Enstrophy dissipation for 2D incompressible flows

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Outline of the talk

- Motivation (Kraichnan-Batchelor theory).
- Weak solutions to Euler equations: vanishing viscosity limit, regularization by mollification.
- Enstrophy defects and balance equations. Eyink's conjecture.
- Finite enstrophy data, $\omega_0 \in L^2$: existence of positive defect, transport of enstrophy density (velocity unbounded).
- Infinite enstrophy data, $\omega_0 \in L^{2,\infty} \cap B^0_{2,\infty}$: existence of a well-defined defect, counterexample to Eyink's conjecture.
- Conclusions.

Motivation

- 2D Turbulence: direct Enstrophy cascade with energy spectrum $E(k) \sim k^{-3}$ + log (Kraichnan-Batchelor, $\sim$ 1967-69).

- Energy dissipation $\sim 2\nu\Omega \Rightarrow$ negligible for small viscosity. Enstrophy dissipation cannot be neglected as $\nu \to 0$.

- As viscosity $\to 0$, turbulent solutions to Navier-Stokes give rise to weak solutions to 2D Euler (Onsager 1949).

- Enstrophy $\Omega(t) = 1/2\|\omega(t)\|^2_{L^2}$ is conserved for regular Euler flows. A paradox.

- At small scales and high Re, coherent vortices appear $\Rightarrow$ exact steady-state solutions to 2D Euler (McWilliams 1984).

Vorticity formulation for 2D Flows

Vorticity ω-velocity u formulation to 2D Euler:

$$\partial_t \omega + u \cdot \nabla \omega = 0, \tag{1a}$$
$$u = K * \omega, \tag{1b}$$

where $K(x) \equiv \frac{x^{\perp}}{2\pi|x|^2}$ is the Biot-Savart kernel. (1a) is a transport equation for ω.

Vorticity -velocity formulation to 2D Navier-Stokes:

$$\partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega, \tag{2a}$$
$$u = K * \omega, \tag{2b}$$

where ν is the viscosity coefficient. (2a) is a transport-diffusion equation for ω.

Weak solutions to 2D Euler

Definition. $\omega \in L^\infty([0,T); L^p(\mathbb{R}^2))$, $p \geq 4/3$, is a weak solution to 2D Euler with initial data $\omega_0 \in L^p_c(\mathbb{R}^2)$, if $\forall \varphi \in C^\infty_c([0,T) \times \mathbb{R}^2)$,

$$\int_0^T \int_{\mathbb{R}^2} \varphi_t \omega + \nabla\varphi \cdot u\,\omega \; dxdt + \int_{\mathbb{R}^2} \varphi(x,0)\omega_0(x)\; dx = 0,$$

and $u \in L^\infty([0,T); L^2(\mathbb{R}^2) + L^\infty(\mathbb{R}^2))$.

$\omega \in L^p(\mathbb{R}^2)$, $p \geq 4/3 \Rightarrow u\,\omega \in L^1(\mathbb{R}^2)$.

Uniqueness is proved for nearly bounded vorticity (Yudovich, Vishik).

Existence holds for measures $\omega_0 \in (\mathcal{B}M_{c,+} + L^1_c) \cap H^{-1}_{\text{loc}}$, e.g. vortex sheets (Delort, Majda, Schochet, Vecchi-Wu), using a *different* weak formulation.

Enstrophy

Define enstrophy $\Omega(t) = \frac{1}{2}\|\omega(t)\|_{L^2}^2$ with enstrophy density $\vartheta(x,t) = \frac{1}{2}|\omega|^2(x,t)$. ϑ describes the space-time distribution of enstrophy.

