Remarks on the blow-up criterion of the 3D incompressible Euler equations

Dongho Chae

Sungkyunkwan University
Korea

1. Introduction

• We are concerned on the Euler equations for the homogeneous incompressible fluid flows in $\mathbb{R}^3 \times (0, \infty)$.

$$(E)\begin{cases} \dfrac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p \\ \text{div } v = 0, \end{cases}$$

where $v = (v^1, v^2, v^3)$, $v^j = v^j(x,t)$, $j = 1, 2, 3$ is the velocity of the fluid flows, $p = p(x,t)$ is the scalar pressure.

• Taking curl of the momentum equation we obtain the vorticity formulation.

$$\begin{cases} \dfrac{\partial \omega}{\partial t} + (v \cdot \nabla)\omega = \omega \cdot \nabla v \\ \text{div } v = 0, \quad \text{curl } v = \omega \cdots\cdots (*) \end{cases}$$

• The elliptic system (*) can be solved to give the Biot-Savart law

$$v(x,t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \times \omega(y,t)}{|x-y|^3} dy.$$

• Construction of local solutions in many function spaces:

◇ Kato, Ebib-Marsden, Bourguignon-Brezis, Temam($H^m, H^s, W^{k,p}$, bounded or whole of $\mathbb{R}^3$, or on Riemannian manifold), Kato-Ponce($L^{s,p}$), Lichtenstein, Chemin(C^s), M. Vishik, C. ($B^s_{p,q}, F^s_{p,q}$), . . .

• Outstanding open question:

Finite time blow-up or not of the local solutions ?

• Beale-Kato-Majda(BKM) criterion:

$$\limsup_{t \nearrow T_*} \|v(t)\|_{H^m} = \infty \Longleftrightarrow \int_0^{T_*} \|\omega(s)\|_{L^\infty} ds = \infty$$

• Refinements using the ‘slightly weaker’ spaces than L^∞ for vorticity:
◇ Kozono-Taniuchi(BMO), C. ($\dot{F}^0_{\infty,\infty}$), C., Kozono-Ogawa-Taniuchi($\dot{B}^0_{\infty,\infty}$)

• In this talk we are concerned on reducing number of components of the vorticity to control blow-up for flows with or without axisymmetry.

Definition of function space $\dot{B}^0_{\infty,1}$.

Given $f \in \mathcal{S}$ its Fourier transform $\widehat{f}$ is defined by

$$\mathcal{F}(f) = \widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x) dx.$$

We consider $\varphi \in \mathcal{S}$ satisfying

(i) $\operatorname{Supp} \widehat{\varphi} \subset \{\xi \in \mathbb{R}^n \,|\, \frac{1}{2} \le |\xi| \le 2\}$,

(ii) $\widehat{\varphi}(\xi) \ge C > 0$ if $\frac{2}{3} < |\xi| < \frac{3}{2}$,

(iii) $\sum_{j\in\mathbb{Z}} \widehat{\varphi}_j(\xi) = 1$, where $\widehat{\varphi}_j = \widehat{\varphi}(2^{-j}\xi)$.

Note that $\widehat{\varphi}_j$ is supported on the annulus of radius about 2^j.

• Then, $\dot{B}^0_{\infty,1}$ is defined by

$$f \in \dot{B}^0_{\infty,1} \Longleftrightarrow \|f\|_{\dot{B}^0_{\infty,1}} = \sum_{j\in\mathbb{Z}} \|\varphi_j * f\|_{L^\infty} < \infty,$$

where $*$ is the standard notation for convolution, $(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$.

• Note that (iii)(partition of unity) above implies immediately that $\dot{B}^0_{\infty,1} \hookrightarrow L^\infty$.

• In fact $\dot{B}^0_{\infty,1}$ can be regarded as 'slightly' regular function class than L^∞, where the Calderon-Zygmund SIO operates well.

2. Main results

(i)Case without any symmetry:

Theorem 1 *Let $m > 5/2$.*
Suppose $v \in C([0, T_1); H^m(\mathbb{R}^3))$ is the local classical solution of (E) for some $T_1 > 0$, corresponding to the initial data $v_0 \in H^m(\mathbb{R}^3)$, and $\omega =$ curl v is its vorticity. We decompose $\omega = \tilde{\omega} + \omega^3 e_3$, where $\tilde{\omega} = \omega^1 e_1 + \omega^2 e_2$, and $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3$. Then,

$$\limsup_{t \nearrow T} \|v(t)\|_{H^m} = \infty \Leftrightarrow \int_0^T \|\tilde{\omega}(t)\|^2_{\dot{B}^0_{\infty,1}} dt = \infty.$$

Remark 1.1. For plane flows $\tilde{\omega} \equiv 0$. Hence, as a trivial corollary of the above theorem we obtain the global regularity for the 2-D Euler equations.

