& Nazarov, Volberg

2D Quasi-Geostrophic Equation:

$\theta_t = u \cdot \nabla\theta - (-\Delta)^{\alpha}\theta, \quad \theta(x,0) = \theta_0(x),$

$\mathbb{T}^2$ (periodic BC).

$\hat{u}(\xi) = \left(-\frac{i\xi_2}{|\xi|}\hat{\theta}(\xi), \frac{i\xi_1}{|\xi|}\hat{\theta}(\xi)\right).$

Maximum principle: $\|\theta\|_\infty$ is non-increasing
(Resnick, Cordoba-Cordoba)

$\alpha > 1/2 \Rightarrow$ global smooth solution for θ_0 sufficiently smooth

$\alpha \le 1/2 \Rightarrow$ only local

$\alpha = 1/2$ – critical, especially interesting.

Constantin, Cordoba, Wu: $\|\theta_0\|_\infty$ & H^2 small $\Rightarrow$ global solution, real analytic $\forall t > 0$.

Theorem Assume $\theta_0(x)$ is smooth, periodic. Then 2D critical QG equation has a global smooth unique solution. Moreover,

$\|\nabla\theta(x,t)\|_\infty \le C\|\nabla\theta_0\|_\infty \exp\exp(C\|\theta_0\|_\infty), \forall t$

Remarks. 1. Can afford $\theta_0 \in H^s$, $s > 1$.

2. $\theta(x,t)$ is real analytic for any $t > 0$.

3. Originally, we worked on Burgers (& Nazarov, Shterenberg): $\theta_t + \theta_x \theta + (-\Delta)^\alpha \theta = 0$, periodic

Complete picture: $\alpha \geq 1/2$ global smooth solutions (analytic in x).

$\alpha < 1/2$ blow up in finite time is possible.

4. Can also prove existence of solutions smooth for $t > 0$ for very rough initial data:

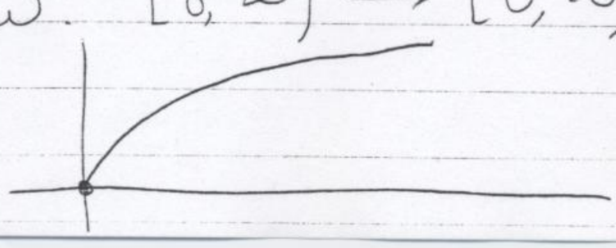
$\theta_0 \in L^p$, $p > 1$. Uniqueness is not clear.

Idea of proof.

New maximum principle (non-local).

Will find a modulus of continuity ω preserved by solutions.

$\omega: [0, \infty) \to [0, \infty)$, concave, continuous, $\omega(0) = 0$



Will also have $\omega'(0) = 1$, $\omega''(0) = -\infty$:

near 0, for ξ small, $\omega(\xi) = \xi - \xi^{3/2}$.

For large ξ, $\omega(\xi) \sim c \log\log \xi$.

Now if solutions with all periods conserve $\omega(\xi)$, they also conserve $\omega_B(\xi) = \omega(B\xi)$ due to scaling invariance of the equation.

If that is true, then:

1. Any smooth periodic $\theta_0(x)$ has ω_B for B large enough.

2. Then the solution corresponding to $\theta_0(x)$ also has ω_B, and so

$|\theta(x,t) - \theta(y,t)| \le \omega_B(|x-y|)$.

Thus $\|\nabla\theta\|_\infty \le B$. Such a-priori bound is sufficient for global existence.

How to construct ω?

Lemma 1 Assume $|\theta_0(x) - \theta_0(y)| < \omega(|x-y|)$, $\forall x, y$. Then the only way solution can lose ω is if there exists t_0 such that

1. $\exists x, y:\ \theta(x,t_0) - \theta(y,t_0) = \omega(|x-y|)$
2. $\forall t < t_0,\quad |\theta(x,t) - \theta(y,t)| < \omega(|x-y|),\ \forall x, y.$

Proof $\|\nabla\theta(x,t_0)\| = 1$ is impossible since $\omega''(0) = -\infty$.