Study transport of ϑ by *irregular* velocity field $u \Rightarrow$ renormalized solutions to *linear* transport equations (DiPerna-Lions):

Definition. $u \in L^1([0,T], W^{1,1}_{\text{loc}})$, $\omega \in L^\infty([0,T], L^0)$.
ω is a renormalized solution to $\partial_t \omega + u \cdot \nabla \omega = 0$ if

$$\partial_t \beta(\omega) + u \cdot \nabla \beta(\omega) = 0,$$

for all β admissible $\in \mathcal{A} = \{\beta \in C^1 \cap L^\infty, \beta \equiv 0 \text{ near } 0\}$.

- Renormalized solutions are unique given u. The distribution function and any rearrangement-invariant norm is preserved if $\operatorname{div} u = 0$.

- 2D Euler solution $\omega \in L^\infty([0,T], L^p)$, $p \geq 2$, is the unique weak and renormalized solution to the linear transport equation $\Rightarrow$ Ω exactly conserved.

- If $p > 2$, then $\beta(s) = s^2$ can be taken as an admissible function $\Rightarrow$ ϑ is also transported by u (Eyink).

- 2D Euler solution $\omega \in L^\infty([0,T], L^p)$, $1 < p < 2$, is the unique renormalized solution if limit of exact smooth solutions.

Formulation of the problem

Reconcile KB theory with solutions to 2D Euler:

May be possible to define non-trivial enstrophy flux as limit of source terms in local balance equation after regularization (Eyink).

- Finite-enstrophy case: $\omega^0 \in L^2_c \Rightarrow u \in$ BMO, $\theta \in L^1$.
  Need to define non-linear term $u\,\theta$ in transport equation.

- Infinite-enstrophy case: $\omega^0 \in L^{2,\infty}_c \cap B^0_{2,\infty}$.
  Meaningful enstrophy defect from *renormalized* enstrophy.

Regularization by vanishing viscosity and mollifying equation.

Balance equations

Define $\Omega_\epsilon(t) = \dfrac{1}{2}\|\omega_\epsilon(t)\|^2_{L^2}$. The density $\vartheta_\epsilon(x,t)$ satisfies:

$$\partial_t \vartheta_\epsilon + \text{div}\,[u_\epsilon \vartheta_\epsilon + \omega_\epsilon\,((u\omega)_\epsilon - u_\epsilon \omega_\epsilon)] = -Z_\epsilon(\omega), \tag{3}$$

where

$$Z_\epsilon(\omega) = -\nabla \omega_\epsilon \cdot ((u\omega)_\epsilon - u_\epsilon \omega_\epsilon)\,.$$

with $(u\,\omega)_\epsilon = j_\epsilon * (u\,\omega)$.

Similarly, $\Omega_\nu(t) = \dfrac{1}{2}\|\omega_\nu(t)\|^2_{L^2}$. The density $\vartheta_\nu(x,t)$ satisfies:

$$\partial_t \vartheta_\nu + u_\nu \cdot \nabla \vartheta_\nu - \nu \Delta \vartheta_\nu = -Z^\nu(\omega_\nu), \tag{4}$$

where

$$Z^\nu(\omega_\nu) = \nu |\nabla \omega_\nu|^2 \geq 0.$$

Enstrophy defects

Transport enstrophy defect: ω any weak solution to Euler

$$Z^T(\omega) \equiv \lim_{\epsilon \to 0} Z_\epsilon(\omega) = \lim_{\epsilon \to 0} [-\nabla \omega_\epsilon \cdot ((u\omega)_\epsilon - u_\epsilon \omega_\epsilon)],$$

enstrophy disspation due to irregular transport.

Viscous enstrophy defect: ω viscosity solution

$$Z^V(\omega) \equiv \lim_{\nu \to 0} Z^\nu(\omega) = \lim_{\nu \to 0} \nu |\nabla \omega_\nu|^2,$$

enstrophy dissipation due to viscosity.

$Z^V(\omega) \geq 0$ if it exists as a distribution.

ω will be called dissipative if Z^T exists and $Z^T(\nu) \geq 0$.

