A Fast Phase Space Method for Computing Creeping Rays

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High Frequency Wave Propagation
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Geometrical optics for high-frequency waves

Consider Helmholtz equation at high frequencies,

$$\Delta u + n(x)^2\omega^2 u = 0, \qquad \omega \gg 1.$$

Look at simple wave solutions of the form

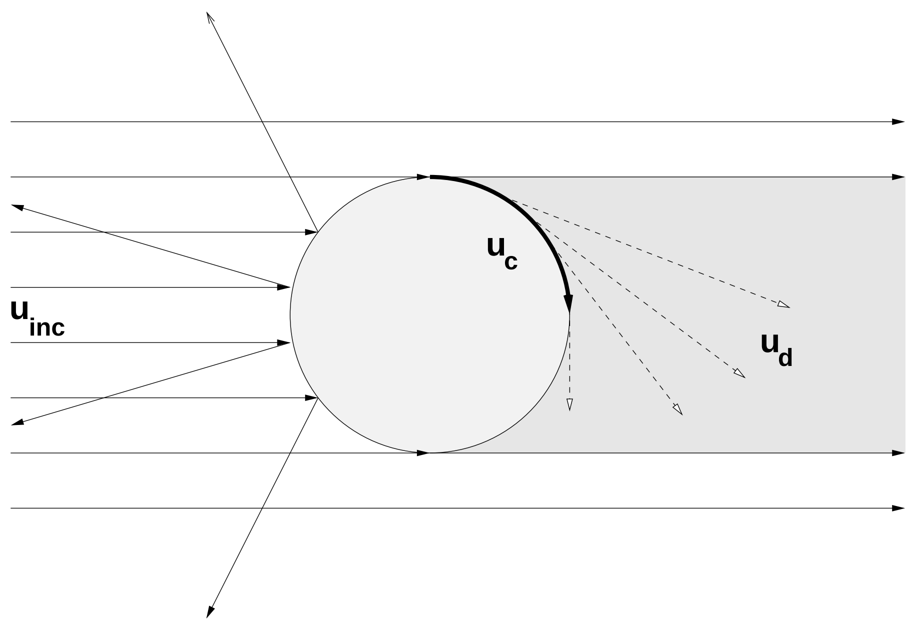
$$u(x) \approx A(\boldsymbol{x})e^{i\omega\phi(\boldsymbol{x})}.$$

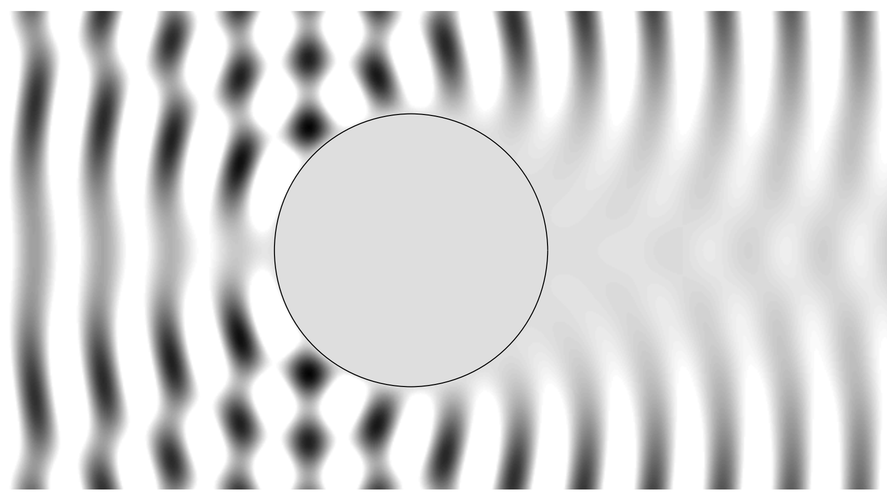
- Amplitude A and phase ϕ vary on a much coarser scale than u.
- Geometrical optics approximation considers A and ϕ as $\omega \to \infty$,

$$|\nabla\phi| = n(x), \qquad 2\nabla\phi \cdot \nabla A + \Delta\phi A = 0.$$

- Waves propagate as rays, c.f. visible light.
- Good accuracy for large ω. Computational cost ω-independent.
- Not all wave effects captured correctly, in particular at *boundaries* (diffracted, creeping waves) and *caustics*.
- Geometrical theory of diffraction [Keller, 62] gives corrections.

Creeping rays

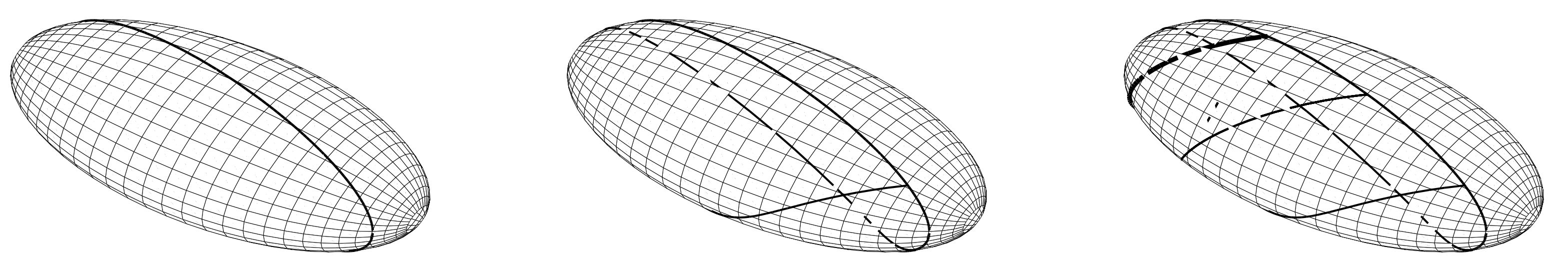




Creeping rays, cont.