Lemma 2 If $\theta(x)$ has ω, then $u(x)$ has a modulus of continuity

$$\Omega(\xi) = A\left(\int_0^\xi \frac{\omega(\eta)}{\eta}\,d\eta + \xi\int_\xi^\infty \frac{\omega(\eta)}{\eta^2}\,d\eta\right).$$

Proof CZ operator estimates.

What is needed for conservation of ω?

Look at "violation scenario" of Lemma 1

$\theta(x,t_0) - \theta(y,t_0) = \omega(|x-y|)$, set $\xi = |x-y|$.

$$\frac{\partial}{\partial t}\left(\theta(x,t) - \theta(y,t)\right)\Big|_{t=t_0} = \text{flow term} + \text{dissipation term}$$

Flow term : $u\cdot\nabla\theta(x) - u\cdot\nabla\theta(y) =$

$$\lim_{h\to 0}\frac{1}{h}\left(\theta(x+u(x)h) - \theta(x) - \theta(y+u(y)h) + \theta(y)\right)$$

$$\leq \lim_{h\to 0}\frac{1}{h}\left(\omega(\xi + |u(x)-u(y)|h) - \omega(\xi)\right) \leq$$

$$\leq \omega'(\xi)\,\Omega(\xi).$$

Dissipation term : $(-\Delta)^{1/2}\theta(y) - (-\Delta)^{1/2}\theta(x)$

$$(-\Delta)^{1/2}\theta(x) = \lim_{h\to 0}\frac{1}{h}\left((P_h * \theta)(x) - \theta(x)\right),$$

P_h is 2D Poisson kernel (in $\mathbb{R}^2$, applied to $\mathbb{R}^2$ periodization of θ).

After some computation, get

$$\text{Dissip. term} \le \frac{1}{\pi}\left(\int_0^{\xi/2} \frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^2}\,d\eta + \right.$$
$$\left. + \int_{\xi/2}^{\infty} \frac{\omega(2\eta+\xi)-\omega(2\eta-\xi)-2\omega(\xi)}{\eta^2}\,d\eta \right).$$

Both terms are negative by concavity.
Have to build $\omega(\xi)$ so that
flow term + dissipation term < 0.

$$\omega(\xi) = \xi + \xi^{3/2}, \quad \text{for } 0 < \xi < \delta$$

$$\omega'(\xi) = \frac{\gamma}{\xi(4+\log \xi/\delta)}, \quad \text{for } \xi > \delta, \ \gamma < \delta,$$

for some small enough δ, γ works.

why log? Look at $\xi > \delta$, use $\overset{\text{dissipative}}{\frown}$ term #2 : $\sim -\frac{c\,\omega(\xi)}{\xi}$

Have to offset $\sim \omega'(\xi)\,\omega(\xi)\cdot\log(\xi) \Rightarrow$ need
$\omega'(\xi) \sim -\frac{c}{\xi\log\xi}$ (just $-\frac{c}{\xi}$ for Burgers).

GLOBAL WELL-POSEDNESS FOR THE CRITICAL $2D$ DISSIPATIVE QUASI-GEOSTROPHIC EQUATION

A. KISELEV, F. NAZAROV, AND A. VOLBERG

ABSTRACT. We give an elementary proof of the global well-posedness for the critical $2D$ dissipative quasi-geostrophic equation. The argument is based on a non-local maximum principle involving appropriate moduli of continuity.

1. INTRODUCTION AND MAIN RESULTS

The $2D$ quasi-geostrophic equation attracted quite a lot of attention lately from various authors. Mainly it is due to the fact that it is the simplest evolutionary fluid dynamics equation for which the problem of existence of smooth global solutions remains unsolved. In this paper we will consider the so-called dissipative quasi-geostrophic equation

$$\begin{cases} \theta_t = u \cdot \nabla\theta - (-\Delta)^\alpha \theta \\ u = (u_1, u_2) = (-R_2\theta, R_1\theta) \end{cases}$$

where $\theta : \mathbb{R}^2 \to \mathbb{R}$ is a scalar function, R_1 and R_2 are the usual Riesz transforms in $\mathbb{R}^2$ and $\alpha > 0$. It is well known (see [5, 8]) that for $\alpha > \frac{1}{2}$ (the so-called subcritical case), the initial value problem $\theta(x, 0) = \theta_0(x)$ with C^∞-smooth periodic initial data θ_0 has a global C^∞ solution.

