Unfolding Complex Singularities for the Euler Equations

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Outline

- Complex singularities: PDE examples
- Numerical construction of complex singularities for Euler
- Unfolding complex singularities

Methods for Construction of Possible Euler Singularities

- Numerical construction
  - Orszag et al (1983), Kerr (1993), Pelz (1996) …
- Similarity solutions
  - Childress et al. (1989), …
- Complex variables
  - Bardos, Benachour & Zerner (1976), REC (1993)
- Unfolding / catastrophe theory
  - Ercolani, Steele & REC (1996)

Canonical Example 1: Cauchy-Riemann Equations

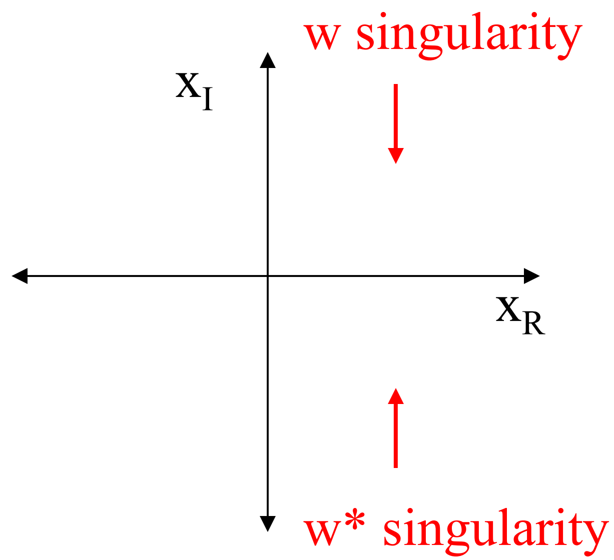
- Laplace equation in x,t

$$u_{tt} + u_{xx} = 0$$

- Complex traveling wave solution

$$u(x,t) = w(x+it) + w^*(x-it)$$

- Singularities in initial data move toward the real axis.



Canonical Example 2: Burgers Equations

- Burgers equation
  - Initial value problem

$$u_t + uu_x = 0$$

$$u(0,x) = u_0(x)$$

  - Characteristic form

$$\partial_t u = 0 \quad on \quad \partial_t x = u = u_0$$

  - Invert initial data

$$x_0(u) : u_0(x_0(u)) = u$$

  - Implicit solution

$$x = x_0(u) + tu$$

- Singularities

$$u_x = \infty \Leftrightarrow x_u = 0$$

i.e. $0 = \partial_u x_0(u) + t$

Canonical Example 2: Burgers Equations (Cont)

- Burgers equation singularity condition

$$0 = \partial_u x_0(u) + t$$

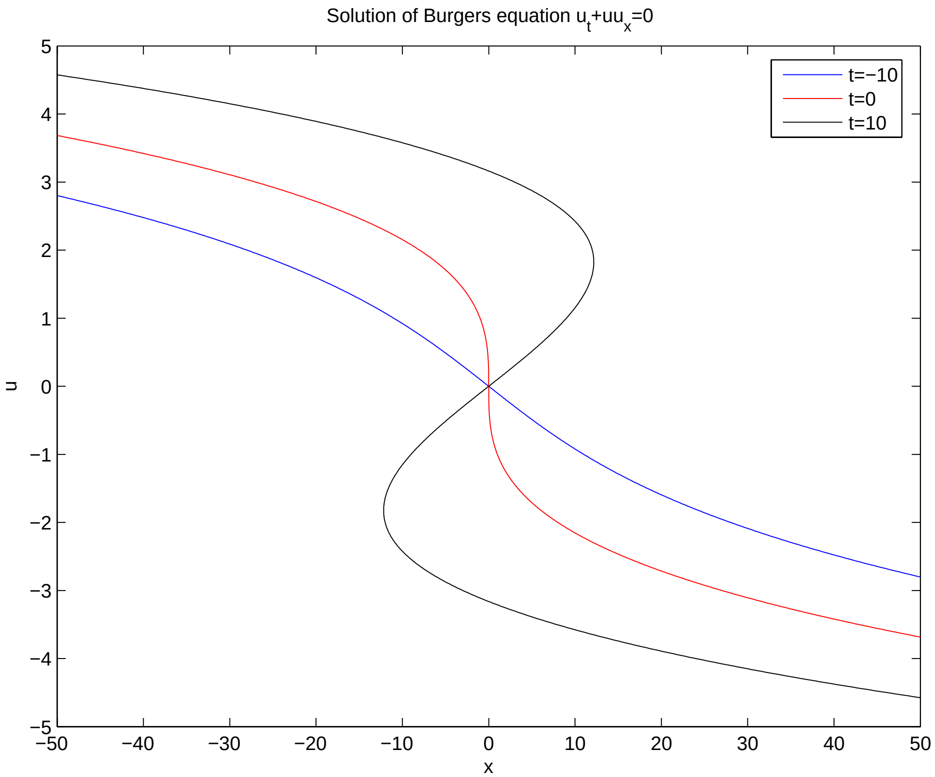
- Example
  - Initial data $\quad u_0(x_0) = -x_0^{1/3}$

