k^{m-1})$$

So we begin by making $p(x, \phi_x(x))$ vanish to high order on $x(s)$.

The first three sets of equations, corresponding to vanishing of orders zero, one and two are (using summation on like indices in products)

$$p = 0 \tag{0}$$

$$p_{x_i} + p_{\xi_j}\phi_{x_j x_i} = 0 \text{ for } 1 \le i \le n \tag{1}$$

$$p_{x_i x_k} + p_{x_i \xi_j}\phi_{x_j x_k} + p_{x_k \xi_j}\phi_{x_j x_i} + p_{\xi_j \xi_l}\phi_{x_j x_i}\phi_{x_l x_k} + p_{\xi_j}\phi_{x_j x_i x_k} = 0 \tag{2}$$

for $1 \le i, k \le n$.

Set $\phi_x(x(s)) = \xi(s)$. Then all the derivatives of p in these equations are evaluated at $(x(s), \xi(s))$. Denote df/ds by $\dot{f}$.

Necessary step Require

$$\dot{x}(s) = p_\xi(x(s), \xi(s)).$$

Since differentiation with respect to s gives

$$\phi_{x_i x_j} \dot{x}_j = \dot{\xi}_i,$$

the equation (1) is equivalent to

$$\dot{\xi} = -p_x(x, \xi)$$

Conclusion: $p(x, \phi_x)$ vanishes to first order on $x(s)$ if $(x(s), \xi(s))$ is a null bicharacteristic for $P(x, D)$.

What about the other equation?

It's a lot simpler in matrix notation. Let

$$M(s) = \left(\phi_{x_i x_j}(x(s))\right), \; A(s) = \left(p_{x_i x_j}(x(s), \xi(s))\right),$$

$$B(s) = \left(p_{x_i \xi_j}(x(s), \xi(s))\right), \; C(s) = \left(p_{\xi_i \xi_j}(x(s), \xi(s))\right)$$

Then (2) becomes

$$A + BM + MB^t + MCM + \dot{M} = 0$$

Matrix Riccati! $M = NY^{-1}$ will be solution when (Y, N) is a matrix solution to the linear system

$$\dot{Y} = CN + B^t Y \qquad \dot{N} = -BN - AY$$

Key to the Construction:

As long as $\dot{x}(s)$ never vanishes, you can choose initial data $(Y(0), N(0))$ so that $Y(s)$ is invertible for all s.

Invariance: Writing $\psi_1 = (y^1(s), \eta^1(s))$ and $\psi_2 = (y^2(s), \eta^2(s))$ for a pair of vector solutions of

$$\dot{y} = C\eta + B^t y \qquad \dot{\eta} = -B\eta - Ay,$$

the bilinear form

$$\sigma(\psi_1, \psi_2) = y^2 \cdot \eta^1 - y^1 \cdot \eta^2 \tag{3}$$

is **constant** in s. Since $\overline{\psi}_2$, where $\bar{\cdot}$ denotes complex conjugate, is also a solution of (3)

$$\sigma(\psi_1, \overline{\psi}_2) = \overline{y}^2 \cdot \eta^1 - y^1 \cdot \overline{\eta}^2$$

is **also** constant in s. This invariance leads to:

Lemma: Choose the initial data $(Y(0), N(0)) = (I, M(0))$, where

i) $M(0) = M(0)^t$ and $M(0)\dot{x}(0) = \dot{\xi}(0)$, and

(ii) Im $M(0)$ is positive definite of the orthogonal complement of $\dot{x}(0)$.

Then $Y(s)$ is invertible for all s, and $M(s) = N(s)Y(s)^{-1}$ satisfies $M(s) = M(s)^t$ and Im $M(s)$ is positive definite of the orthogonal complement of $\dot{x}(s)$ for all s.

Conclusion: Since $M(s)$ is the Hessian of $\phi(x)$ at $x(s)$, we now have the gradient and Hessian of ϕ on the ray.