For $\alpha = \frac{1}{2}$, this equation arises in geophysical studies of strongly rotating fluid flows (see e.g. [2] for further references). Therefore, a significant amount of research focused specifically on the critical $\alpha = \frac{1}{2}$ case. In particular, Constantin, Cordoba, and Wu in [3] showed that the global smooth solution exists provided that $\|\theta_0\|_\infty$ is small enough. Cordoba and Cordoba [6] proved that the viscosity solutions are smooth on time intervals $t \leqslant T_1$ and $t \geqslant T_2$. The aim of this paper is to demonstrate that, in the critical case, smooth global solutions exist for any C^∞ periodic initial data θ_0, with no additional qualifications or assumptions. What happens in the supercritical case $0 \leqslant \alpha < \frac{1}{2}$ remains an open question.

The main idea of our proof is quite simple: we will construct a special family of moduli of continuity that are preserved by the dissipative evolution, which will allow us to get an a priori estimate for $\|\nabla\theta\|_\infty$ independent of time. More precisely, we will prove the following theorem.

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critical

Theorem. *The quasi-geostrophic equation with periodic smooth initial data $\theta_0(x)$ has a unique global smooth solution. Moreover, the following estimate holds:*

$$\|\nabla\theta\|_\infty \leqslant C\|\nabla\theta_0\|_\infty \exp\exp\{C\|\theta_0\|_\infty\}\,. \tag{1}$$

At this moment we do not know how sharp the upper bound (1) is: On the other hand, any a-priori bound for $\|\nabla\theta\|_\infty$ is sufficient for the proof of well-posedness. Indeed, local existence and regularity results then allow to extend the unique smooth solution indefinitely. For the critical and supercritical quasi-geostrophic equation; such results can be found for example in [9] (Theorems 3.1 and 3.3). Hence, the rest of the paper is devoted to the proof of (1).

We remark that this paper is built upon the ideas discovered in a related work on the dissipative Burgers equation [7].

2. Moduli of continuity

Let us remind the reader that a modulus of continuity is just an arbitrary increasing continuous concave function $\omega : [0,+\infty) \to [0,+\infty)$ such that $\omega(0) = 0$. Also, we say that a function $f : \mathbb{R}^n \to \mathbb{R}^m$ has modulus of continuity ω if $|f(x) - f(y)| \leqslant \omega(|x-y|)$ for all $x, y \in \mathbb{R}^n$.

Singular integral operators like Riesz transforms do not preserve moduli of continuity in general but they do not spoil them too much either. More precisely, we have

Lemma. *If the function θ has modulus of continuity ω, then $u = (-R_2\theta, R_1\theta)$ has modulus of continuity*

$$\Omega(\xi) = A\left(\int_0^\xi \frac{\omega(\eta)}{\eta}\,d\eta + \xi\int_\xi^\infty \frac{\omega(\eta)}{\eta^2}\,d\eta\right)$$

with some universal constant $A > 0$.

The proof of this result is elementary but since we could not readily locate it in the literature, we provide a sketch in the appendix.

The flow term $u \cdot \nabla\theta$ in the dissipative quasi-geostrophic equation tends to make the modulus of continuity of θ worse while the dissipation term $(-\Delta)^\alpha\theta$ tends to make it better. Our aim is to construct some special moduli of continuity for which the dissipation term always prevails and such that every periodic C^∞-function θ_0 has one of these special moduli of continuity.

Note that the critical ($\alpha = \frac{1}{2}$) equation has a simple scaling invariance: if $\theta(x,t)$ is a solution, then so is $\theta(Bx, Bt)$. This means that if we manage to find one modulus of continuity ω that is preserved by the dissipative evolution for all periodic solutions (i.e., with arbitrary lengths and spacial orientations of the periods), then the whole family $\omega_B(\xi) = \omega(B\xi)$ of moduli of continuity will also be preserved for all periodic solutions.