Creeping rays are created at the *shadowline*, where the incident plane wave direction is tangential to the surface.

They then follow geodesics on the (dark side of the) scatterer, continuously shedding diffracted rays in their tangential direction.



Governing equations

Suppose $X(u,v) \in \mathbb{R}^3$ describes the scatterer surface, with (u,v) belonging to a bounded set $\Omega \subset \mathbb{R}^2$,

The geodesics are then given by the ODEs in parameter space

$$\ddot{u} + \Gamma^1_{11}\dot{u}^2 + 2\Gamma^1_{12}\dot{u}\dot{v} + \Gamma^1_{22}\dot{v}^2 = 0,$$
$$\ddot{v} + \Gamma^2_{11}\dot{u}^2 + 2\Gamma^2_{12}\dot{u}\dot{v} + \Gamma^2_{22}\dot{v}^2 = 0,$$

where $\Gamma^k_{ij}(u,v)$ are the Christoffel symbols for the surface.

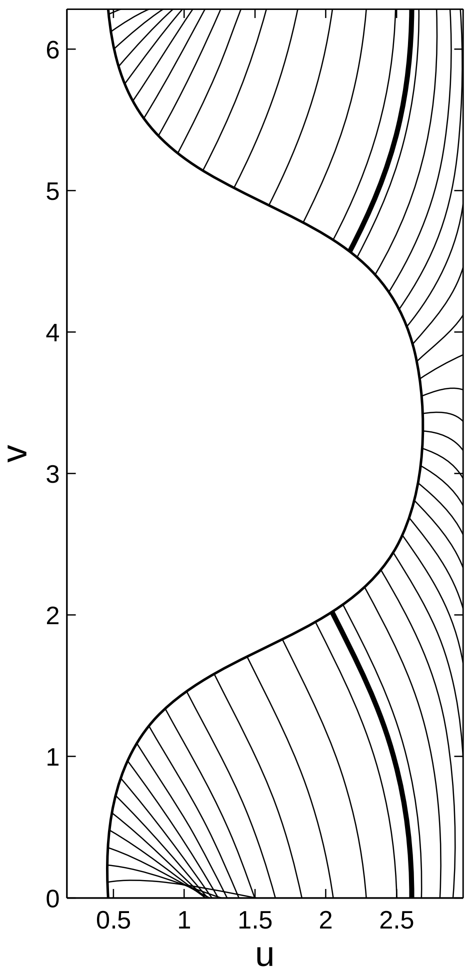
By changing parameterization along the geodesic, ODEs reduce to

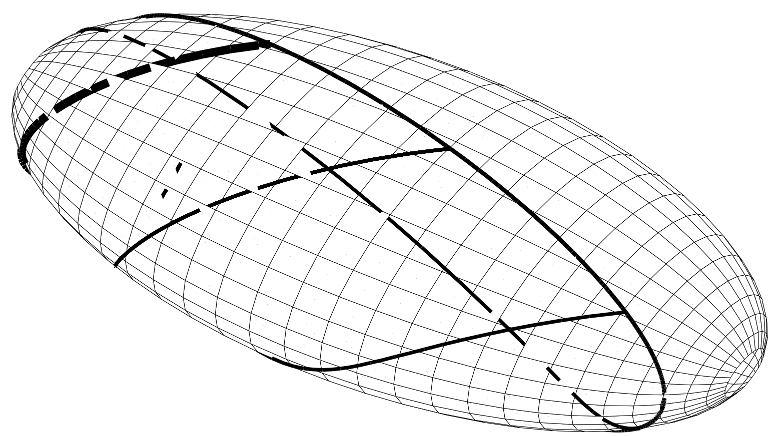
$$\frac{du}{d\tau} = \cos\theta,$$
$$\frac{dv}{d\tau} = \sin\theta,$$
$$\frac{d\theta}{d\tau} = V(u,v,\theta).$$

V depends on the Γ^k_{ij}.

θ is the direction of geodesic in parameter space.

Creeping rays in parameter space





May solve surface eikonal equation

$$\nabla\phi^T (J^T J)^{-1} \nabla\phi = n(u,v)^2, \quad J = [X_u X_v]$$

However, creeping rays may cross $\Rightarrow$ caustics and multivaluedness.

