Random Geometric Acoustics: from Waves to Diffusion

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Refs: 1. BKR, Self-averaging of Wigner transforms in random media, Comm. Math. Phys., 242, 2003, 81-135.

2. KR, Diffusion in a weakly random Hamiltonian flow, to appear in Comm. Math. Phys., 2006.

3. KR, The stochastic acceleration problem in two dimensions, to appear in Israel Jour. Math., 2006.

Main sujet: mixing in time-independent random media as a source of kinetics.

1. Wave equation and kinetic models – a brief review.
2. Diffusion in the regime of random geometric optics:
from waves to rays,
from rays to ray direction diffusion,
from ray direction diffusion to position diffusion.

The Main Culprits

The wave equation:

$$\frac{1}{c^2(x)}\phi_{tt} - \Delta\phi = 0$$

The radiative transport equation:

$$w_t + c_0\hat{k}\cdot\nabla_x w = \int \sigma(k,p)\delta(c_0|k| - c_0|p|)[w(t,x,p) - w(t,x,k)]dp$$

The Liouville equation (geometrical optics):

$$w_t + \nabla_k\omega(x,k)\cdot\nabla_x w - \nabla_x\omega(x,k)\cdot\nabla_k w = 0,\ \omega(x,k) = c(x)|k|.$$

The Fokker-Planck equation (ray diffusion):

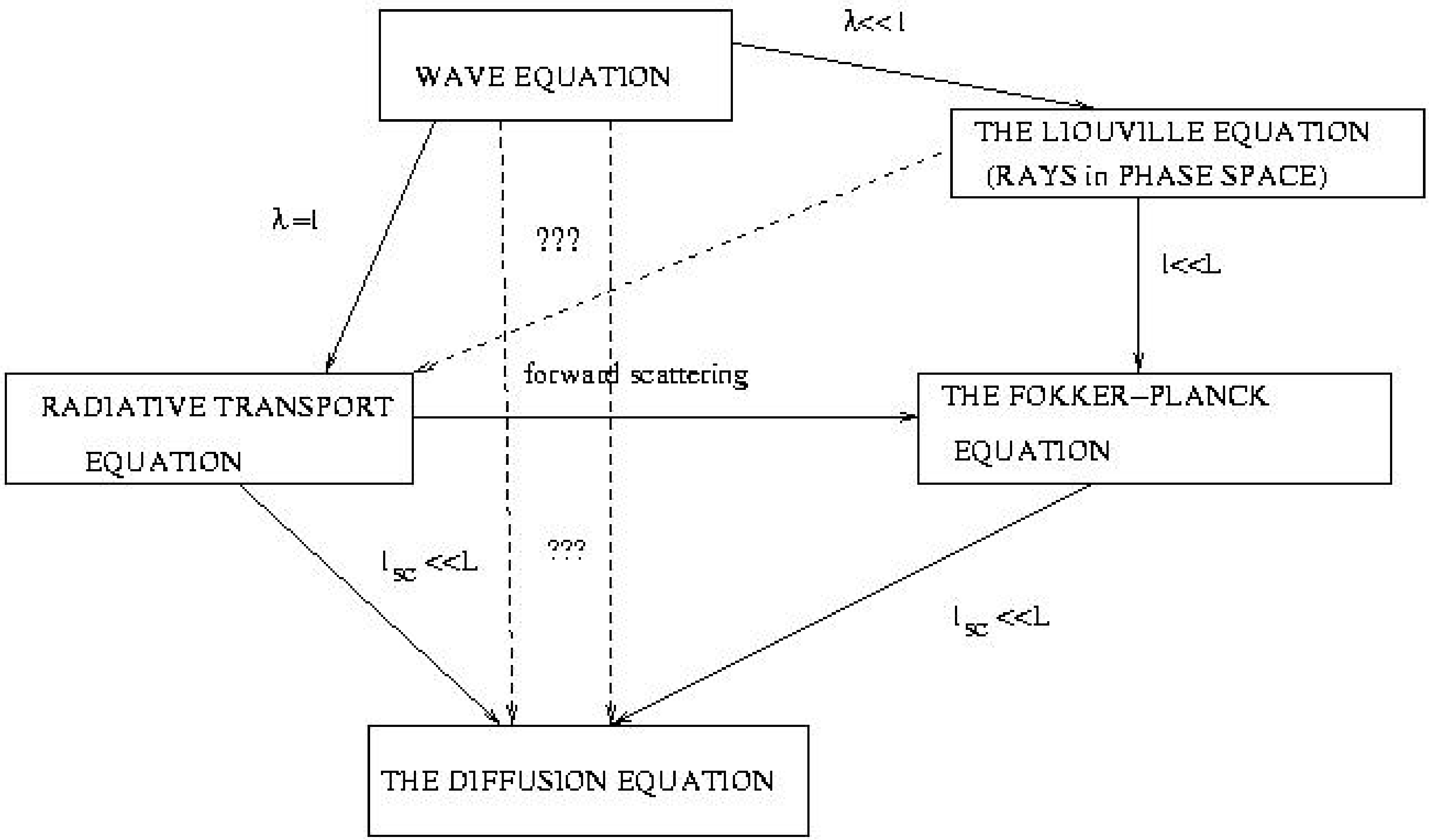
$$w_t + c_0\hat{k}\cdot\nabla_x w = \frac{\partial}{\partial k_m}\left(D_{mn}(k)\frac{\partial w}{\partial k_n}\right).$$

The diffusion equation (spatial diffusion):

$$w_t = D(|k|)\Delta_x w.$$

Scales: λ – wave length, l_{cor} – correlation length, l_{sc} – scattering mean free path, L – propagation distance.