Observe now that if ω is unbounded, then any given C^∞ periodic function has modulus of continuity ω_B if $B > 0$ is sufficiently large. Also, if the modulus of continuity ω has finite derivative at 0, it can be used to estimate $\|\nabla\theta\|_\infty$. Thus, our task reduces to constructing an unbounded modulus of continuity with finite derivative at 0 that is preserved by the dissipative evolution.

From now on, we will also assume that, in addition to unboundedness and the condition $\omega'(0) < +\infty$, we have $\lim_{\xi\to 0+}\omega''(\xi) = -\infty$. Then, if a C^∞ periodic function f has modulus of continuity ω, we have

$$\|\nabla f\|_\infty < \omega'(0).$$

Indeed, take a point $x \in \mathbb{R}^2$ at which $\max|\nabla f|$ is attained and consider the point $y = x + \xi e$ where $e = \frac{\nabla f}{|\nabla f|}$. Then we must have $f(y) - f(x) \leqslant \omega(\xi)$ for all $\xi \geqslant 0$. But the left hand side is at least $|\nabla f(x)|\xi - C\xi^2$ where $C = \frac{1}{2}\|\nabla^2 f\|_\infty$ while the right hand side can be represented as $\omega'(0)\xi - \rho(\xi)\xi^2$ with $\rho(\xi) \to +\infty$ as $\xi \to 0+$. Thus $|\nabla f(x)| \leqslant \omega'(0) - (\rho(\xi) - C)\xi$ for all $\xi > 0$ and it remains to choose some $\xi > 0$ satisfying $\rho(\xi) > C$.

3. The breakthrough scenario

Now assume that θ has modulus of continuity ω for all times $t < T$. Then θ remains C^∞ smooth up to T and, according to the local regularity theorem, for a short time beyond T. By continuity, we see that θ must also have modulus of continuity ω at the moment T. Suppose that $|\theta(x,T) - \theta(y,T)| < \omega(|x-y|)$ for all $x \neq y$. We claim that then θ has modulus of continuity ω for all $t > T$ sufficiently close to T. Indeed, by the remark above, at the moment T we have $\|\nabla\theta\|_\infty < \omega'(0)$. By continuity of derivatives, this also holds for $t > T$ close to T, which immediately takes care of the inequality $|\theta(x,t) - \theta(y,t)| < \omega(|x-y|)$ for small $|x-y|$. Also, since ω is unbounded and $\|\theta\|_\infty$ doesn't grow with time, we automatically have $|\theta(x,t) - \theta(y,t)| < \omega(|x-y|)$ for large $|x-y|$. The last observation is that, due to periodicity of θ, it suffices to check the inequality $|\theta(x,t) - \theta(y,t)| < \omega(|x-y|)$ for x belonging to some compact set $K \subset \mathbb{R}^2$. Thus, we are left with the task to show that, if $|\theta(x,T) - \theta(y,T)| < \omega(|x-y|)$ for all $x \in K$, $\delta \leqslant |x-y| \leqslant \delta^{-1}$ with some fixed $\delta > 0$, then the same inequality remains true for a short time beyond T. But this immediately follows from the uniform continuity of θ.

This implies that the only scenario in which the modulus of continuity ω may be lost by θ is the one in which there exists a moment $T > 0$ such that ω has modulus of continuity ω for all $t \in [0,T]$ and there are two points $x \neq y$ such that $\theta(x,T) - \theta(y,T) = \omega(|x-y|)$. We shall rule this scenario out by showing that, in such case, the derivative $\frac{\partial}{\partial t}(\theta(x,t) - \theta(y,t))\big|_{t=T}$ must be negative, which, clearly, contradicts the assumption that the modulus of continuity ω is preserved up to the time T.