Phase space & Escape equations

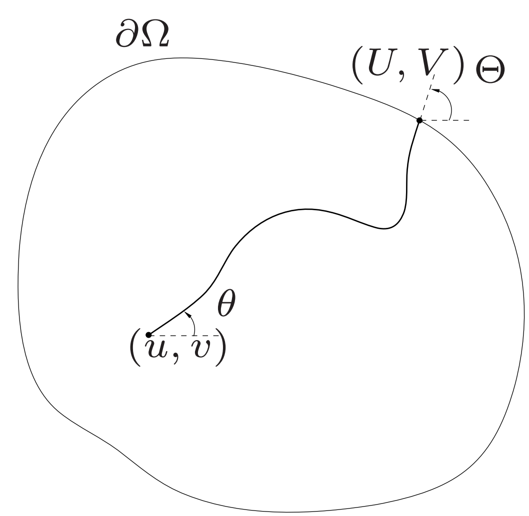
C.f. [Fomel, Sethian, PNAS 2002]

We introduce the phase space $\mathbb{P} = \mathbb{R}^2 \times \mathbb{S}$, and consider the triplet (u, v, θ) as a point in this space.

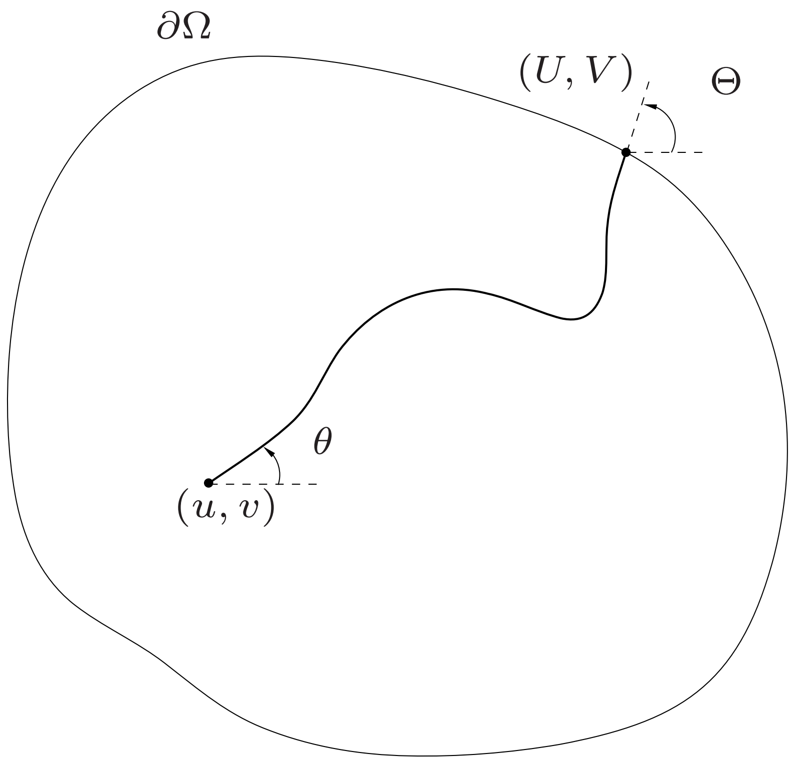
The geodesic locations/directions on the scatterer is then confined to a subdomain $\Omega_p = \Omega \times \mathbb{S} \subset \mathbb{P}$ in phase space.

New unknown: $F : \mathbb{P} \to \mathbb{P}$

Let $F(u, v, \theta) = (U, V, \Theta)$ be the point where the geodesic starting at (u, v) with direction θ will eventually cross the boundary of Ω_p.



Phase space & Escape equations



Since the value of F is constant along a geodesic we have

$$0 = \frac{d}{d\tau} F(u(\tau), v(\tau), \theta(\tau)) = \frac{du}{d\tau} F_u + \frac{dv}{d\tau} F_v + \frac{d\theta}{d\tau} F_\theta$$

$$= \cos\theta F_u + \sin\theta F_v + V(u, v, \theta) F_\theta.$$

Escape equation

Hence, F satisfies the *escape* PDE

$$\cos\theta F_u + \sin\theta F_v + V(u,v,\theta)F_\theta = 0, \quad (u,v,\theta) \in \Omega_p,$$

with boundary condition at inflow points

$$F(u,v,\theta) = (u,v,\theta), \qquad (u,v,\theta) \in \partial\Omega_p^{\text{inflow}}.$$

- $F(u_1,v_1,\theta_1) = F(u_2,v_2,\theta_2)$ implies that (u_1,v_1,θ_1) and (u_2,v_2,θ_2) are points on the same creeping ray.
- PDE can be solved by a version of Fast Marching at a $O(N^3 \log N)$ cost for N discretization points in each dimension.

Geodesic length

We can also get an ODE for the length L of the geodesic,

$$\frac{dL}{d\tau} = \rho(u, v, \theta)$$

for some (complicated) ρ.

We define $\phi : \mathbb{P} \to \mathbb{R}$ as

> $\phi(u, v, \theta)$ is the distance traveled by a geodesic starting at the point (u, v) with direction θ before it hits the boundary of Ω_p.