From the wave equation to kinetic models



The Wigner transform

$$W_\varepsilon(t,x,k) = \int e^{ik\cdot y}\phi_\varepsilon\left(t, x-\frac{\varepsilon y}{2}\right)\bar{\phi}_\varepsilon\left(t, x+\frac{\varepsilon y}{2}\right)\frac{dy}{(2\pi)^d}$$

Basic properties: $\phi_\varepsilon \to 0$ weakly, but W_ε is less oscillatory

(i) $\int W_\varepsilon(t,x,k)dk = |\phi_\varepsilon(t,x)|^2 dx$.

(ii) $W_\varepsilon(t,x,k)$ converges to a measure $W(t,x,k) \geq 0$ in $\mathcal{S}'(\mathbb{R}^{2d})$.

(iii) If ϕ_ε oscillates on scales not smaller than ε then energy is captured correctly: $|\phi_\varepsilon(t,x,)|^2 \to \int W(t,x,k)dk$.

Examples.

(i) WKB: $\phi_\varepsilon(x) = A(x)e^{iS(x)/\varepsilon}$, then $W(x,k) = |A(x)|^2\delta(k - \nabla S)$

(ii) Localized data: $\phi_\varepsilon(x) = \frac{1}{\varepsilon^{d/2}}\phi\left(\frac{x}{\varepsilon}\right)$ then $W(x,k) = |\widehat{\phi}(k)|^2\delta(x)$.

Path #1: Waves-RTE-Diffusion

Quantum waves to radiative transport

$$i\varepsilon\frac{\partial\phi}{\partial t} + \frac{\varepsilon^2}{2}\Delta\phi - \sqrt{\varepsilon}V\left(\frac{x}{\varepsilon}\right)\phi = 0, \quad \varepsilon \ll 1.$$

$V(y)$ – spatially homogeneous mean-zero random process with a correlation function $R(x) = \langle V(y)V(x+y)\rangle$.

Weak fluctuations: $\sigma \sim \sqrt{\varepsilon}$ and $l_{cor} \approx \lambda \ll L$, $\varepsilon = \lambda/L$.

The Wigner transform:

$$W_\varepsilon(t,x,k) = \int e^{ik\cdot y}\phi\left(t, x - \frac{\varepsilon y}{2}\right)\bar{\phi}\left(t, x + \frac{\varepsilon y}{2}\right)\frac{dy}{(2\pi)^d} \to W(t,x,k) \geq 0.$$

The radiative transport equation:

$$\frac{\partial W}{\partial t} + k\cdot\nabla_x W = \int \hat{R}(p-k)\delta\left(\frac{k^2}{2} - \frac{p^2}{2}\right)(W(t,x,p) - W(t,x,k))\frac{dp}{(2\pi)^{d-1}}.$$

Spohn (1977) – small times convergence,

Erdös-Yau (2001), 65pp. – global in time convergence

All methods based on diagrammatic expansions and Duhamel's formula.

Classical waves to radiative transport

Start with the wave equation

$$\frac{1}{c^2(x)}\frac{\partial^2\phi}{\partial t^2} - \Delta\phi = 0$$

with $c(x) = c_0(1 + \sqrt{\varepsilon}c(x/\varepsilon))$ – again $\sigma = \sqrt{\varepsilon}$, $\lambda/L = l_{cor}/L = \varepsilon = \sigma^2$.
End up with the radiative transport equation

$$W_t + c_0\hat{k} \cdot \nabla_x W = \int \sigma(k,p)\delta(c_0|k| - c_0|p|)(W(t,x,p) - W(t,x,k))dp.$$

Physical literature – too numerous (1960's-1990's)

Formal: R., Papanicolaou, Keller (1996); Powell, Vanneste (2005); Bal (2005) – no rigorous results.

Mathematical: Lukkarinen, Spohn (2005), 71 pp. – the discrete case, $\omega(k) \geq \omega_0 > 0$ – rigorous but the true wave equation is excluded – based on diagrammatic expansions

Radiative transport to diffusion

Solution of the radiative transport equation

$$W_t + c_0 \widehat{k} \cdot \nabla_x W = \int \sigma(k,p)\delta(c_0|k| - c_0|p|)(W(t,x,p) - W(t,x,k))dp.$$

converges in the large time – large distance limit to solution of the diffusion equation

$$\bar{W}_t = D\Delta_x \bar{W}.$$

Can we go in one step?

Quantum waves directly to diffusion

Erdös, Salmhofer and Yau (2005), 124 pp.: Start with

$$i\frac{\partial \phi}{\partial t} + \frac{1}{2}\Delta\phi - \sqrt{\varepsilon}V(x)\phi = 0$$