4. Estimate of the derivative: the flow term

Assume that the above scenario takes place. Let $\xi = |x-y|$. Observe that $(u \cdot \nabla\theta)(x) = \frac{d}{dh}\theta(x + hu(x))\big|_{h=0}$ and similarly for y. But

$$\theta(x + hu(x)) - \theta(y + hu(y)) \leqslant \omega(|x-y| + h|u(x) - u(y)|) \leqslant \omega(\xi + h\Omega(\xi))$$

where, as before,

$$\Omega(\xi) = A\left(\int_0^\xi \frac{\omega(\eta)}{\eta}\,d\eta + \xi\int_\xi^\infty \frac{\omega(\eta)}{\eta^2}\,d\eta\right).$$

Since $\theta(x) - \theta(y) = \omega(\xi)$, we conclude that

$$(u \cdot \nabla\theta)(x) - (u \cdot \nabla\theta)(y) \leqslant \Omega(\xi)\omega'(\xi).$$

5. Estimate of the derivative: the dissipation term

Recall that the dissipative term can be written as $\frac{d}{dh}\mathcal{P}_h * \theta\big|_{h=0}$ where $\mathcal{P}_h$ is the usual Poisson kernel in $\mathbb{R}^2$, (again, this formula holds for all smooth periodic functions regardless of the lengths and spatial orientation of the periods, which allows us to freely use the scaling and rotation tricks below). Thus, our task is to estimate $(\mathcal{P}_h * \theta)(x) - (\mathcal{P}_h * \theta)(y)$ under the assumption that θ has modulus of continuity ω. Since everything is translation and rotation invariant, we may assume that $x = (\frac{\xi}{2}, 0)$ and $y = (-\frac{\xi}{2}, 0)$.

Write

$$
\begin{aligned}
(\mathcal{P}_h * \theta)(x) - (\mathcal{P}_h * \theta)(y) &= \iint_{\mathbb{R}^2} [\mathcal{P}_h(\tfrac{\xi}{2} - \eta, \nu) - \mathcal{P}_h(-\tfrac{\xi}{2} - \eta, \nu)]\theta(\eta, \nu)\, d\eta d\nu \\
&= \int_{\mathbb{R}} d\nu \int_0^\infty [\mathcal{P}_h(\tfrac{\xi}{2} - \eta, \nu) - \mathcal{P}_h(-\tfrac{\xi}{2} - \eta, \nu)][\theta(\eta, \nu) - \theta(-\eta, \nu)]\, d\eta \\
&\leqslant \int_{\mathbb{R}} d\nu \int_0^\infty [\mathcal{P}_h(\tfrac{\xi}{2} - \eta, \nu) - \mathcal{P}_h(-\tfrac{\xi}{2} - \eta, \nu)]\omega(2\eta)\, d\eta \\
&= \int_0^\infty [P_h(\tfrac{\xi}{2} - \eta) - P_h(-\tfrac{\xi}{2} - \eta)]\omega(2\eta)\, d\eta \\
&= \int_0^\xi P_h(\tfrac{\xi}{2} - \eta)\omega(2\eta)\, d\eta + \int_0^\infty P_h(\tfrac{\xi}{2} + \eta)[\omega(2\eta + 2\xi) - \omega(2\eta)]\, d\eta
\end{aligned}
$$

where P_h is the 1-dimensional Poisson kernel. Here we used symmetry and monotonicity of the Poisson kernels together with the observation that $\int_{\mathbb{R}} \mathcal{P}_h(\eta, \nu)\, d\nu = P_h(\eta)$. The last formula can also be rewritten as

$$
\int_0^{\frac{\xi}{2}} P_h(\eta)[\omega(\xi + 2\eta) + \omega(\xi - 2\eta)]\, d\eta + \int_{\frac{\xi}{2}}^\infty P_h(\eta)[\omega(2\eta + \xi) - \omega(2\eta - \xi)]\, d\eta\,.
$$

Recalling that $\int_0^\infty P_h(\eta)\, d\eta = \frac{1}{2}$, we see that the difference $(\mathcal{P}_h * \theta)(x) - (\mathcal{P}_h * \theta)(y) - \omega(\xi)$ can be estimated from above by

$$
\begin{aligned}
&\int_0^{\frac{\xi}{2}} P_h(\eta)[\omega(\xi + 2\eta) + \omega(\xi - 2\eta) - 2\omega(\xi)]\, d\eta \\
&+ \int_{\frac{\xi}{2}}^\infty P_h(\eta)[\omega(2\eta + \xi) - \omega(2\eta - \xi) - 2\omega(\xi)]\, d\eta\,.
\end{aligned}
$$