Continuing to set derivatives of $p(x, \phi(x))$ equal to zero on $x(s)$ one determines the Taylor series of ϕ along $x(s)$ to all orders. However, the systems of equations for the partial derivatives of order greater than two are **linear**. Thus one can solve them for all s for any initial data that satisfies equality of mixed partials and is compatible with $\partial_x \phi(x(s)) = \xi(s)$. Thus – if we work hard enough – we can make $p(x, \partial_x \phi(x))$ vanish to any desired order on $x(s)$. [In applications so far we've been happy with order 2.]

Just as setting the coefficient of k^m in $P(x,D)u$ to zero leads to the eichonal equation in geometric optics, the coefficients of lower powers of k correspond to the transport equations. Since these equations are linear, solving them to any order on $x(s)$ presents no problems. Note, however, that the we are limited by the number of terms in the Taylor series of ϕ that we have computed, since derivatives of ϕ appear in the coefficients of the transport equations.

The simplest example of this construction is $P = \partial_t^2 - \partial_x^2 - \partial_y^2$, the wave equation in two space dimensions. For the bicharacteristic given by $t(s), x(s), y(s) = (s, 0, s)$, for any positive constants a and b the phase

$$\phi(t,x,y) =$$

$$\frac{y-t}{2} + \frac{a^2x^2t}{1+4a^2t^2} + i\left(\left(\frac{a}{1+4a^2t^2}\right)\frac{x^2}{2} + b\frac{(y-t)^2}{2}\right),$$

and the amplitude

$$a_0 = (1+2iat)^{-1/2}$$

give $u = e^{ik\phi}a_0$ satisfying $(\partial_{tt} - \Delta)u = O(k^{1/2})$ in k.

Possible Extensions

a) Operators of the form

$$P(x, D; h) \doteq P_0(x, hD) + hP(x, hD) + h^2 P(x, hD) + \cdots$$

To construct asymptotic solutions to $P(x, D; h)u = 0$ as $h \downarrow 0$ one can use the same *Ansatz* with $k = 1/h$. For such operators the symbol $p(x, \xi; h)$ is defined by

$$P(x, D; h) \int e^{ix\cdot\xi/h} \hat{f}(\xi) d\xi = \int e^{ix\cdot\xi/h} p(x, \xi; h) \hat{f}(\xi) d\xi$$

Note that for the operator above

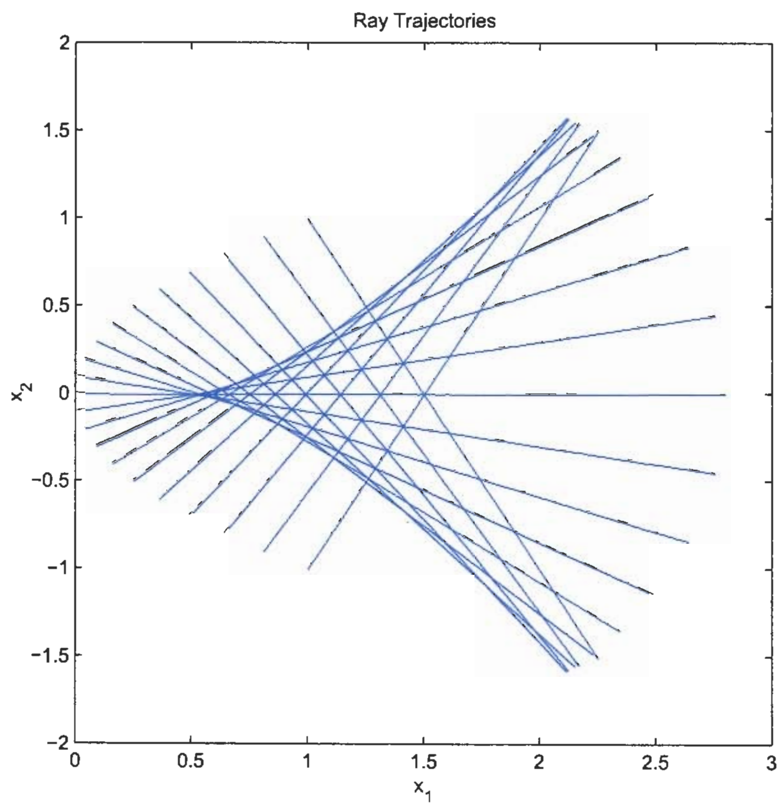
$$p(x, \xi; h) = p_0(x, \xi) + hp_1(x, \xi) + \cdots$$

where p_j is the **full** symbol of $P_j(x, D)$. Examples here (with $P_j = 0$ for $j > 0$