Remark 1.2. It would be interesting to improve the above result by replacing $\|\tilde{\omega}\|_{\dot{B}^0_{\infty,1}}$ by $\|\tilde{\omega}\|_{L^\infty}$, or even $\|\tilde{\omega}\|_{\dot{B}^0_{\infty,\infty}}$.

For the 3D Navier-Stokes case it was possible to obtain the Serrin type of regularity criterion by $\|\tilde{\omega}\|_{L^{p,q}_T}$, where $L^{p,q}_T = L^p(0,T : L^q(\mathbb{R}^3))$, $\frac{2}{p} + \frac{3}{q} = 1$ is the scale invariant space.
(C. and H-J. Choe, '99)

(ii) Axisymmetric case(with nonzero swirl):

Theorem 2 *Let v be the local classical axisymmetric solution of the 3-D Euler equations, corresponding to an axisymmetric initial data $v_0 \in H^m(\mathbb{R}^3)$. We decompose $\omega = \tilde{\omega} + \vec{\omega}_\theta$, where $\tilde{\omega} = \omega^r e_r + \omega^3 e_3$ and $\vec{\omega}_\theta = \omega^\theta e_\theta$. Then,*

$$\limsup_{t \nearrow T} \|v(t)\|_{H^m} = \infty \Leftrightarrow \int_0^T \|\vec{\omega}_\theta(t)\|_{\dot{B}^0_{\infty,1}} dt = \infty.$$

Remark 1.3. Similar remarks to Remark 1.2. We note that for the axisymmetric 3-D Navier-Stokes equations with swirl it is possible to control the regularity only by $\|\vec{\omega}_\theta\|_{L_T^{p,q}}$ with $\frac{2}{p}+\frac{3}{q}=1$.(C. and J. Lee, '02)

Remark 1.4. Compare with the previous result(C. and N. Kim, '96):
The blow-up is controlled by the integral,

$$\int_0^T \|\vec{\omega}_\theta(t)\|_{L^\infty}\left[1+\log^+(\|\vec{\omega}_\theta(t)\|_{C^{0,\gamma}})\right]dt.$$

2. Ouline of Proofs

(i)Case without any symmetry:

• Multiply the vorticity equation by e_3,

$$\frac{\partial \omega^3}{\partial t} + (v \cdot \nabla)\omega^3 = (\omega \cdot \nabla)v \cdot e_3 \qquad (E_3)$$

• We consider the particle trajectory mapping $X(\alpha, t)$ defined by

$$\frac{\partial X(\alpha, t)}{\partial t} = v(X(\alpha, t), t), \quad X(\alpha, 0) = \alpha \in \mathbb{R}^3.$$

• Integrating (E_3) along $X(\alpha, t)$, we have

$$\omega^3(X(\alpha, t), t) = \omega_0^3(\alpha) + \int_0^t [(\omega \cdot \nabla)v \cdot e_3](X(\alpha, s), s)ds.$$

• Taking supremum over $\alpha \in \mathbb{R}^3$ yields

$$\|\omega^3(t)\|_{L^\infty} \le \|\omega_0^3\|_{L^\infty} + \int_0^t \|(\omega \cdot \nabla)v \cdot e_3](s)\|_{L^\infty} ds.$$

• Below we estimate the vortex stretching term,

$$(\omega \cdot \nabla) v \cdot e_3$$

pointwise.

• From the Biot-Savart law we compute

$$\begin{aligned}
\frac{\partial v^i}{\partial x_j}(x,t) &= \frac{1}{4\pi} \sum_{l,m=1}^{3} \epsilon_{jlm} PV \int_{\mathbb{R}^3} \left\{ \frac{\delta_{il}}{|y|^3} - 3\frac{y_i y_l}{|y|^5} \right\} \cdot \\
&\qquad \cdot \omega_m(x+y,t)\, dy \\
&\qquad -\frac{1}{3} \sum_{l=1}^{3} \epsilon_{ijl} \omega_l(x,t) \\
&:= \mathcal{P}_{ij}(\omega)(x,t),
\end{aligned}$$

where PV denotes the principal value of the integrals, and ϵ_{jlm} is the skew symmetric tensor with the normalization $\epsilon_{123} = 1$.