    $$x_0(u) = -u^3$$
  - Singularity condition

    $$0 = -3u^2 + t$$
  - Real singularities for t>0

    $$u = \pm\sqrt{t/3}$$
  - Complex singularities for t<0

    $$u = \pm\, i\sqrt{|\,t\,|/3}$$



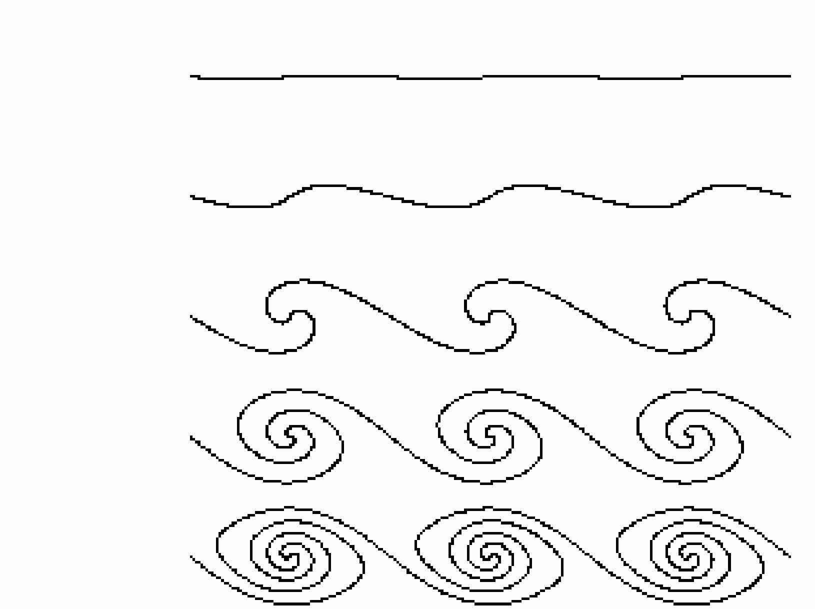
- Complex singularities collide forming shock