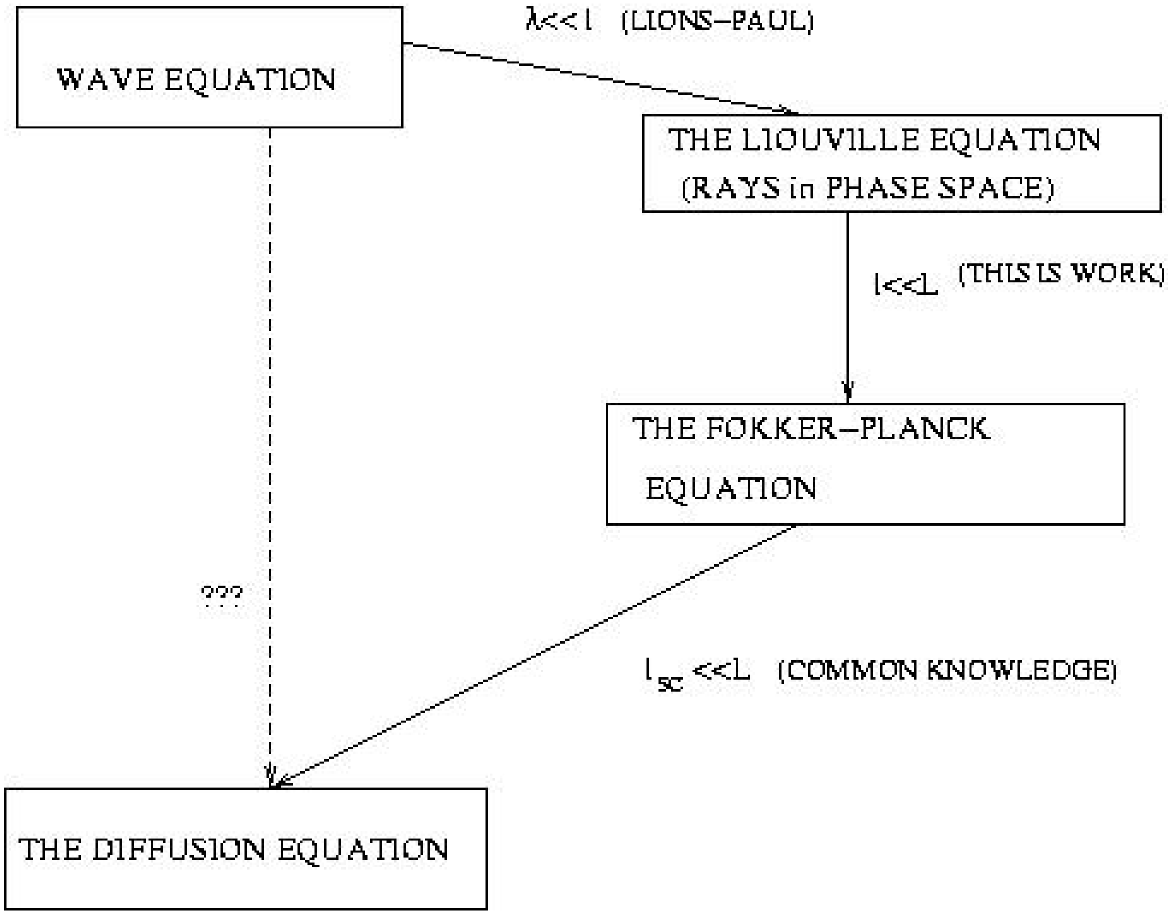
then on time-scales $T \sim \varepsilon^{-1-2\alpha}$ and $X \sim \varepsilon^{-1-\alpha}$, $0 < \alpha < 1/2000$, the Wigner transform converges to the solution of the diffusion equation. Better control of diagrammatic expansions.

Difficulties of path #1:
(i) time-independent media – correlations with the past;
(ii) waves are not local objects.

Remedies:
(i) adopt the parabolic approximation – waves always march forward and see a new medium – math becomes easy but approximation is uncontrolled.

(ii) do random geometric optics – waves become local objects, can use characteristics and particle methods – math is not very easy but easier and approximation is controlled. Price: where is my phase?

Route #2 from waves to diffusion - each step with an error bound



$\varepsilon = \lambda/L$, $\delta = l_{cor}/L$, $\gamma = l_{sc}/L$: need $\varepsilon \ll \delta \ll \gamma \ll 1$.

Level 1. Wave equation

$$\frac{1}{c^2}\phi_{tt} - \Delta\phi = 0.$$

High frequency limit $\Rightarrow$

Level 2. The Liouville equation

$$w_t + \nabla_k \omega(x,k) \cdot \nabla_x w - \nabla_x \omega(x,k) \cdot \nabla_k w = 0$$

with $\omega(x,k) = [c_0 + \sqrt{\delta} c_1(x/\delta)]|k|$. Large propagation distance $\delta \to 0 \Rightarrow$

Level 3. The Fokker-Planck equation (ray diffusion)

$$\bar{w}_t + c_0 \hat{k} \cdot \nabla_x \bar{w} = \frac{\partial}{\partial k_n}\left(D_{nm}(k)\frac{\partial \bar{w}}{\partial k_m}\right).$$

Large propagation distance – large time limit $\Rightarrow$

Level 4. The diffusion equation

$$u_t = D\Delta_x u.$$

Step 1: from the wave equation to the Liouville equations (Lions-Paul, GMMP, KPR)

Start with the acoustic system in dimension $d \geq 3$ with the sound speed $c_\delta(\mathbf{x}) = c_0 + \sqrt{\delta} c_1 \left(\frac{\mathbf{x}}{\delta}\right)$.

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \left(c_\delta(\mathbf{x}) p\right) = 0$$
$$\frac{\partial p}{\partial t} + c_\delta(\mathbf{x}) \nabla \cdot \mathbf{u} = 0.$$

Denote $\mathbf{v} = (\mathbf{u}, p) \in \mathbb{R}^{d+1}$ and write

$A_\delta(\mathbf{x}) = \mathrm{diag}(1, 1, 1, c_\delta(\mathbf{x}))$, and $D^j = \mathbf{e}_j \otimes \mathbf{e}_{d+1} + \mathbf{e}_{d+1} \otimes \mathbf{e}_j$, $j = 1, \ldots, d$.