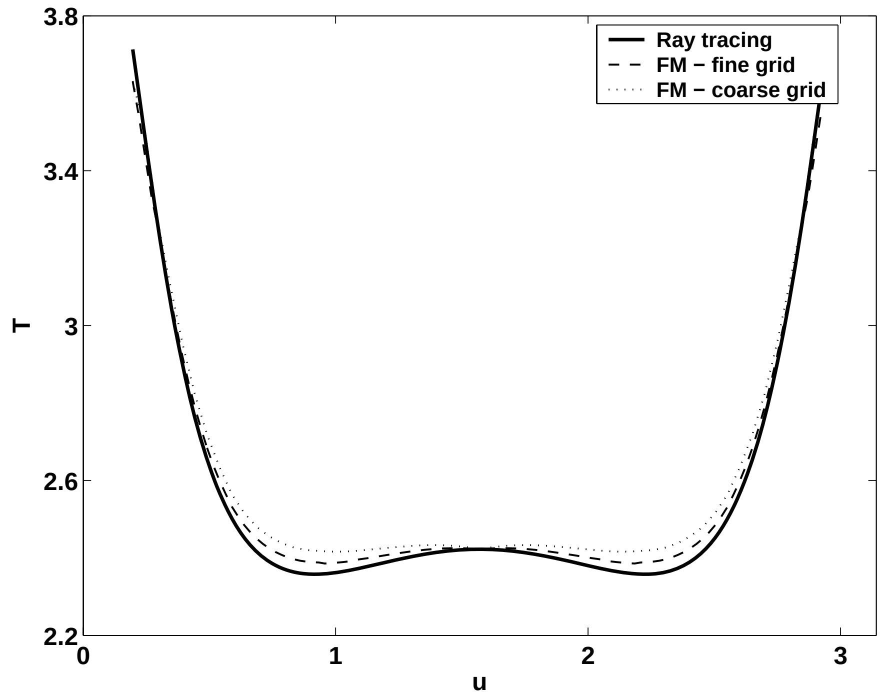
ϕ will also satisfy an escape PDE

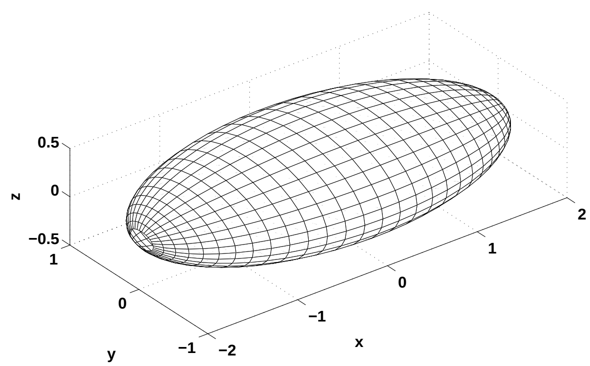
$$\cos\theta\phi_u + \sin\theta\phi_v + V(u, v, \theta)\phi_\theta = \rho(u, v, \theta), \quad (u, v, \theta) \in \Omega_p,$$

with boundary condition at inflow points

$$\phi(u, v, \theta) = 0, \qquad (u, v, \theta) \in \partial\Omega_p^{\text{inflow}}.$$

Example





$$\phi(u,\ v=\pi,\ \theta=\pi/2)$$

Postprocessing

To get things like traveltime, wavefronts, amplitudes, etc. the PDE solution must be postprocessed.

Example: Phase (length) from one illumination angle.

Assume shadowline is known: $(u(s), v(s), \theta(s)) =: \gamma(s)$.

For each point (u, v) find $\theta^*(u, v)$ and s such that

$$F(\gamma(s)) = F(u, v, \theta^*).$$

Then phase is given by

$$|\phi(\gamma(s)) - \phi(u, v, \theta^*(u, v))|.$$

Postprocessing, cont.