Recalling the explicit formula for P_h, dividing by h and passing to the limit as $h \to 0+$, we finally conclude that the contribution of the dissipative term to our derivative is bounded from above by

$$
\begin{aligned}
&\frac{1}{\pi}\int_0^{\frac{\xi}{2}} \frac{\omega(\xi + 2\eta) + \omega(\xi - 2\eta) - 2\omega(\xi)}{\eta^2}\, d\eta \\
&+\frac{1}{\pi}\int_{\frac{\xi}{2}}^\infty \frac{\omega(2\eta + \xi) - \omega(2\eta - \xi) - 2\omega(\xi)}{\eta^2}\, d\eta\,.
\end{aligned}
\tag{2}
$$

Note that due to concavity of ω, both terms are strictly negative.

6. The explicit formula for the modulus of continuity

We will construct our special modulus of continuity as follows. Choose two small positive numbers $\delta > \gamma > 0$ and define the continuous function ω by

$$\omega(\xi) = \xi - \xi^{\frac{3}{2}} \qquad \text{when } 0 \leqslant \xi \leqslant \delta$$

and

$$\omega'(\xi) = \frac{\gamma}{\xi(4 + \log(\xi/\delta))} \qquad \text{when } \xi > \delta\,.$$

Note that, for small δ, the left derivative of ω at δ is about 1 while the right derivative equals $\frac{\gamma}{4\delta} < \frac{1}{4}$. So ω is concave if δ is small enough. It is clear that $\omega'(0) = 1$, $\lim_{\xi\to 0+}\omega''(\xi) = -\infty$ and that ω is unbounded (it grows at infinity like double logarithm). The hard part, of course, is to show that, for this ω, the negative contribution to the time derivative coming from the dissipative term prevails over the positive contribution coming from the flow term. More precisely, we have to check the inequality

$$\begin{aligned} A\left[\int_0^\xi \frac{\omega(\eta)}{\eta}\,d\eta + \xi\int_\xi^\infty \frac{\omega(\eta)}{\eta^2}\,d\eta\right]\omega'(\xi) + \frac{1}{\pi}\int_0^{\frac{\xi}{2}} \frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^2}\,d\eta \\ + \frac{1}{\pi}\int_{\frac{\xi}{2}}^\infty \frac{\omega(2\eta+\xi)-\omega(2\eta-\xi)-2\omega(\xi)}{\eta^2}\,d\eta < 0 \qquad \text{for all } \xi > 0\,. \end{aligned}$$

7. Checking the inequality: case $0 \leqslant \xi \leqslant \delta$

Let $0 \leqslant \xi \leqslant \delta$. Since $\omega(\eta) \leqslant \eta$ for all $\eta \geqslant 0$, we have $\int_0^\xi \frac{\omega(\eta)}{\eta}\,d\eta \leqslant \xi$ and $\int_\xi^\delta \frac{\omega(\eta)}{\eta^2}\,d\eta \leqslant \log\frac{\delta}{\xi}$. Now,

$$\int_\delta^\infty \frac{\omega(\eta)}{\eta^2}\,d\eta = \frac{\omega(\delta)}{\delta} + \gamma\int_\delta^\infty \frac{1}{\eta^2(4+\log(\eta/\delta))}\,d\eta \leqslant 1 + \frac{\gamma}{4\delta} < 2\,.$$

Observing that $\omega'(\xi) \leqslant 1$, we conclude that the positive part of the left hand side is bounded by $A\xi(3 + \log\frac{\delta}{\xi})$.