Here $\mathbf{e}_m \in \mathbb{R}^{d+1}$ is the standard orthonormal basis: $(\mathbf{e}_m)_k = \delta_{mk}$, the acoustic system is

$$\frac{\partial \mathbf{v}}{\partial t} + \sum_{j=1}^{d} A_\delta(\mathbf{x}) D^j \frac{\partial}{\partial x^j} \left(A_\delta(\mathbf{x}) \mathbf{v}(\mathbf{x})\right) = 0.$$

The initial data: a mixture of states, oscillating on scale $\varepsilon \ll \delta$, e.g. WKB $\mathbf{v}(0, \mathbf{x}) = A(\mathbf{x})e^{iS(\mathbf{x})/\varepsilon}$ or localized – $\mathbf{v}_\varepsilon(0, \mathbf{x}) = \varepsilon^{-d/2}\mathbf{v}_0(\mathbf{x}/\varepsilon)$. The wave length ε is much smaller than the correlation length δ of the medium: $\varepsilon \ll \delta \ll 1$.

The dispersion matrix:

$$P_0^\delta(\mathbf{x}, \mathbf{k}) = i \sum_{j=1}^{d} A_\delta(\mathbf{x}) k_j D^j A_\delta(\mathbf{x}) = i \sum_{j=1}^{d} c_\delta(\mathbf{x}) k_j D^j.$$

The self-adjoint matrix $(-iP_0^\delta)$ has an eigenvalue $\omega_0 = 0$ of the multiplicity $d-1$, and two simple eigenvalues $\omega_\pm^\delta(\mathbf{x}, \mathbf{k}) = \pm c_\delta(\mathbf{x})|\mathbf{k}|$. Its eigenvectors are

$$\mathbf{b}_m^0 = \left(\mathbf{k}_m^\perp, 0\right), \ m = 1, \ldots, d-1; \quad \mathbf{b}_\pm = \frac{1}{\sqrt{2}}\left(\frac{\tilde{\mathbf{k}}}{|\mathbf{k}|} \pm \mathbf{e}_{d+1}\right),$$

where $\mathbf{k}_m^\perp \in \mathbb{R}^d$ is the orthonormal basis of vectors orthogonal to $\mathbf{k}$.

The $(d+1)\times(d+1)$ Wigner matrix of a mixture of solutions is

$$W_\varepsilon^\delta(t,\mathbf{x},\mathbf{k}) = \frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d}\int\limits_{S} e^{i\mathbf{k}\cdot\mathbf{y}}\mathbf{v}_\varepsilon^\delta(t,\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2};\zeta)\otimes\mathbf{v}_\varepsilon^{\delta *}(t,\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2};\zeta)d\mathbf{y}\mu(d\zeta).$$

It is well-known that for each fixed $\delta>0$ $W_\varepsilon^\delta(t)$ converges weakly to

$$U^\delta(t,\mathbf{x},\mathbf{k}) = u_+^\delta(t,\mathbf{x},\mathbf{k})\mathbf{b}_+(\mathbf{k})\otimes\mathbf{b}_+(\mathbf{k}) + u_-^\delta(t,\mathbf{x},\mathbf{k})\mathbf{b}_-(\mathbf{k})\otimes\mathbf{b}_-(\mathbf{k}).$$

The scalar amplitudes $u^\delta = u_+^{(\delta)}$ satisfy the Liouville equations:

$$\partial_t u^\delta + \nabla_{\mathbf{k}}H^\delta\cdot\nabla_{\mathbf{x}}u^\delta - \nabla_{\mathbf{x}}H^\delta\cdot\nabla_{\mathbf{k}}u^\delta = 0, \qquad H^\delta(\mathbf{x},\mathbf{k}) = c_\delta(\mathbf{x})|\mathbf{k}|.$$

One may obtain an L^2-error estimate for this convergence with a mixture of states. In order to make the scale separation $\varepsilon \ll \delta \ll 1$ precise we define the set

$$\mathcal{K}_\mu := \left\{(\varepsilon,\delta):\quad |\ln\varepsilon|^{-2/3+\mu}\le\delta\le 1\right\}.$$

The parameter μ is a fixed number in the interval $(0,2/3)$.

Proposition. (Mundane estimates) *Let the acoustic speed* $c_\delta(\mathbf{x}) = c_0 + \sqrt{\delta} c_1(x/\delta)$ *with a nice function* $c_1(y)$. *Assume that the initial data* $W_\varepsilon^\delta(0, \mathbf{x}, \mathbf{k}) \to W_0(\mathbf{x}, \mathbf{k})$ *strongly in* $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ *as* $\mathcal{K}_\mu \ni (\varepsilon, \delta) \to 0$. *Also assume that the limit* $W_0 \in C_c^2(\mathbb{R}_*^{2d})$ *with a support inside* $0 < \alpha \leq |k| \leq \beta < +\infty$, *and is of the form*

$$W_0(\mathbf{x}, \mathbf{k}) = \sum_{q=\pm} u_q^0(\mathbf{x}, \mathbf{k}) \Pi_q(\mathbf{k}), \quad \Pi_q(\mathbf{k}) = \mathbf{b}_q(\mathbf{k}) \otimes \mathbf{b}_q(\mathbf{k}).$$

Let $U^\delta(t, \mathbf{x}, \mathbf{k}) = \sum_{p=\pm} u_p^\delta(t, \mathbf{x}, \mathbf{k}) \Pi_p(\mathbf{k})$, *where the functions* u_p^δ *satisfy the Liouville equations. Then there exists a constant* $C_1 > 0$ *that is independent of* δ *so that*

$$\|W_\varepsilon^\delta(t, \mathbf{x}, \mathbf{k}) - U^\delta(t, \mathbf{x}, \mathbf{k})\|_2 \leq C(\delta) \left(\varepsilon \|W_0\|_{H^2} e^{C_1 t/\delta^{3/2}} + \varepsilon^2 \|W_0\|_{H^3} e^{C_1 t/\delta^{3/2}} \right)$$
$$+ \|W_\varepsilon^\delta(0) - W_0\|_2,$$

where $C(\delta)$ *is a rational function of* δ *with deterministic coefficients.*