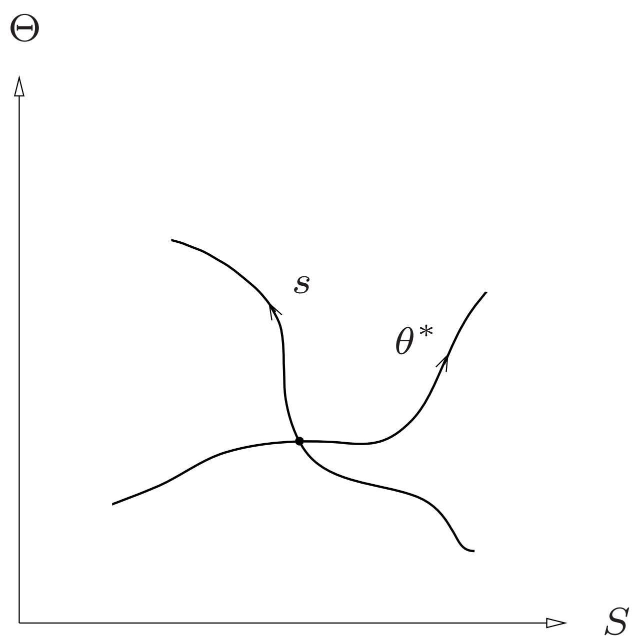
F is a point on the Ω_p boundary

$\Rightarrow$ can be reduced to a point in $\mathbb{R}^2$, say $F = (S, \Theta)$

LHS and RHS of

$$F(\gamma(s)) = F(u, v, \theta^*). \tag{1}$$

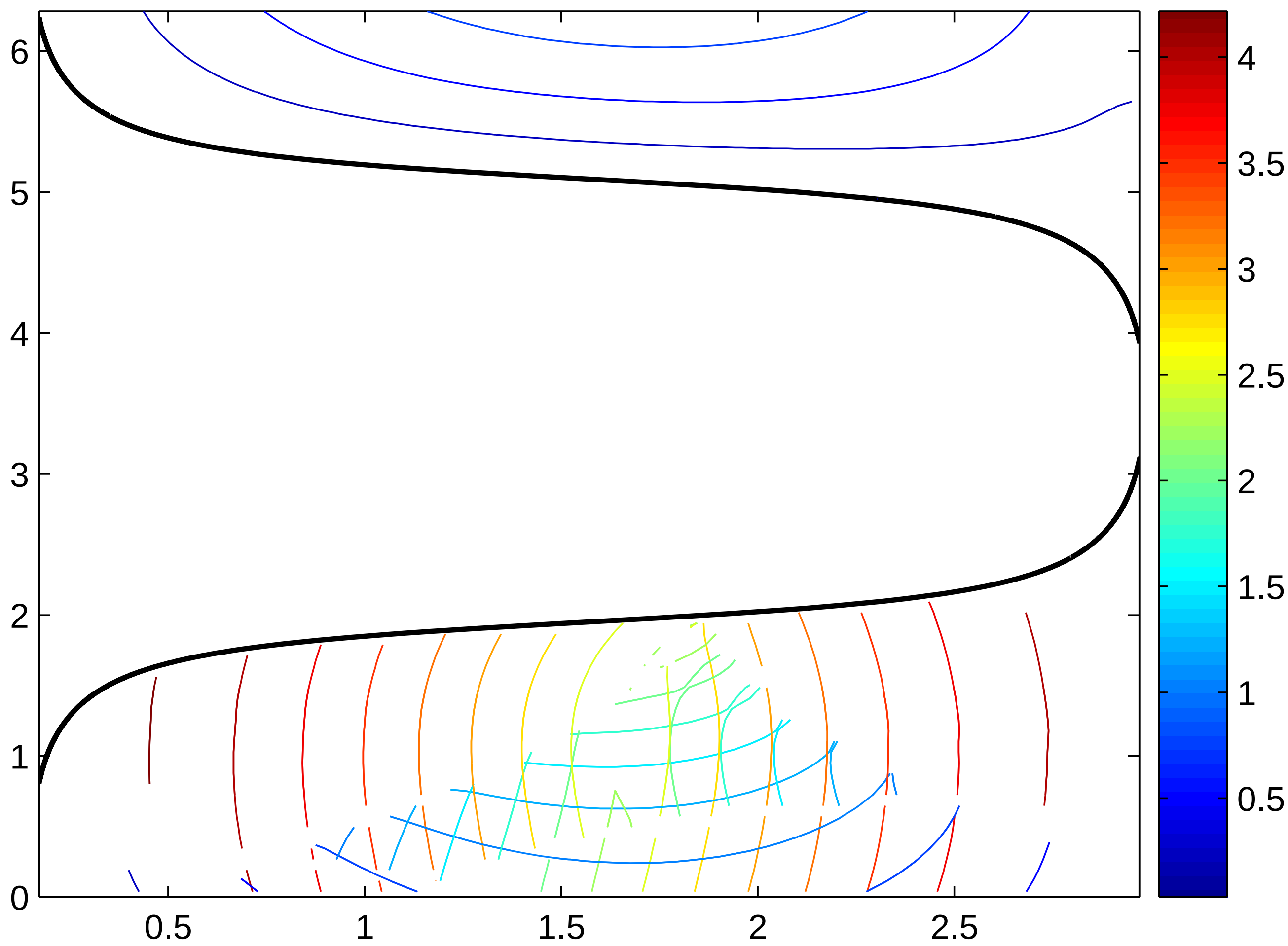
are curves in $\mathbb{R}^2$ parameterized by s and θ^*.



Solving (1) amounts to finding crossing points of these curves.

After discretization: Crossing point of polylines. $O(N)$ algorithms available.