Kelvin-Helmholtz Instability

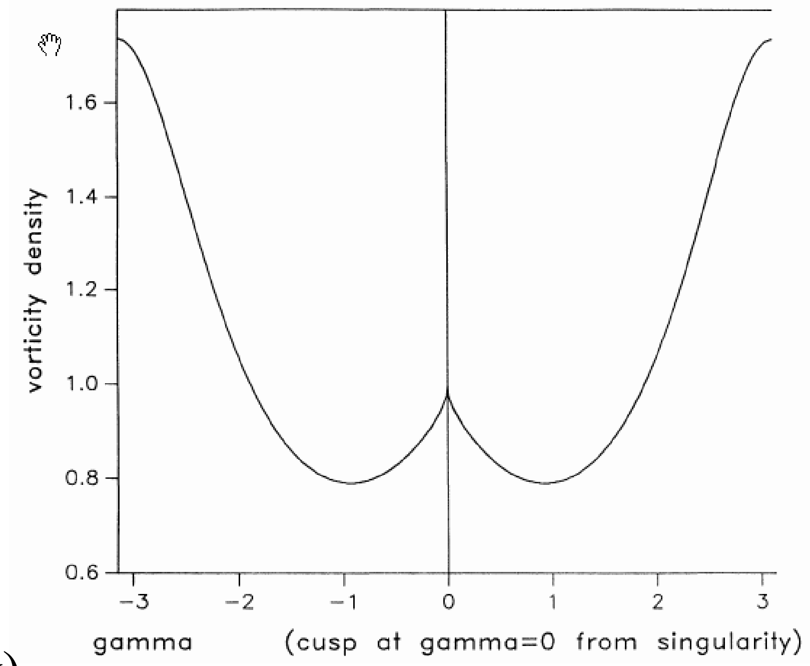
- Moore (1979) constructed singularities through asymptotics, as traveling waves in complex plane
  - $z = x+iy \approx \gamma + (1+i)\,\varepsilon\,(\sin \gamma)^{3/2}$
  - γ =circulation variable
  - Curvature singularity in sheet
- REC and Orellana (1989) constructed solutions, including solutions with singularities and ill-posedness, starting from analytic initial data.
- Wu (2005) showed that any solution, satisfying some mild regularity conditions, is analytic for $t > 0$.

Vortex Sheet Singularity for Kelvin-Helmholtz

- Moore (1979)
  - $z = x+iy \approx \gamma + (1+i)\,\varepsilon\,(\sin \gamma)^{3/2}$
  - γ =circulation variable
  - Curvature in shape of sheet
  - Cusp in sheet strength $(z_\gamma)^{-1}$



Sheet position at various times (Krasny)



Moore's Construction

(REC & Orellana interpretation)

- Birkhoff-Rott Equation

$$\partial_t z^*(\gamma,t) = BR(z) = \frac{1}{2\pi i} PV \int (z(\gamma,t) - z(\gamma',t))^{-1} d\gamma'$$

- Look for $z = \gamma + z_+ + z_-$
  - upper analytic z_+
  - lower analytic z_-
- Ignore interactions between z_+ and z_- (Moore's approx)
  - Evaluate BR for lower analytic functions z_-, z_+^* by contour integration

$$\partial_t z_+^*(\gamma,t) = BR(z_-) = \frac{1}{1+\partial_\gamma z_-}$$

$$\partial_t z_-(\gamma,t) = BR(z_+^*) = \frac{-1}{1+\partial_\gamma z_+^*}$$

  - Nonlinearization of CR eqtns, complex characteristics construction of solutions with singularities

Generalizations of Moore's Construction

- Rayleigh-Taylor
  - Siegel, Baker, REC (1993)
- Muskat problem (2-sided Hele-Shaw, porous media)
  - REC, Howison, Siegel (2004)
  - Cordoba

Complex Euler Singularities: Numerical Construction

- Axisymmetric flow with swirl
  - REC (1993)
- 2D Euler
  - Pauls, Matsumoto, Frisch & Bec (2006)
- 3D Euler (Pelz and related initial data) – talk by Siegel
  - Siegel & REC (2006)
- Singularity detection via asymptotics of fourier components

Singularity Analysis

- Fit to asymptotic form of fourier components in 1D

$$\hat{u}_k \approx c k^{-\alpha} e^{-ikz_*} \longrightarrow u \approx c(z - z_*)^{\alpha - 1}$$

- Apply 3-point fit, to get singularity parameters c, α, z_* as function of k
  - Successful fit has c, α, z_* nearly independent of k

Axisymmetric flow with swirl

REC (1993)

- Solution method
  - Moore's approximation: $u = u_+ + u_-$
    - u_+ upper analytic in z, $u_- = u_+^*$ lower analytic, no interaction between them
  - Traveling wave ansatz (Siegel's thesis 1989 for Rayleigh-Taylor)