The main result: from the Liouville equation to the spatial diffusion equation

$$\frac{\partial w}{\partial t} = \sum_{m,n=1}^{d} a_{mn}(k) \frac{\partial^2 w}{\partial x_n \partial x_m},$$

with the diffusion matrix $a_{nm}(k) = \frac{c_0}{\Gamma_{d-1}} \int_{\mathbb{S}^{d-1}} l_n \chi_m(k\mathbf{l}) d\Omega(\mathbf{l})$, and the functions χ_j above are the mean-zero solutions of

$$\sum_{m,n=1}^{d} \frac{\partial}{\partial k_m} \left(k^2 D_{mn}(\hat{\mathbf{k}}) \frac{\partial \chi_j}{\partial k_n} \right) = -c_0 \hat{k}_j.$$

Suppose also that $u_0^{\pm} \in C_c^3(\mathbb{R}_*^{2d})$ and supp $u_0^{\pm} \subseteq \mathcal{A}(M)$. Let

$$W^0(\mathbf{x}, \mathbf{k}) := u_+^0(\mathbf{x}, \mathbf{k}) \mathbf{b}_+(\mathbf{k}) \otimes \mathbf{b}_+(\mathbf{k}) + u_-^0(\mathbf{x}, \mathbf{k}) \mathbf{b}_-(\mathbf{k}) \otimes \mathbf{b}_-(\mathbf{k}),$$
$$W(t, \mathbf{x}, \mathbf{k}) := w_+(t, \mathbf{x}; \mathbf{k}) \mathbf{b}_+(\mathbf{k}) \otimes \mathbf{b}_+(\mathbf{k}) + w_-(t, \mathbf{x}; \mathbf{k}) \mathbf{b}_-(\mathbf{k}) \otimes \mathbf{b}_-(\mathbf{k}).$$

Theorem. *Let* $\mathcal{K}_{\mu,\rho} := \left\{(\varepsilon,\delta,\gamma) : \quad \delta \geq |\ln\varepsilon|^{-2/3+\mu} \text{ and } \gamma \geq \delta^{\rho}\right\}$, *with* $0 < \mu < 2/3$, $\rho \in (0,1)$. *Assume that the dimension* $d \geq 3$. *Suppose for some* $0 < \mu < 2/3$, $\rho \in (0,1)$ *we have initially*

$$\int_{\mathbb{R}^{2d}} \left| \mathbb{E}W_{\varepsilon}^{\delta}\left(0, \frac{\mathbf{x}}{\gamma}, \mathbf{k}\right) - W^{0}(\mathbf{x},\mathbf{k}) \right|^{2} d\mathbf{x}d\mathbf{k} \to 0, \text{ as } (\varepsilon,\delta,\gamma) \to 0 \text{ and } (\varepsilon,\delta,\gamma) \in \mathcal{K}_{\mu,\rho}.$$

Then, there exists $\rho_1 \in (0,\rho]$ *such that for any* $T > T_* > 0$ *we have*

$$\sup_{t\in[T_*,T]} \int \left| \mathbb{E}W_{\varepsilon}^{\delta}\left(\frac{t}{\gamma^2}, \frac{\mathbf{x}}{\gamma}, \mathbf{k}\right) - W(t,\mathbf{x},\mathbf{k}) \right|^{2} d\mathbf{x}d\mathbf{k} \to 0, \text{ as } (\varepsilon,\delta,\gamma) \to 0$$

and $(\varepsilon,\delta,\gamma) \in \mathcal{K}_{\mu,\rho_1}$. *Here*

$$W(t,\mathbf{x},\mathbf{k}) := w_{+}(t,\mathbf{x};\mathbf{k})\mathbf{b}_{+}(\mathbf{k}) \otimes \mathbf{b}_{+}(\mathbf{k}) + w_{-}(t,\mathbf{x};\mathbf{k})\mathbf{b}_{-}(\mathbf{k}) \otimes \mathbf{b}_{-}(\mathbf{k}).$$

with the functions $w_{\pm}$ *that satisfy the spatial diffusion equation with the initial data* $w_{\pm}(0,\mathbf{x},\mathbf{k}) = \bar{u}^{0}_{\pm}(\mathbf{x},\mathbf{k})$.

Step 3. From the Fokker-Planck to diffusion – very straightforward.
Start with the Fokker-Planck equation

$$\frac{\partial \bar{\phi}}{\partial t} + c_0 \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} \bar{\phi} = \sum_{m,n=1}^{d} \frac{\partial}{\partial k_m} \left(D_{mn}(\hat{\mathbf{k}}, k) \frac{\partial \bar{\phi}}{\partial k_n} \right).$$

Then as $\gamma \to 0$, $\bar{\phi}_\gamma(t/\gamma^2, \mathbf{x}/\gamma, k) \to w(t, \mathbf{x}, k)$ that satisfies

$$\frac{\partial w}{\partial t} = \sum_{m,n=1}^{d} a_{mn}(k) \frac{\partial^2 w}{\partial x_n \partial x_m}$$

with the diffusion matrix $a_{nm}(k) = \frac{1}{\Gamma_{d-1}} \int_{\mathbb{S}^{d-1}} H_0'(k) l_n \chi_m(k\mathbf{l}) d\Omega(\mathbf{l})$. The functions χ_j are the mean-zero solutions of

$$\sum_{m,n=1}^{d} \frac{\partial}{\partial k_m} \left(D_{mn}(\hat{\mathbf{k}}, k) \frac{\partial \chi_j}{\partial k_n} \right) = c_0 \hat{k}_j.$$

with the error estimate

$$\|w(t) - \bar{\phi}_\gamma(t)\|_{L^\infty(\mathcal{A}(M))} \le C\left(\gamma T + \gamma^{1/2}\right) \|\phi_0\|_{2,0}, \quad 0 \le t \le T.$$

Step 2. From the Liouville equation to the Fokker-Planck equation.