Example



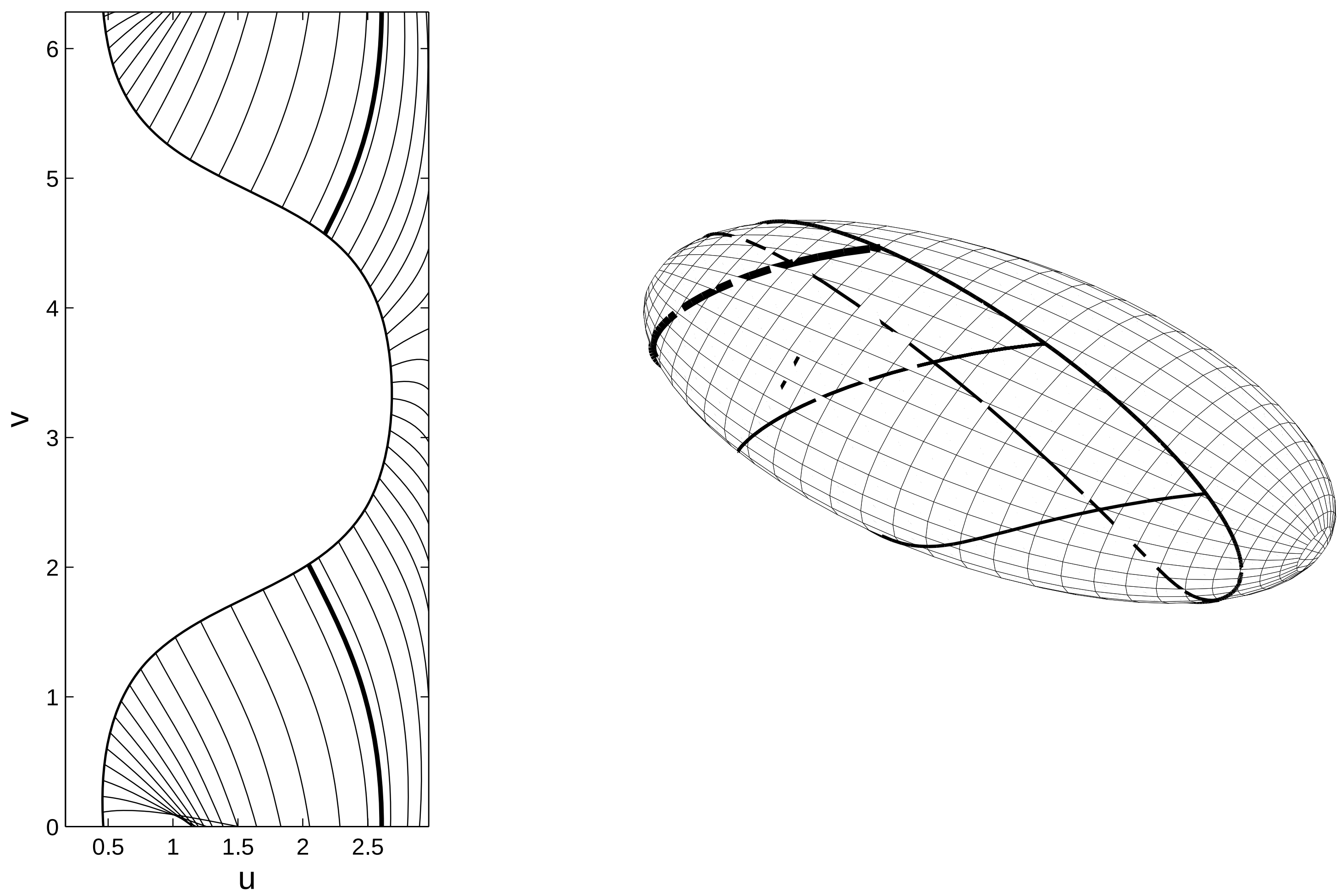
Computational cost

- PDE can be solved in $O(N^3 \log N)$
- Computing one incoming wavefront could in principle be done cheaper, e.g. $O(N^2)$ with wave front tracking/ray tracing/surface eikonal eq. (If shadowline discretized by N points.)
- Computing wavefronts for *all* (N^2) incoming wave front angles by these methods would be more expensive, $O(N^4)$.

Monostatic Radar Cross Section (RCS)

- Monostatic RCS = how much energy is reflected back in the direction of incident wave.
- Most by direct reflection.
- For low observable objects at not too high frequencies, creeping rays can give important contribution.
- Find *backscattered creeping rays* — rays that propagate on the surface and return in the opposite direction of incident wave.

Backscattered creeping ray



Finding backscattered creeping rays

Use postprocessing similar to before.