Eyink's conjecture (full plane)

Consider initial data with locally infinite enstrophy:

ω_0 is in the Besov space $B^0_{2,\infty} \supset L^p$, $p \geq 2 \Rightarrow$ velocity $u_0 = K * \omega_0$ has Kraichnan-Batchelor energy spectrum:

$$E(k) \sim k^{-3}.$$

Viscosity solutions $\omega = \lim_{\nu \to 0} \omega^\nu$ exist such that

$$\sup_{\nu > 0} \|\omega^\nu\|_{L^2([0,T],B^0_{2,\infty})} < C.$$

Conjecture : Let ω be a viscosity solution with data ω_0. Then:

$$Z(\omega) = Z^T(\omega) = Z^V(\omega) \geq 0, \qquad \text{in } \mathcal{D}'.$$

Moreover there exist initial data for which $Z(\omega) > 0$.

The L^2 case

Enstrophy density ϑ is renormalized solution to $\vartheta_t + \mathrm{div}(u\,\vartheta) = 0$, but not necessarily a weak solution $\Rightarrow$ non-zero enstrophy defect may exist even if Ω conserved.

$Z^V(\omega) \equiv 0$: by positivity enough to prove

$$\lim_{\nu\to 0} \int_0^T \int_{\mathbb{R}^2} Z_\nu(\omega)\, dx\, dt = \lim_{\nu\to 0} \nu \int_0^T \|\nabla\omega_\nu\|_{L^2}^2\, dt = 0,$$

Direct consequence of energy conservation plus strong convergence $\omega_\nu \to \omega$ in L^2.

But $Z^T(\omega) = \lim_\epsilon Z_\epsilon$ and $Z\epsilon = [-\nabla\omega_\epsilon \cdot ((u\omega)_\epsilon - u_\epsilon\omega_\epsilon)]$ does not have distinguish sign.

$\exists\ \omega \in L^2(\mathbb{R}^2)$ such that $u\,\theta = (K * \omega)\,\omega^2 \notin \mathcal{D}'(\mathbb{R}^2)$.

Look at logarithmic refinement of L^2, rearrangement-invariant space where $u\,\vartheta$ can be defined.

Choose $\omega \in L^2 \log L^{1/4}$ so that $\vartheta \in L^1 \log L^{1/2} \hookrightarrow H^{-1}_{\text{loc}}$ continuously.

Definition. $\omega \in L^2 \log L^{1/4} \cap L^1$, $u = K * \omega$, $\Phi \in C_0^\infty$,

$$\langle u\vartheta, \Phi\rangle = -\int_{\mathbb{R}^2} \omega(y) \int_{\mathbb{R}^2} K(y-x) \cdot \Phi(x)\vartheta(x)\, dx\, dy.$$

Use antisymmetry of Biot-Savart kernel (cf. Schochet's proof of Delort theorem).

Theorem 1 (H. Lopes, M. Lopes, A. M.).
Consider a viscosity solution $\omega \in L^\infty([0,T); L^2(\log L)^{1/4}(\mathbb{R}^2))$ *to 2D Euler. Then the following equation holds in the sense of distributions:*

$$\partial_t(|\omega|^2) + \mathrm{div}(u|\omega|^2) = 0, \qquad u = K * \omega.$$

Theorem 2 (H. Lopes, M. Lopes, A. M.).
Let $\omega \in L^\infty([0,T); L^2(\log L)^{1/4}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2))$ *be a weak solution of 2D Euler. Then* $Z^T(\omega)$ *exists (as a distribution). If* ω *is a viscosity solution, then* $Z^T(\omega) \equiv 0$.

If $\exists\, \omega \in L^\infty([0,T], L^2 \log L^{1/4})$ with $Z^T(\omega) \neq 0$, nonuniqueness of solutions to 2D Euler follows.