Schrödinger: $ihu_t + h^2 \Delta u - V(x)u = 0$ and

Helmholz: $c^2(x)\Delta u + k^2 u = 0$

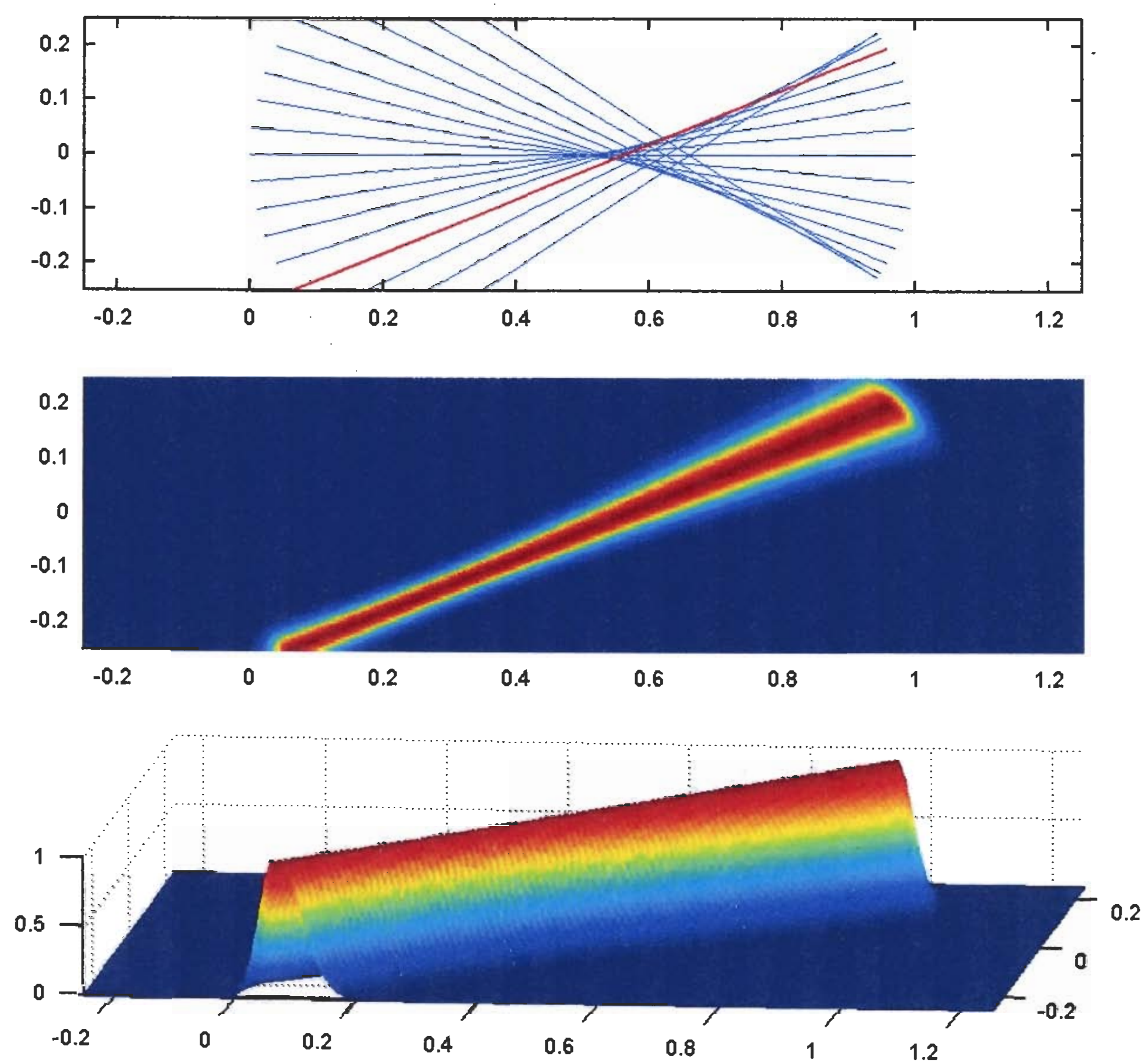
Wave Equation Bicharacteristics



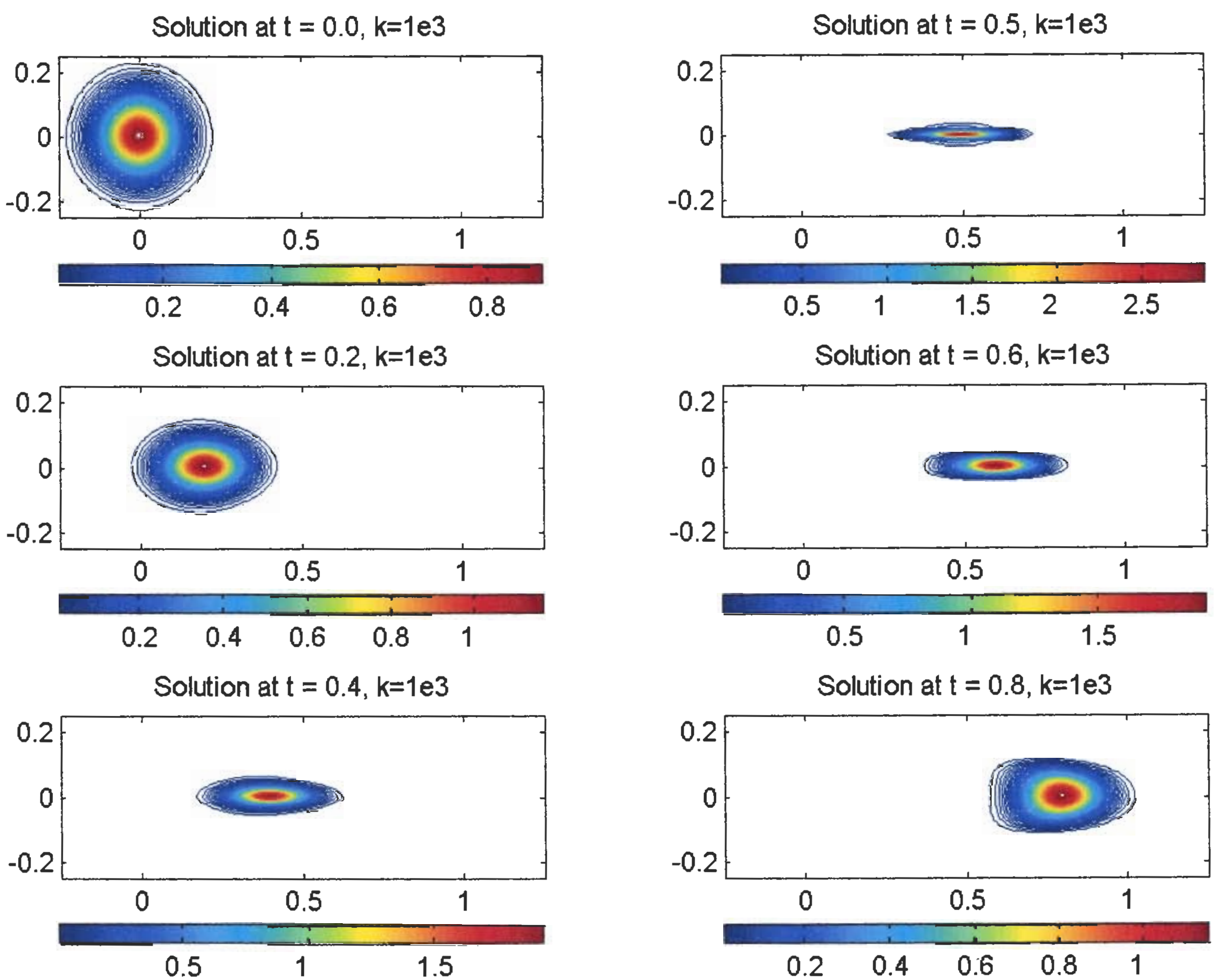
$$
\begin{aligned}
t(s) &= -2\sqrt{1+4y_2^2}\; s \\
x_1(s) &= 2s + y_1 \\
x_2(s) &= -4y_2 s + y_2 \\
\tau(s) &= -\sqrt{1+4y_2^2} \\
\xi_1(s) &= 1 \\
\xi_2(s) &= -2y_2
\end{aligned}
$$