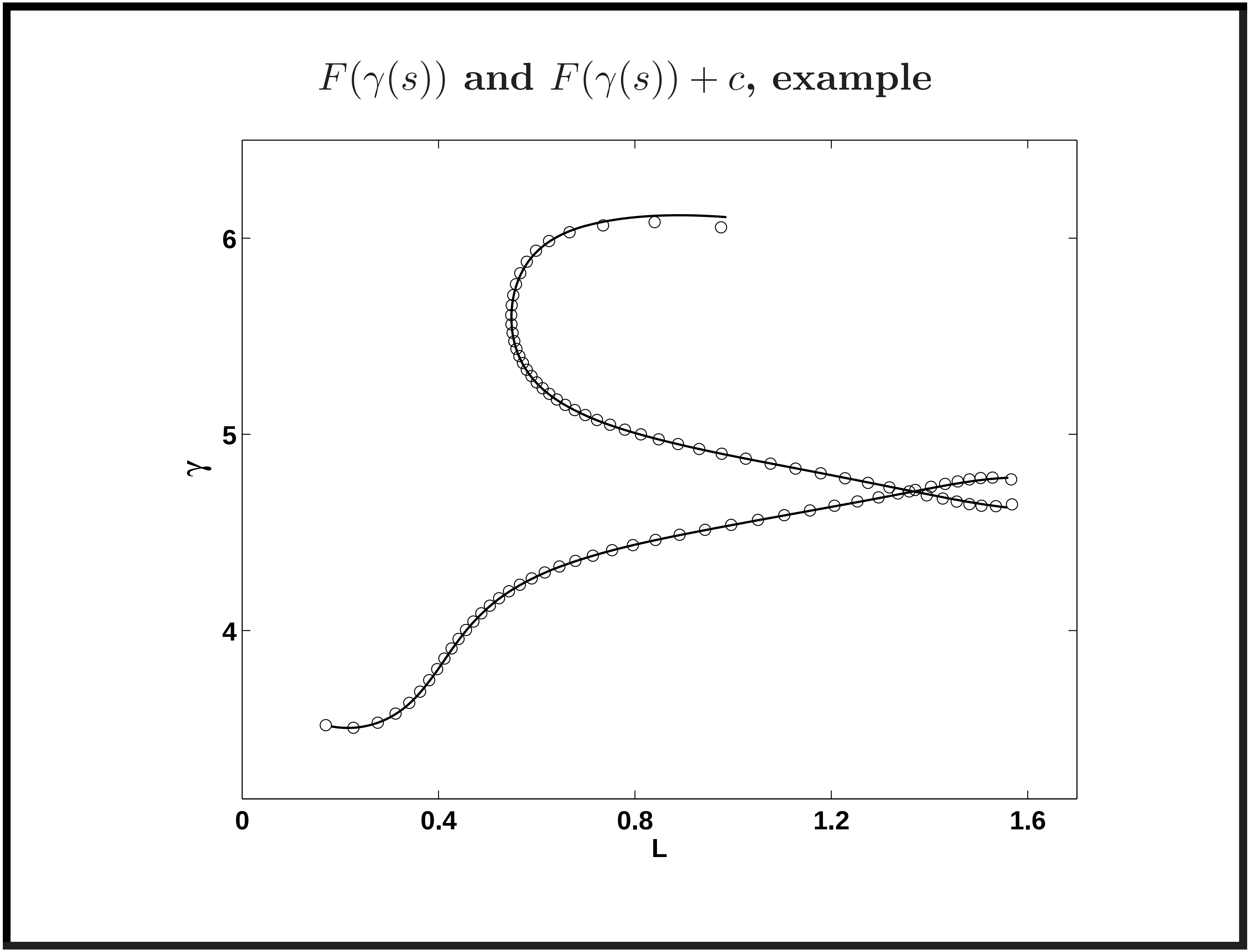
Assume shadowline is known: $(u(s), v(s), \theta(s)) =: \gamma(s)$.

A backscattered ray starting at point s_1 and ending at point s_2 on shadowline should satisfy

$$F(\gamma(s_1)) = F(\gamma(s_2)) + c.$$

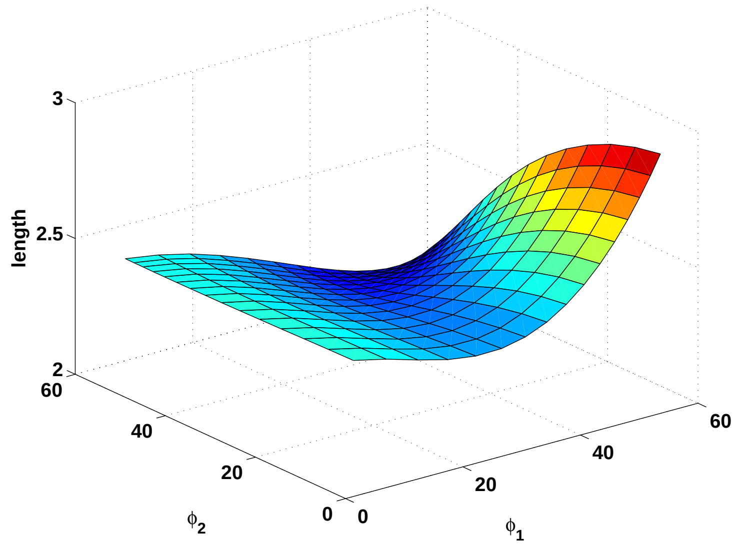
$F(\gamma(s))$ a curve in $\mathbb{R}^2$ parameterized by s.

Find intersection points! Same kind of problem as before.

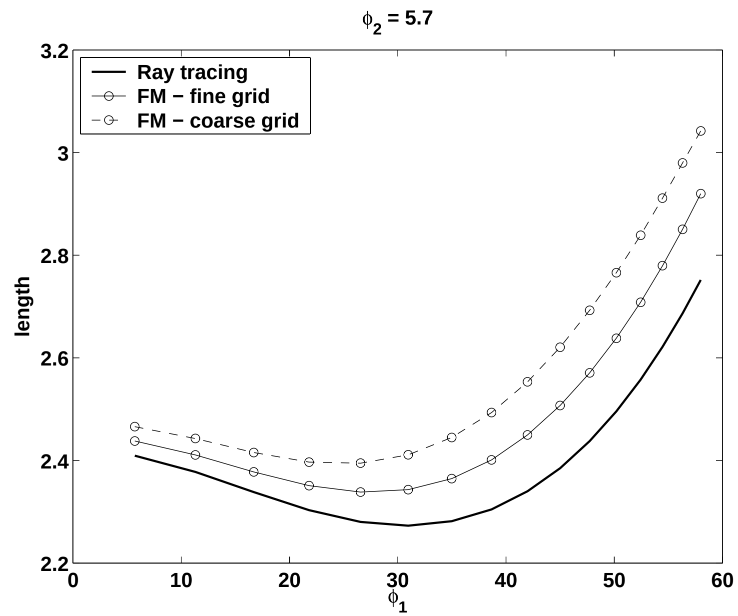


$F(\gamma(s))$ and $F(\gamma(s))+c$, example
6
5
4
γ
0
0.4
0.8
1.2
1.6
L

Length of backscatterd rays, ellipsoid



Length



Convergence

Length for all angles of incoming field (symmetric part).