• We note that $\mathcal{P}_{ij}(\cdot)$ is a matrix valued singular integral operator of the Calderon-Zygmund type.

• We compute explicitly the vortex stretching term:

$$[(\omega\cdot\nabla)v\cdot e_3](x,t) = \sum_{i,j=1}^{3}\omega_i(x,t)\frac{\partial v^i}{\partial x_j}(x,t)(e_3)_j$$

$$= \frac{1}{4\pi}PV\int_{\mathbb{R}^3}\Big\{\frac{\omega(x,t)\times\omega(x+y,t)}{|y|^3}\cdot e_3$$

$$-3\frac{y\times\omega(x+y,t)}{|y|^5}\cdot e_3\,(y\cdot\omega(x,t))\Big\}\,dy$$

(Set $\omega = \tilde{\omega} + \omega^3 e_3$)

$$= \frac{1}{4\pi}PV\int_{\mathbb{R}^3}\Big\{\frac{\tilde{\omega}(x,t)\times\tilde{\omega}(x+y,t)}{|y|^3}\cdot e_3$$

$$-3\frac{y\times\tilde{\omega}(x+y,t)}{|y|^5}\cdot e_3\; y_3\omega_3(x,t)$$

$$-3\frac{y\times\tilde{\omega}(x+y,t)}{|y|^5}\cdot e_3\,(y\cdot\tilde{\omega}(x,t))\Big\}\;dy$$

$$= \sum_{i,j=1}^{3}\tilde{\omega}_i(x,t)\mathcal{P}_{ij}(\tilde{\omega})(x,t)(e_3)_j$$

$$+\sum_{i,j=1}^{3}\omega^3(x,t)(e_3)_i\mathcal{P}_{ij}(\tilde{\omega})(x,t)(e_3)_j.$$

• We have the pointwise estimate:

$$\begin{aligned}|[(\omega\cdot\nabla)v\cdot e_3](x,t)| &\leq C|\tilde{\omega}(x,t)||\mathcal{P}(\tilde{\omega})(x,t)| \\ &\quad +C|\omega^3(x,t)||\mathcal{P}(\tilde{\omega})(x,t)|.\end{aligned}$$

• From $\dot{B}^0_{\infty,1} \hookrightarrow L^\infty$, we obtain

$$\begin{aligned}\|(\omega\cdot\nabla)v\cdot e_3]\|_{L^\infty} &\leq C\|\tilde{\omega}\|_{L^\infty}\|\mathcal{P}(\tilde{\omega})\|_{L^\infty} \\ &\quad +C\|\omega^3\|_{L^\infty}\|\mathcal{P}(\tilde{\omega})\|_{L^\infty} \\ &\leq C\|\tilde{\omega}\|_{L^\infty}\|\mathcal{P}(\tilde{\omega})\|_{\dot{B}^0_{\infty,1}} \\ &\quad +C\|\omega^3\|_{L^\infty}\|\mathcal{P}(\tilde{\omega})\|_{\dot{B}^0_{\infty,1}} \\ &\leq C\|\tilde{\omega}\|^2_{\dot{B}^0_{\infty,1}} \\ &\quad +C\|\omega^3\|_{L^\infty}\|\tilde{\omega}\|_{\dot{B}^0_{\infty,1}}.\end{aligned}$$

• Substituting this into the inequality for ω^3, we obtain the estimate:

$$\begin{aligned}\|\omega^3(t)\|_{L^\infty} &\leq \|\omega^3_0\|_{L^\infty} \\ &\quad +C\int_0^t \|\omega^3(s)\|_{L^\infty}\|\tilde{\omega}(s)\|_{\dot{B}^0_{\infty,1}}ds \\ &\quad +C\int_0^t \|\tilde{\omega}(s)\|^2_{\dot{B}^0_{\infty,1}}ds.\end{aligned}$$