- solution develops a “cusp” caustic

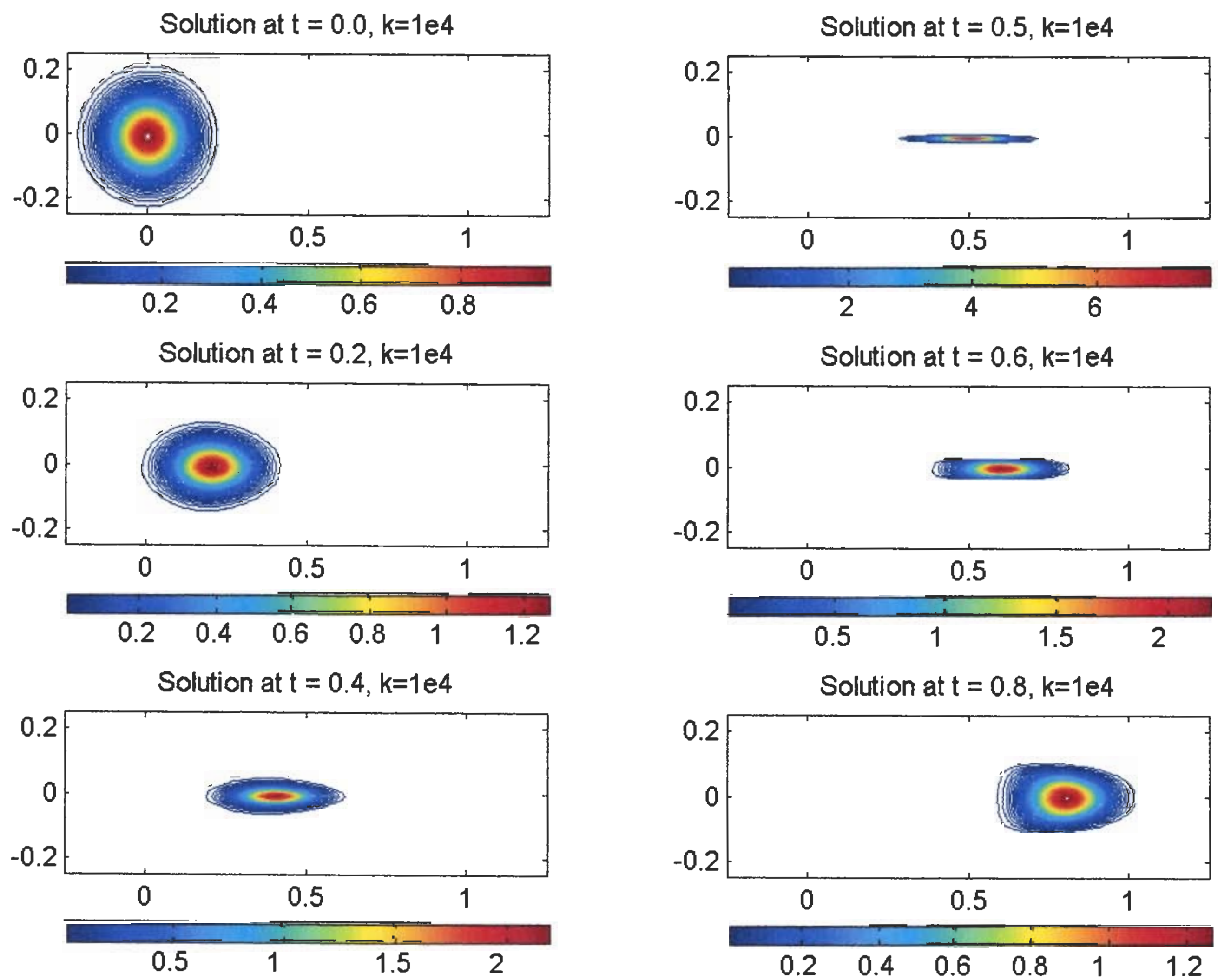
Gaussian Beam



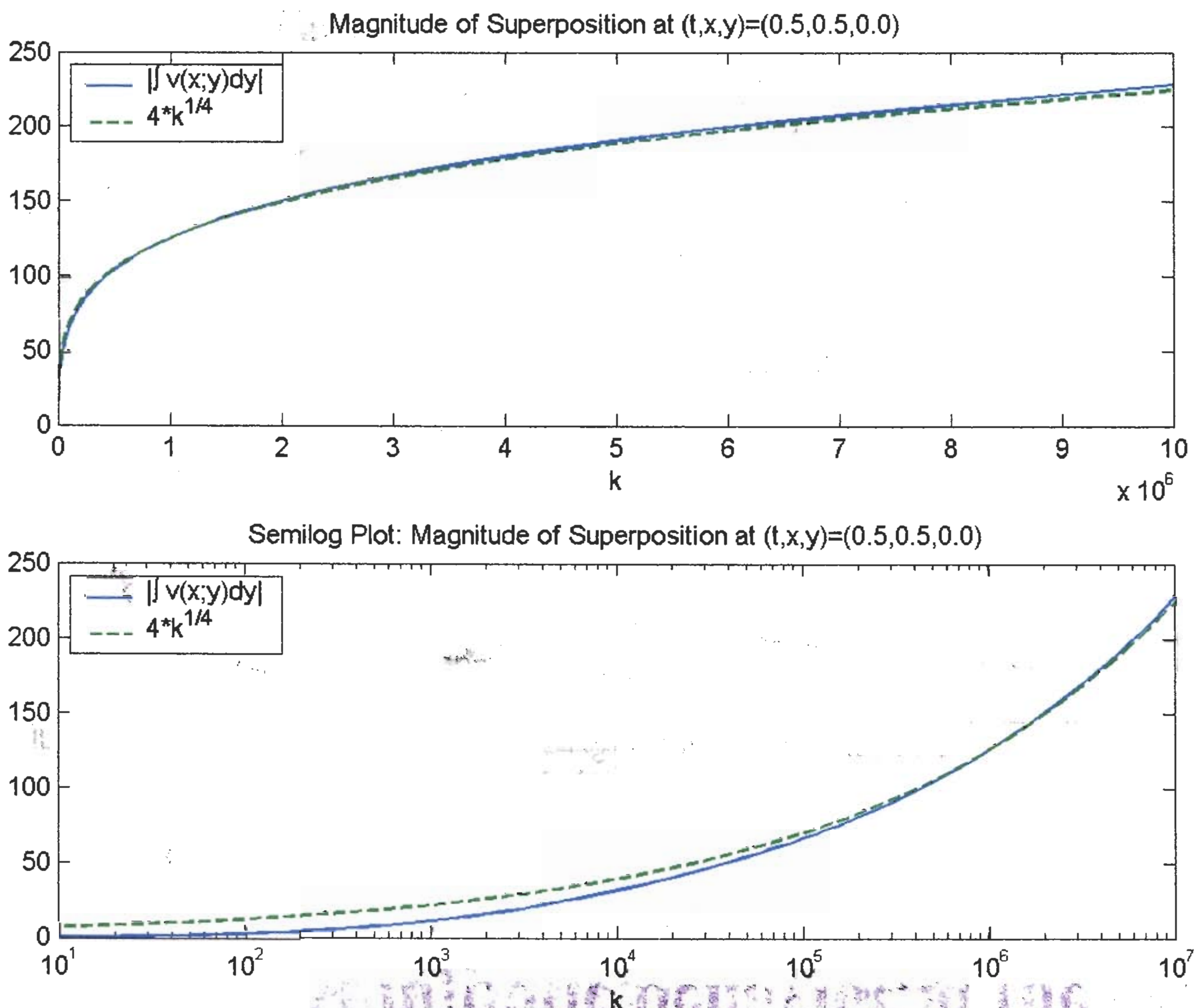
Superposition for $k = 10^3$



Superposition for $k = 10^4$



Asymptotic behavior at the caustic



Quasimodes

When the curve γ traced by $(x(s), \xi(s))$ is a stable periodic orbit of the bicharacteristic flow, one can use Gaussian beams to build sequences of functions u_l and corresponding sequences of numbers $\lambda_l \to \infty$ such that $P(x, D)u_l - \lambda_l u_l = O(\lambda_l^{-M})$. To do this assume that $p(x, \phi_x) = 1$ on γ, and use exactly the same *Ansatz* to build beams which satisfy

$$P(x, D)u = (k^m + c_1 k^{m-1} + \dots)u.$$

[The constants $c_1, c_2, \dots$ are to be determined.]

Now

$$\dot{y} = C\eta + B^t y \qquad \dot{\eta} = -B\eta - Ay$$

has periodic coefficients. If S is the period (smallest positive number such that $(x(0), \xi(0)) = (x(S), \xi(S))$, one has the