Proof of Theorem 1:

- ω viscosity solution. Pass to the limit $\nu \to 0$ in

$$\int_0^T \int_{\mathbb{R}^2} \varphi_t \vartheta_\nu \, dxdt + \int_0^T \int_{\mathbb{R}^2} \nabla\varphi \cdot u_\nu \vartheta_\nu \, dxdt = \int_0^T \int_{\mathbb{R}^2} \nu \Delta\varphi \, \vartheta_\nu \, dxdt - \int_0^T \int_{\mathbb{R}^2} \varphi Z^\nu(\omega_\nu) \, dxdt.$$

- $\omega_\nu(t) \to \omega(t)$ strongly in $L^2(\mathbb{R}^2)$ from energy estimate:

$$\|\omega(t)\|_{L^2} - \|\omega_\nu(t)\|_{L^2} = \nu \int_0^t \int_{\mathbb{R}^2} |\nabla\omega_\nu|^2 \, dx \, dt.$$

- $\vartheta_\nu \to \vartheta$ in $L^1([0,T) \times \mathbb{R}^2)$, $Z^\nu(\omega_\nu) \to 0$ in $L^1([0,T) \times \mathbb{R}^2)$
  $\Rightarrow$ in the limit:

$$\int_0^T \int_{\mathbb{R}^2} \varphi_t \vartheta \, dxdt + \lim_{\nu \to 0} \int_0^T \int_{\mathbb{R}^2} \nabla\varphi \cdot u_\nu \, \vartheta_\nu \, dxdt = 0$$

- Analyze behavior of non-linear term as $\nu \to 0$:

$$-\int_0^T \int_{\mathbb{R}^2} \omega_\nu(y,t) \int_{\mathbb{R}^2} K(y-x) \cdot \nabla\varphi(x,t)\vartheta_\nu(x,t)\, dx\, dydt.$$

- Product estimate:

$$\|f\, g\|_{L \log L^{1/2}} \le 4(\max\{\|f\|_{L^2 \log L^{1/4}}; \|g\|_{L^2 \log L^{1/4}}\})^2.$$

Use Luxemburg norm:

$$\|f\|_{L^p \log L^\alpha} = \inf\left\{k > 0 \mid \int A_{p,\alpha}(|f(x)|)\, dx \le 1\right\}.$$

Exploit that $A_{p,\alpha}(s) = [s \log^\alpha(2+s)]^p$ is non-decreasing and convex.

- $\vartheta_\nu \to \vartheta$ strongly in $L^1([0,T)\times\mathbb{R}^2) \Rightarrow K*(\nabla\phi\vartheta_\nu) \to K*(\nabla\phi\vartheta)$ weakly.

- Uniform bound for $\omega_\nu(t)$ in $L^2 \log L^{1/4}$ from divergence-free condition on u and convexity of $A_{2,1/4}$:

$$\|\omega_\nu(t)\|_{L^2 \log L^{1/4}} \le \|\omega_0\|_{L^2 \log L^{1/4}}.$$

  Product estimate + uniform bound $\Rightarrow$ $\{\nabla\phi\vartheta_\nu\}$ bounded in $L^\infty((0,T); L(\log L)^{1/2})$.

- Biot-Savart operator K smoothing of order 1 and $L^\infty((0,T), L(\log L)^{1/2}) \hookrightarrow L^\infty((0,T), H^{-1})_{\text{loc}} \Rightarrow$

$$\{K*\nabla\phi\vartheta_\nu\} \text{ is bounded in } L^\infty((0,T); L^2_{\text{loc}}).$$

- If Supp $\phi \subset B_R$, show $|K * (\nabla\phi\vartheta_\nu)(y,t)| \lesssim \Omega/|y|$, $|y| > 2R \Rightarrow$

  $K * \nabla\phi\vartheta_\nu \to K * (\nabla\phi\vartheta)$ weakly in $L^\infty((0,T) \times B(0,2R))$.

- $\omega_\nu \to \omega$ strongly in $L^1([0,T] \times \mathbb{R}^2)$ from uniform bound on L^1 norm (maximum principle) and strong convergence in L^2.