• The Gronwall lemma yields

$$\|\omega^3(t)\|_{L^\infty} \leq \|\omega_0^3\|_{L^\infty} \exp\left(C\int_0^t \|\tilde\omega(s)\|_{\dot B^0_{\infty,1}} ds\right)$$
$$+C\int_0^t \|\tilde\omega(s)\|^2_{\dot B^0_{\infty,1}} \exp\left(C\int_s^t \|\tilde\omega(\tau)\|_{\dot B^0_{\infty,1}} d\tau\right) ds$$
$$\leq \left(\|\omega_0^3\|_{L^\infty} + \int_0^t \|\tilde\omega(s)\|^2_{\dot B^0_{\infty,1}}\right) \times$$
$$\times \exp\left(C\int_0^t \|\tilde\omega(s)\|_{\dot B^0_{\infty,1}} ds\right).$$

• Set $\left(\int_0^T \|\tilde\omega(t)\|^2_{\dot B^0_{\infty,1}} dt\right)^{\frac{1}{2}} = A_T$, then we deduce that

$$\int_0^T \|\omega(t)\|_{L^\infty} dt \leq \int_0^T \|\tilde\omega(t)\|_{L^\infty} dt + \int_0^T \|\omega^3(t)\|_{L^\infty} dt$$
$$\leq \sqrt{T} A_T + \left[\|\omega_0^3\|_{L^\infty} + CA_T^2\right] T \exp\left(C\sqrt{T} A_T\right).$$

implying the necessity part of the criterion.

• The sufficiency part easily follows by trivial application of the imbedding, $H^m(\mathbb{R}^3) \hookrightarrow B^0_{\infty,1}(\mathbb{R}^3)$ for $m > \frac{5}{2}$.

(ii) The case of Axisymmetry :

• The velocity field $v(r, x_3, t)$ has the representation:

$$v(r, x_3, t) = v^r(r, x_3, t)e_r + v^\theta(r, x_3, t)e_\theta + v^3(r, x_3, t)e_3,$$

where $r = \sqrt{x_1^2 + x_2^2}$, and

$$e_r = (\frac{x_1}{r}, \frac{x_2}{r}, 0), \quad e_\theta = (-\frac{x_2}{r}, \frac{x_1}{r}, 0), \quad e_3 = (0, 0, 1).$$

• The vorticity $\omega =$ curl v is computed,

$$\omega = \omega^r e_r + \omega^\theta e_\theta + \omega^3 e_3,$$

where

$$\omega^r = -\partial_{x_3} v^\theta, \quad \omega^\theta = \partial_{x_3} v^r - \partial_r v^3, \quad \omega^3 = \frac{1}{r}\partial_r(r v^\theta).$$

• We recall the notations:

$$\tilde{v} = v^r e_r + v^3 e_3, \qquad \tilde{\omega} = \omega^r e_r + \omega^3 e_3,$$

and $\omega = \tilde{\omega} + \vec{\omega}_\theta$ with $\vec{\omega}_\theta = \omega^\theta e_\theta$.

• The Euler equations for the axisymmetric solution:

$$\frac{\partial v^r}{\partial t} + (\tilde{v} \cdot \tilde{\nabla}) v^r = -\frac{\partial p}{\partial r},$$

$$\frac{\partial v^\theta}{\partial t} + (\tilde{v} \cdot \tilde{\nabla}) v^\theta = -\frac{v^r v^\theta}{r},$$

$$\frac{\partial v^3}{\partial t} + (\tilde{v} \cdot \tilde{\nabla}) v^3 = -\frac{\partial p}{\partial x_3},$$

$$\text{div } \tilde{v} = 0,$$

$$v(r, x_3, 0) = v_0(r, x_3),$$

where we set $\tilde{\nabla} = e_r \frac{\partial}{\partial r} + e_3 \frac{\partial}{\partial x_3}$.

• In the axisymmetry the Euler equations in the vorticity formulation becomes

$$\frac{\partial \omega^r}{\partial t} + (\tilde{v} \cdot \tilde{\nabla})\omega^r (\tilde{\omega} \cdot \tilde{\nabla}) v^r$$

$$\frac{\partial \omega^3}{\partial t} + (\tilde{v} \cdot \tilde{\nabla})\omega^3 (\tilde{\omega} \cdot \tilde{\nabla}) v^3$$

$$\left[\frac{\partial}{\partial t} + \tilde{v} \cdot \tilde{\nabla}\right] \left(\frac{\omega^\theta}{r}\right) = (\tilde{\omega} \cdot \tilde{\nabla}) \left(\frac{v^\theta}{r}\right)$$

$$\text{div } \tilde{v} = 0, \quad \text{curl } \tilde{v} = \vec{\omega}^\theta \cdots\cdots (*)$$