Floquet Map $\Phi : (y(0), \eta(0)) \to (y(S), \eta(S))$.

Φ maps real vectors to real vectors and preserved the (symplectic) form σ. That implies that it has 1 as an eigenvalue of multiplicity at least two, and its eigenvalues come in both conjugate and reciprocal pairs.

Strong Stability: The eigenvalues of Φ are

$$\{1, 1, e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_{n-1}}, e^{-i\theta_{n-1}}\}$$

and the θ_j's and π are rationally independent.

Under this hypothesis one can use the eigenvectors of Φ to build $M(0)$ with all the properties required in the **Lemma** so that the resulting $M(s)$ satisfies $M(0) = M(S)$. This can be done if the eigenvalues of Φ are just simple except for 1 which is double, but one needs rational independence to get periodic solutions for the higher derivatives of ϕ.

Construction of the amplitude $a(x,k)$ – and the determination of the c_j's – is a long story. For self-adjoint $P(x,D)$ one gets an amplitude for **each multi-index** α such that $a(x(S)) = \exp(-i\beta(\alpha))a(x(0))$, and in the simplest case ("subprincipal symbol zero")

$$\beta(\alpha) = (\alpha_1 + \frac{1}{2})\theta_1 + \cdots + (\alpha_{n-1} + \frac{1}{2})\theta_{n-1} \mathrm{mod}\ \pi.$$

Introducing the "action" around γ, i.e.

$$\phi(x(S)) - \phi(x(0)) = \int_0^S \dot{x}(s) \cdot \phi_x(x(s))ds$$

$$= \int_0^S \dot{x}(s) \cdot \xi(s)ds = mS,$$

the condition that determines the sequence of approximate eigenvalues λ_l is just

$$kmS - \beta = 2\pi l \text{ or}$$

$$k_l = \frac{2\pi l + \beta(\alpha)}{mS}$$

To leading order $\lambda_l = k_l^m$. These formulas say that for l large quite a few eigenvalues of P are associated with γ – assuming that the coefficients of P grow at infinity so that the spectrum of P consists only of eigenvalues. Actually more is true: a positive fraction of *all* the large eigenvalues of P are associated with γ – see Popov (2000).