In a "random geometric acoustics" medium $c(x) = c_0 + \sqrt{\delta} c_1(x/\delta)$ and

$$\frac{\partial w}{\partial t} + (c_0 + \sqrt{\delta} c_1(x/\delta))\hat{k} \cdot \nabla_x w - \frac{|k|}{\sqrt{\delta}} \nabla c_1 \left(\frac{x}{\delta}\right) \cdot \nabla_k w = 0.$$

Trajectories: $\frac{dX}{dt} = (c_0 + \sqrt{\delta} c_1(X/\delta))\hat{K}$, $\frac{dK}{dt} = -\frac{|K|}{\sqrt{\delta}} \nabla c_1 \left(\frac{X}{\delta}\right)$ – a weakly perturbed random Hamiltonian flow with $H(\mathbf{x}, \mathbf{k}) = [c_0 + \sqrt{\delta} c_1(\mathbf{x}/\delta)]|k|$.

Diffusion in a random Hamiltonian flow

The Hamiltonian $H_\delta(\mathbf{x}, \mathbf{k}) = H_0(|k|) + \sqrt{\delta} H_1(\mathbf{x}, |k|)$

1. The background Hamiltonian $H_0(|k|) \in C^3_{loc}(\mathbb{R}^d)$ depends only on $|k|$, and is uniform in space. Moreover, $H_0 : [0, +\infty) \to \mathbb{R}$ is a strictly increasing C^3-function satisfying $H_0(0) \geq 0$ and $H_0'(k) > 0$ for all $k > 0$,

Examples: the quantum Hamiltonian $H_0(k) = k^2/2$ (Kesten-Papanicolaou without the error estimate), the acoustic wave Hamiltonian $H_0(k) = c_0 k$.

2. The random medium.

(a) The realizations of $H_1(\mathbf{x},k)$ are $\mathbb{P}$–a.s. C^2-smooth in and
$D_{i,j}(M) := \max_{|\alpha|=i} \operatorname{ess-sup}_{(\mathbf{x},k,\omega)\in\mathbb{R}^d\times[M^{-1},M]\times\Omega} |\partial_{\mathbf{x}}^{\alpha}\partial_k^j H_1(\mathbf{x},k;\omega)| < +\infty$, $i,j=0,1,2$.

(b) Mixing. The random field is strongly mixing in the uniform sense: for any $R>0$ let $\mathcal{C}_R^i$ and $\mathcal{C}_R^e$ be the σ–algebras generated by random variables $H_1(\mathbf{x},k)$ with $k\in[0,+\infty)$, $\mathbf{x}\in\mathbb{B}_R$ and $\mathbf{x}\in\mathbb{B}_R^c$ respectively. The uniform mixing coefficient between the σ–algebras is

$\phi(\rho) := \sup[\,|\mathbb{P}(B)-\mathbb{P}(B|A)|:\ R>0,\ A\in\mathcal{C}_R^i,\ B\in\mathcal{C}_{R+\rho}^e],$

for all $\rho>0$. We suppose that $\phi(\rho)$ decays faster than any power: for each $p>0$ $h_p := \sup_{\rho\ge 0}\rho^p\phi(\rho) < +\infty$.

3. Two-point correlations: $H_1(\mathbf{x},|k|)$ is a mean-zero stationary (in $\mathbf{x}$) random field with the correlation function $R(\mathbf{y},k) := \mathbb{E}[H_1(\mathbf{y},k)H_1(\mathbf{z},k)] \in C^\infty$. Example: $H_1(\mathbf{x},k) = c_1(\mathbf{x})h(k)$, where $c_1(\mathbf{x})$ is a stationary uniformly mixing random field with a smooth correlation function, and $h(k)$ is a smooth deterministic function.

The main theorem. Let $\phi_\delta(t,\mathbf{x},\mathbf{k})$ satisfy the Liouville equation

$$\frac{\partial \phi^\delta}{\partial t} + \nabla_{\mathbf{k}} H_\delta\left(\mathbf{x},\mathbf{k}\right)\cdot\nabla_{\mathbf{x}}\phi^\delta - \nabla_{\mathbf{x}} H_\delta\left(\mathbf{x},\mathbf{k}\right)\cdot\nabla_{\mathbf{k}}\phi^\delta = 0,$$

$$\phi^\delta(0,\mathbf{x},\mathbf{k}) = \phi_0(\delta\mathbf{x},\mathbf{k}) \in C_c^4.$$

$\mathrm{supp}\phi_0 \subseteq \mathcal{A}(M) = \{(\mathbf{x},\mathbf{k}) : M^{-1} < |\mathbf{k}| < M\}$ for some positive $M > 0$.