Computational cost $O(N^3 \log N)$. (Postprocessing an $O(N)$ operation.)

Note: Even if initial data for all N^2 backscattered rays were known, ray tracing would cost $O(N^3)$.

Amplitude of backscattered rays

By geometrical theory of diffraction (GTD, Levy and Keller, 1959) the shedded diffracted ray has the form

$$u(t) \sim \left(\frac{d\sigma_0}{d\sigma}\right)^{1/2} e^{i\omega L(t)-\int_0^t \alpha(s)ds}.$$

Here α is the *attenuation factor*,

$$\alpha = \frac{q_0}{\rho_g} e^{i\frac{\pi}{6}} \left(\frac{\omega\rho_g}{2}\right)^{1/3},$$

where q_0 is the smallest positive zero of the Airy function and $\rho_g(u, v, \theta)$ is the radius of curvature in direction θ.

The factor $d\sigma_0/d\sigma$ is the geometrical spreading: the change in length of an infinitesimal initial creeping wavefront.

Amplitude of backscattered rays, cont

Letting β solve

$$\frac{d\beta}{d\tau} = \tilde{\alpha}(u, v, \theta), \qquad \alpha =: \omega^{1/3}\tilde{\alpha}$$

the amplitude would then be

$$|u(t)| \sim \left(\frac{d\sigma_0}{d\sigma}\right)^{1/2} e^{-\omega^{1/3}\beta}.$$

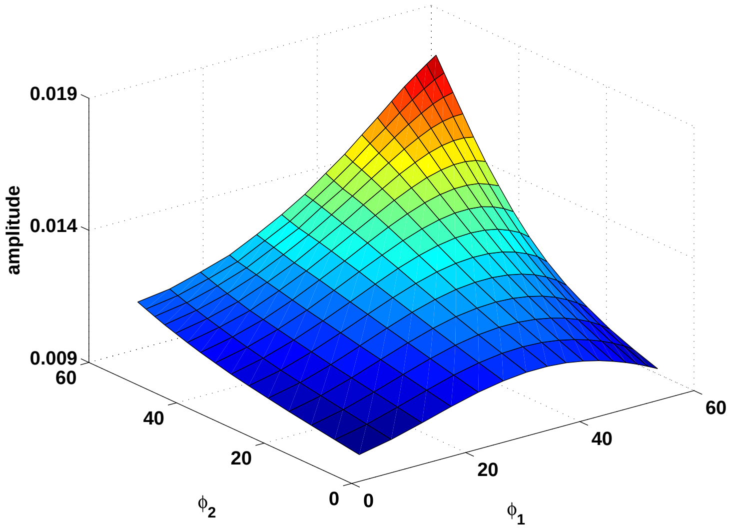
Letting B be the corresponding Eulerian variable, we get the PDE

$$\cos\theta B_u + \sin\theta B_v + V(u, v, \theta)B_\theta = \rho\tilde{\alpha}(u, v, \theta), \quad (u, v, \theta) \in \Omega_p.$$

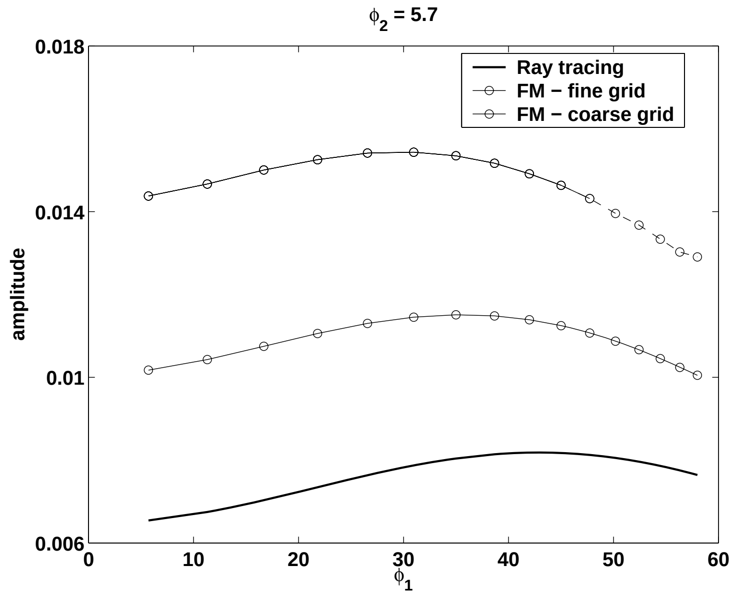
The geometrical spreading can be computed by postprocessing. Qualitatively, as

$$„ \ \frac{d\sigma_0}{d\sigma} = \frac{dX(\gamma(s))}{dX(F(\gamma(s)))}. \ "$$

Amplitude of backscatterd rays, ellipsoid



Length



Convergence

Backscattered amplitude by creeping rays for all angles of incoming field (symmetric part).