- Non-linear term:

$$\int_0^T \int_{B_{2R}} \omega_\nu \, K * (\nabla\phi\vartheta_\nu)\, dx\, dt + \int_0^T \int_{B_{2R}^c} \omega_\nu \, K * (\nabla\phi\vartheta_\nu)\, dx\, dt$$
$$\to \langle \omega, K * (\nabla\phi\,\vartheta)\rangle = \langle u\,\omega, \nabla\phi\rangle,$$

  since each integral forms a "weak-strong" pair.

Theorem 1 is nearly optimal $\Rightarrow$ there exist $\omega \in L^2 \log L^{1/6}$ such that $(K * \omega)\, \omega^2 = u\, \vartheta \notin \mathcal{D}'$.

Define $\omega^{\pm}(x) = \dfrac{1}{|x||\log|x||^{\alpha}}\, \chi_{D^{\pm}(0;1/3)}(x)$, $u^{\pm} = K * \omega^{\pm}$, where

$$1/2 < \alpha < 1, \qquad \begin{cases} D^{+}(0;1/3) = D(0,1/3) \cap \{x_2 > 0\}, \\ D^{-}(0;1/3) = D(0,1/3) \cap \{x_2 < 0\}. \end{cases}$$

Show $|u^{+}(x)| \geq C|\log|x||^{1-\alpha}$ near origin.
Note: $u^{+} = u - u^{-}$, where u radial and u^{-} harmonic in $\mathbb{H}^{+}$.

$u = K * \omega$ bounded, because $\omega \in L^2$ *radial.*
Obtain growth of u^{-} by evaluating $K * \omega^{-}$ on real axis and using potential estimates on harmonic extension.

Initial data with infinite enstrophy

Look at data in $L^{2,\infty} \cap B^0_{2,\infty}$, rearrangement-invariant space of functions with KB spectrum.

Mildest behavior of u is for *radial* vorticity (K is odd $\Rightarrow$ cancellations in $K * \omega$): $\omega = \phi(x)\dfrac{1}{|x|}$, ϕ cut-off near the origin.

Construct exact steady viscosity solution ω to 2D Euler such that

$$Z^T(\omega) = 0, \qquad Z^V(\omega) = \frac{4\pi^3}{t}\delta_o, \qquad t > 0.$$

Strictly dissipative solutions exist.

Counterexample to Conjecture: notion of enstrophy defect *depends* on the approx sequence.

ω radial $\Rightarrow u = K * \omega$ exact steady solution to 2D Euler, since $u \perp \nabla\omega$. Example of coherent vortex (DiPerna-Majda).

$u_\epsilon \perp \omega_\epsilon$ so that $Z_\epsilon(\omega) = -\nabla\omega_\epsilon \cdot ((u\omega)_\epsilon - u_\epsilon\omega_\epsilon) = 0$.

ω_ν solves heat equation.

Heat kernel estimates and homogeneity of ω give

$$\|Z^\nu(\omega)\|_{L^1([0,t]\times\mathbb{R}^2)} = \frac{4\pi^3}{t} + o(1), \qquad \nu \to 0^+, \quad t > 0.$$

$\Rightarrow Z^\nu(\omega)$ uniformly bounded in $L^1 \Rightarrow$
$\exists\, \nu_k$ and a Radon measure μ such that $Z^{\nu_k}(\omega) \rightharpoonup \mu$.

Study support properties of μ to identify limit.

Conclusions

- Solutions in L^2 such that $Z^T(\omega) > 0$ would suggest that non-linear interactions are responsible for enstrophy dissipation at very high Re.

- Counterexample indicates that when the enstrophy is infinite, it is not necessary to have non-linear interactions to sustain the cascade picture.

- The behavior of radial vorticity should be the weakest among the same regularity class. Use comparison estimates for solutions to parabolic equations with spherically symmetrized data.