The diffusion matrix:

$$D_{mn}(\hat{\mathbf{k}},l) = -\frac{1}{2}\int_{-\infty}^{\infty}\frac{\partial^2 R(H_0'(l)s\hat{\mathbf{k}},l)}{\partial x_n \partial x_m}ds = -\frac{1}{2H_0'(l)}\int_{-\infty}^{\infty}\frac{\partial^2 R(s\hat{\mathbf{k}},l)}{\partial x_n \partial x_m}ds$$

Theorem Let $\bar\phi$ satisfy

$$\frac{\partial\bar\phi}{\partial t} + H_0'(k)\,\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}}\bar\phi = \sum_{m,n=1}^{d}\frac{\partial}{\partial k_m}\left(D_{mn}(\hat{\mathbf{k}},k)\frac{\partial\bar\phi}{\partial k_n}\right)$$

and $\bar\phi(0,\mathbf{x},\mathbf{k}) = \phi_0(\mathbf{x},\mathbf{k})$. Suppose that $M \geq M_0 > 0$ and $T \geq T_0 > 0$. Then, there exist two constants C, $\alpha_0 > 0$ such that for all $T \geq T_0$

$$\sup_{(t,\mathbf{x},\mathbf{k})\in[0,T]\times K}\left|\mathbb{E}\phi^\delta\left(\frac{t}{\delta},\frac{\mathbf{x}}{\delta},\mathbf{k}\right) - \bar\phi(t,\mathbf{x},\mathbf{k})\right| \leq CT(1+\|\phi_0\|_{1,4})\delta^{\alpha_0}$$

for all compact sets $K \subset \mathcal{A}(M)$.

Outline of the proof. Characteristics:

$$\frac{dX}{dt} = -(c_0 + \sqrt{\delta} c_1(X(t)/\delta))\widehat{K}(t),$$
$$\frac{dK}{dt} = \frac{|K(t)|}{\sqrt{\delta}} \nabla c_1(X(t)/\delta), \quad X(0) = \mathbf{x}, \quad K(0) = \mathbf{k}.$$

Then solution of the Liouville equation is $w^\delta(t, \mathbf{x}, \mathbf{k}) = W_0(X(t), K(t))$.

The strategy is a modification of the idea of Kesten and Papanicolaou.

1. The problem is self-intersections – create correlations with the past.
2. Define a stopping time τ_δ s.t. no self-intersections until $t = \tau_\delta$.
3. Introduce an augmented process that coincides with the true trajectories until $t = \tau_\delta$ but becomes the limit diffusion after the stopping time.
4. Show that the modified process converges to the right limit.
5. The stopping time for the diffusion tends to infinity as $\delta \to 0$.
6. Hence $\tau_\delta \to \infty$ for the modified process as $\delta \to 0$.
7. Hence the original process does not do self-intersections.
8. Hence the original process converges to the right limit.

The stopping time:

returning back to a sausage around $X(s)$, $0 \leq s \leq t_{k-1}$, $t_k = k/p$ - mesh of times, $1/q$ – sausage width, $q \gg p$

$$U_\delta(\pi) := \inf \left[t \geq 0 : \exists\, k \geq 1 \text{ and } t \in [t_k^{(p)}, t_{k+1}^{(p)}) \text{ for which } X(t) \in \mathfrak{X}_{t_{k-1}^{(p)}}(q) \right].$$

The "violent turn" stopping time (prevents turning around quickly)

$$S_\delta(\pi) := \inf \Bigg[t \geq 0 : \text{ for some } k \geq 0 \text{ we have } t \in \left[t_k^{(p)}, t_{k+1}^{(p)} \right) \text{ and }$$
$$\widehat{K}(t_{k-1}^{(p)}) \cdot \widehat{K}(t) \leq 1 - \frac{1}{N}, \text{ or } \widehat{K}\left(t_k^{(p)} - \frac{1}{N_1} \right) \cdot \widehat{K}(t) \leq 1 - \frac{1}{N} \Bigg],$$

we set the stopping time

$$\tau_\delta(\pi) := S_\delta(\pi) \wedge U_\delta(\pi).$$

The modified dynamics

We set

$$F_\delta(t, \boldsymbol{y}, \mathrm{l}; \pi, \omega) = \Theta(t, \delta \boldsymbol{y}, \mathrm{l}; \pi) \nabla_{\boldsymbol{y}} c_1 \left(\boldsymbol{y}, |\mathbf{l}|; \omega\right) |\mathbf{l}|.$$

The modified process $(\boldsymbol{y}^{(\delta)}(t; \mathbf{x}, \mathbf{k}, \omega), \boldsymbol{l}^{(\delta)}(t; \mathbf{x}, \mathbf{k}, \omega))_{t \geq 0}$ solves

$$\begin{cases} \frac{d\boldsymbol{y}^{(\delta)}(t;\mathbf{x},\mathbf{k})}{dt} = \left[c_0 + \sqrt{\delta}\, c_1 \left(\frac{\boldsymbol{y}^{(\delta)}(t;\mathbf{x},\mathbf{k})}{\delta}, \right) \right] \hat{\boldsymbol{l}}^{(\delta)}(t; \mathbf{x}, \mathbf{k},) \\ \frac{d\boldsymbol{l}^{(\delta)}(t;\mathbf{x},\mathbf{k})}{dt} = -\frac{1}{\sqrt{\delta}} F_\delta \left(t, \frac{\boldsymbol{y}^{(\delta)}(t;\mathbf{x},\mathbf{k})}{\delta}, \boldsymbol{l}^{(\delta)}(t; \mathbf{x}, \mathbf{k}); \mathbf{y}^{(\delta)}(\cdot; \mathbf{x}, \mathbf{k}), \boldsymbol{l}^{(\delta)}(\cdot; \mathbf{x}, \mathbf{k}) \right) \\ \boldsymbol{y}^{(\delta)}(0; \mathbf{x}, \mathbf{k}) = \mathbf{x}, \quad \boldsymbol{l}^{(\delta)}(0; \mathbf{x}, \mathbf{k}) = \mathbf{k}. \end{cases}$$

The cut-off Θ does two things:

(i) You shall not turn back violently.