To estimate the negative part, we just use the first integral in (2). Note that $\omega(\xi+2\eta) \leqslant \omega(\xi) + 2\omega'(\xi)\eta$ due to concavity of ω, and $\omega(\xi-2\eta) \leqslant \omega(\xi) - 2\omega'(\xi)\eta - 2\omega''(\xi)\eta^2$ due to the second order Taylor formula and monotonicity of ω'' on $[0,\xi]$. Plugging these inequalities into the integral, we get the bound

$$\frac{1}{\pi}\int_0^{\frac{\xi}{2}} \frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^2}\,d\eta \leqslant \frac{1}{\pi}\xi\omega''(\xi) = -\frac{3}{4\pi}\xi\xi^{-\frac{1}{2}}\,.$$

But, obviously, $\xi\left(A(3+\log\frac{\delta}{\xi}) - \frac{3}{4\pi}\xi^{-\frac{1}{2}}\right) < 0$ on $(0,\delta]$ if δ is small enough.

8. Checking the inequality: case $\xi \geqslant \delta$

In this case, we have $\omega(\eta) \leqslant \eta$ for $0 \leqslant \eta \leqslant \delta$ and $\omega(\eta) \leqslant \omega(\xi)$ for $\delta \leqslant \eta \leqslant \xi$. Hence

$$\int_0^\xi \frac{\omega(\eta)}{\eta}\,d\eta \leqslant \delta + \omega(\xi)\log\frac{\xi}{\delta} \leqslant \omega(\xi)\left(2 + \log\frac{\xi}{\delta}\right)$$

because $\omega(\xi) \geqslant \omega(\delta) > \frac{\delta}{2}$ if δ is small enough.

Also

$$\int_{\xi}^{\infty}\frac{\omega(\eta)}{\eta^2}\,d\eta=\frac{\omega(\xi)}{\xi}+\gamma\int_{\xi}^{\infty}\frac{d\eta}{\eta^2(4+\log(\eta/\delta))}\leqslant\frac{\omega(\xi)}{\xi}+\frac{\gamma}{\xi}\leqslant\frac{2\omega(\xi)}{\xi}$$

if $\gamma<\frac{\delta}{2}$ and δ is small enough.

Thus, the positive term on the left hand side is bounded from above by the expression $A\omega(\xi)\left(4+\log\frac{\xi}{\delta}\right)\omega'(\xi)=A\gamma\frac{\omega(\xi)}{\xi}$.

To estimate the negative term, note that, for $\xi\geqslant\delta$, we have

$$\omega(2\xi)\leqslant\omega(\xi)+\frac{\gamma}{4}\leqslant\frac{3}{2}\omega(\xi)$$

under the same assumptions on γ and δ as above. Also, due to concavity, we have $\omega(2\eta+\xi)-\omega(2\eta-\xi)\leqslant\omega(2\xi)$ for all $\eta\geqslant\frac{\xi}{2}$. Therefore,

$$\frac{1}{\pi}\int_{\frac{\xi}{2}}^{\infty}\frac{\omega(2\eta+\xi)-\omega(2\eta-\xi)-2\omega(\xi)}{\eta^2}\,d\eta\leqslant-\frac{1}{2\pi}\int_{\frac{\xi}{2}}^{\infty}\frac{\omega(\xi)}{\eta^2}\,d\eta=-\frac{1}{\pi}\frac{\omega(\xi)}{\xi}\,.$$

But $\frac{\omega(\xi)}{\xi}(A\gamma-\frac{1}{\pi})<0$ if γ is small enough.

9. Concluding remarks

We'll start with quoting (with necessary minor modifications) a paragraph from [6]. Note that it was written just 2 years ago.

The case $\alpha=\frac{1}{2}$ is specially relevant because the viscous term $(-\Delta)^{\frac{1}{2}}\theta$ models the so-called Eckmann's pumping, which has been observed in quasi-geostrophic flows. On the other hand, several authors have emphasized the deep analogy existing between the dissipative quasi-geostrophic equation with $\alpha=\frac{1}{2}$ and the 3D incompressible Navier-Stokes equations.

This paper provides an elementary treatment of the $\alpha=\frac{1}{2}$ case. Unfortunately, the argument does not seem to extend to the Navier-Stokes equations due to the different structure of nonlinearity. So, while our paper resolves the global existence and regularity question in a physically relevant model, it also suggests that there is a significant structural difference between the critical $2D$ quasi-geostrophic equation and $3D$ Navier-Stokes equations.