• We use the notation:

$$\tilde{\nabla}\tilde{v} = \begin{pmatrix} \frac{\partial v^r}{\partial r} & \frac{\partial v^r}{\partial x_3} \\ \frac{\partial v^3}{\partial r} & \frac{\partial v^3}{\partial x_3} \end{pmatrix}, \ \nabla \tilde{v} = \left(\frac{\partial \tilde{v}_j}{\partial x_k}\right)_{j,k=1}^3 .$$

• We can check easily

$$|\tilde{\nabla}\tilde{v}(x)| \leq |\nabla\tilde{v}(x)| \qquad \forall x \in \mathbb{R}^3.$$

• The elliptic system (*) implies

$$\nabla\tilde{v}(x) = \mathcal{P}(\vec{\omega}_\theta)(x) + C_0\vec{\omega}_\theta(x),$$

where $\mathcal{P}(\cdot)$ is a matrix valued singular integral operator of the Calderon-Zygmund type, and C_0 is a constant matrix.

• We also consider the particle trajectory mapping $\tilde{X}(\alpha, t)$ defined by

$$\frac{\partial\tilde{X}(\alpha,t)}{\partial t} = \tilde{v}(\tilde{X}(\alpha,t),t), \quad \tilde{X}(\alpha,0) = \alpha.$$

• Then, integrating the vorticity equations along $\tilde{X}(\alpha, t)$, we find that

$$\omega^r(\tilde{X}(\alpha,t),t) = \omega_0^r(\alpha) + \int_0^t (\tilde{\omega}\cdot\tilde{\nabla}) v^r(\tilde{X}(\alpha,s),s)ds,$$

$$\omega^3(\tilde{X}(\alpha,t),t) = \omega_0^3(\alpha) + \int_0^t (\tilde{\omega}\cdot\tilde{\nabla}) v^3(\tilde{X}(\alpha,s),s)ds.$$

• Taking supremum over $\alpha \in \mathbb{R}^3$,

$$\begin{aligned} \|\tilde{\omega}(t)\|_{L^\infty} &\le \|\tilde{\omega}_0\|_{L^\infty} + \int_0^t \|\tilde{\omega}(s)\|_{L^\infty} \|\tilde{\nabla}\tilde{v}(s)\|_{L^\infty} ds \\ &\le \|\tilde{\omega}_0\|_{L^\infty} + \int_0^t \|\tilde{\omega}(s)\|_{L^\infty} \|\nabla\tilde{v}(s)\|_{L^\infty} ds. \end{aligned}$$

• By Gronwall's lemma,

$$\begin{aligned} \|\tilde{\omega}(t)\|_{L^\infty} &\le \|\tilde{\omega}_0\|_{L^\infty} \exp\left(\int_0^t \|\nabla\tilde{v}(s)\|_{L^\infty} ds\right) \\ &\le \|\tilde{\omega}_0\|_{L^\infty} \exp\left(C\int_0^t \|\nabla\tilde{v}(s)\|_{\dot{B}^0_{\infty,1}} ds\right) \\ &\le \|\tilde{\omega}_0\|_{L^\infty} \exp\left(C\int_0^t \|\vec{\omega}_\theta(s)\|_{\dot{B}^0_{\infty,1}} ds\right). \end{aligned}$$

• Combining this with

$$\|\vec{\omega}_\theta(t)\|_{L^\infty} \le C\|\vec{\omega}_\theta(t)\|_{\dot{B}^0_{\infty,1}},$$

we find

$$\begin{aligned}\int_0^T \|\omega(t)\|_{L^\infty} dt \le & \int_0^T \|\tilde{\omega}(t)\|_{L^\infty} dt \\ & + \int_0^T \|\vec{\omega}_\theta(t)\|_{L^\infty} dt \\ \le \ & T\|\tilde{\omega}_0\|_{L^\infty} \exp\left(C\int_0^T \|\vec{\omega}_\theta(t)\|_{\dot{B}^0_{\infty,1}} dt\right) \\ & + C\int_0^T \|\vec{\omega}_\theta(t)\|_{\dot{B}^0_{\infty,1}} dt.\end{aligned}$$

• Thus, the BKM criterion implies the necessity part.

• Similarly to the previous proofthe sufficiency part easily follows from the imbedding,
$H^m(\mathbb{R}^3) \hookrightarrow B^0_{\infty,1}(\mathbb{R}^3)$ for $m > \frac{5}{2}$.