(ii) You shall go across the past sausage along a straight line. This prevents correlation gain.

The Kesten-Papanicolaou mixing lemma: the closure problem – how to split $\langle VVW\rangle$?

Suppose Z, g_1, g_2, are $\mathcal{F}_t$-measurable, while $\tilde{X}_1, \tilde{X}_2$ are random fields of the form $\tilde{X}_i(\mathbf{x}, k) = X_i\left(\left(c_1(\mathbf{x}), \nabla_{\mathbf{x}} c_1(\mathbf{x}), \nabla^2_{\mathbf{x}} c_1(\mathbf{x})\right)\right)$.

We also let $U(\theta_1, \theta_2) := \mathbb{E}\left[\tilde{X}_1(\theta_1)\tilde{X}_2(\theta_2)\right]$.

The mixing lemma. *Assume that* $r, t \geq 0$ *and* $\inf_{u \leq t} \left| g_i - \frac{\mathbf{y}^{(\delta)}(u)}{\delta} \right| \geq \frac{r}{\delta}$, $\mathbb{P}$*–a.s. on the set* $Z \neq 0$ *for* $i = 1, 2$*. Then, we have*

$$\left|\mathbb{E}\left[\tilde{X}_1(g_1)\tilde{X}_2(g_2)Z\right] - \mathbb{E}\left[U(g_1, g_2)Z\right]\right| \leq C\phi\left(\frac{r}{2\delta}\right)\|X_1\|_{L^\infty}\|X_2\|_{L^\infty}\|Z\|_{L^1(\Omega)}.$$

This allows to avoid infinite Duhamel expansions in the weak coupling regime – replaces the diagrams. You shall iterate only twice and then use mixing to show "$\langle VVW\rangle = \langle VV\rangle\langle W\rangle +$ small".

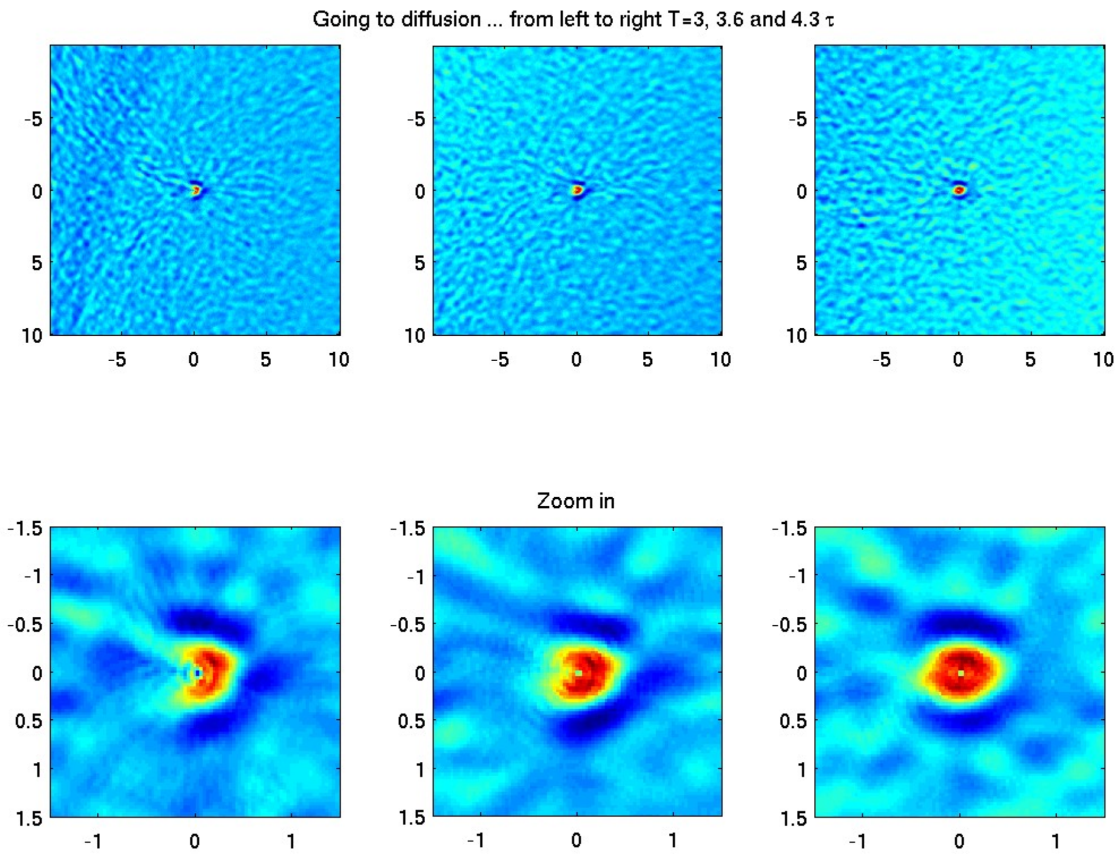
The dream: mixing lemma without trajectories – but waves ain't local objects.

The two-dimensional case

Can show convergence of characteristics to the Fokker-Planck momentum diffusion – did not try to extend to the spatial diffusion but should be possible.

The key point. The main difficulty in 2D – self-intersections – so allow them but prohibit non-transverse self-intersections. This produces a controlled small correlation gain that vanishes in the limit.

A beautiful picture



Some open questions.

1. The radiative transport regime – the mixing lemma?
2. Convergence in probability and self-averaging – doable but tedious.
3. Low-dimensional noise in a higher dimensional Hamiltonian system.
4. Bounded domains – uniformization of eigenfunctions.
5. Application of the kinetic equations to the inverse problems.