Remark. After this article has been submitted, we learned of a preprint by Caffarelli and Vasseur [1], where the global regularity of solutions of the critical dissipative quasi-geostrophic equation was established by a completely different method using the DiGiorgi type techniques. The results of that paper differ from ours in two main respects: they start with just L^2 initial data and they do not use the smoothing out of the drift to establish the smoothing out of the solution. To reduce technicalities, in this paper we did not attempt to treat most general initial data. However, perhaps it is worth mentioning that our "good moduli of continuity" method, properly modified and combined with a few fairly simple and well-known ideas, allows to recapture at least the first of those advantageous features of [1].

Currently, the strongest existence theorem for the solutions of the critical dissipative quasi-geostrophic equation with periodic initial data we can prove seems to be the following: if $\theta_0\in L^p$ with $1<p<+\infty$, then there exists a function $\theta(x,t)$ that is real analytic in x and C^∞ in t for all $t>0$ and such that it satisfies the equation for all $t>0$ in the classical sense and $\lim_{t\to0+}\|\theta(\cdot,t)-\theta_0\|_{L^p}=0$ A very interesting question we still cannot answer is whether such a solution is always unique. Another unsolved problem is whether, for every initial data in H^s $(0 < s < 1)$, there exists a solution such that $\lim_{t\to 0+} \|\theta(\cdot,t) - \theta_0\|_{H^s} = 0$.

10. Appendix

Here we provide a sketch of the proof of the Lemma.

Proof. The Riesz transforms are singular integral operators with kernels $K(r,\zeta) = r^{-2}\Omega(\zeta)$, where (r,ζ) are the polar coordinates. The function Ω is smooth and $\int_{S^1} \Omega(\zeta)d\sigma(\zeta) = 0$. Assume that the function f satisfies $|f(x) - f(y)| \leq \omega(|x-y|)$ for some modulus of continuity ω. Take any x, y with $|x - y| = \xi$, and consider the difference

$$P.V.\int K(x-t)f(t)\,dt - P.V.\int K(y-t)f(t)\,dt \tag{3}$$

with integrals understood in the principal value sense. Note that

$$\left|P.V.\int_{|x-t|\leq 2\xi} K(x-t)f(t)\,dt\right| = \left|P.V.\int_{|x-t|\leq 2\xi} K(x-t)(f(t)-f(x))\,dt\right| \leq C\int_0^{2\xi} \frac{\omega(r)}{r}\,dr.$$

Since ω is concave, we have

$$\int_0^{2\xi} \frac{\omega(r)}{r}\,dr \leq 2\int_0^{\xi} \frac{\omega(r)}{r}\,dr.$$

A similar estimate holds for the second integral in (3). Next, let $\bar{x} = \frac{x+y}{2}$. Then

$$\left|\int_{|x-t|\geq 2\xi} K(x-t)f(t)\,dt - \int_{|y-t|\geq 2\xi} K(y-t)f(t)\,dt\right| =$$

$$\left|\int_{|x-t|\geq 2\xi} K(x-t)(f(t)-f(\bar{x}))\,dt - \int_{|y-t|\geq 2\xi} K(y-t)(f(t)-f(\bar{x}))\,dt\right|$$

$$\leq \int_{|\bar{x}-t|\geq 3\xi} |K(x-t) - K(y-t)||f(t)-f(\bar{x})|\,dt +$$

$$\int_{3\xi/2\leq|\bar{x}-t|\leq 3\xi} (|K(x-t)| + |K(y-t)|)|f(t)-f(\bar{x})|\,dt.$$

Since

$$|K(x-t) - K(y-t)| \leq C\frac{|x-y|}{|\bar{x}-t|^3}$$

when $|\bar{x} - t| \geq 3\xi$, the first integral is estimated by $C\xi\int_{3\xi}^{\infty} \frac{\omega(r)}{r^2}\,dr$. The second integral is estimated by $C\omega(3\xi)$, and hence is controlled by $3C\int_0^{\xi} \frac{\omega(r)}{r}\,dr$. □

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