$$u_+(r,z,t) = u_+(r, z - i\sigma t)$$

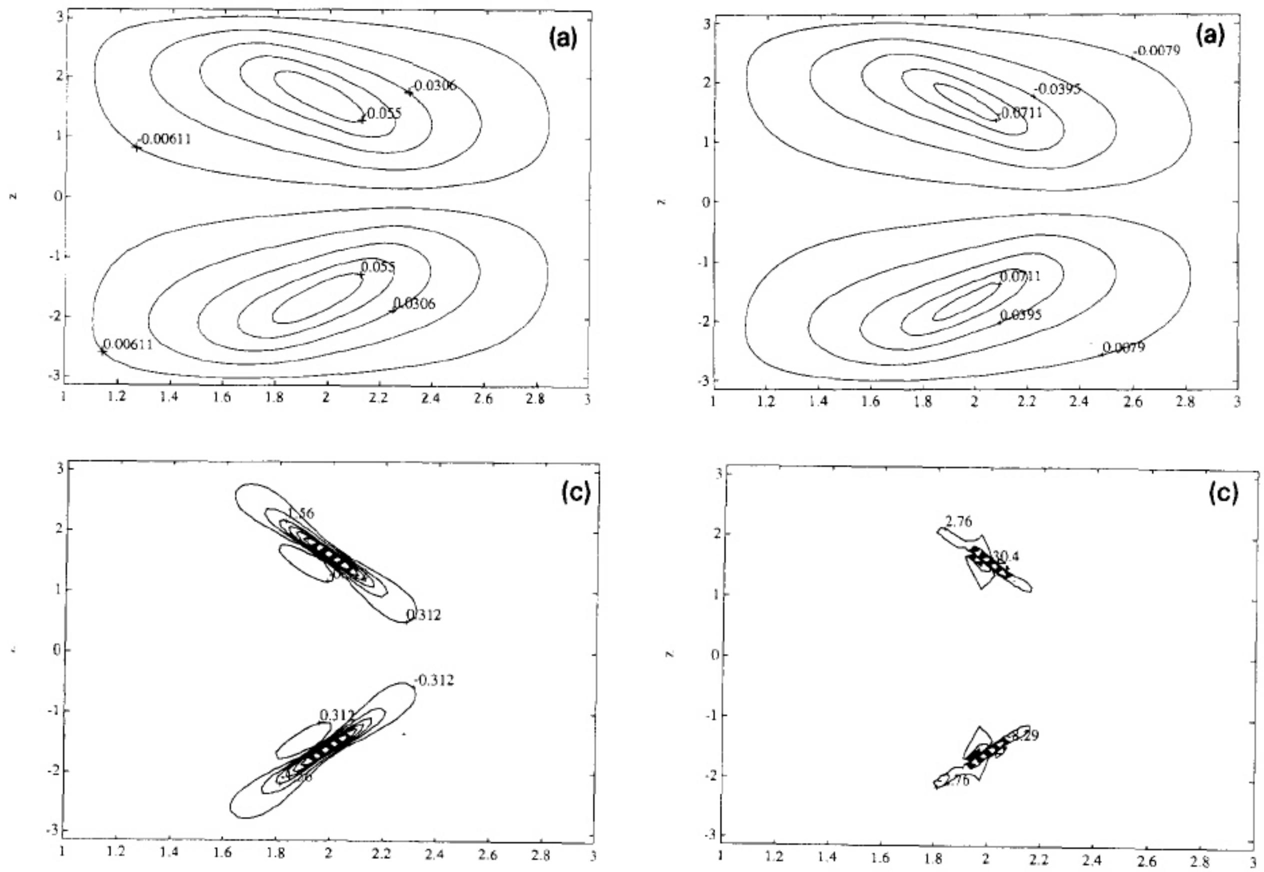
  - Ultra-high precision,
    - needed to control amplification of round-off error
- Singularity type
  - $u \approx x^{-1/3}$
  - $\omega \approx x^{-4/3}$
- Real singularity? No
  - Violates Deng-Hou-Yu (DHY) criterion, restricted directionality

Complex upper analytic solution: pure swirling flow

- Flow in periodic anulus,
  - $r_1 < r < r_2$ (no normal flow BCs)
  - $0 < z < 2\pi$ (periodic BCs)

u_θ

ω_θ



$t < t_{sing}$ $t = t_{sing}$

2D Euler

Pauls, Matsumoto, Frisch & Bec (2006)

- Solution method
  - Small time asymptotics, spectral computation
  - Ultra-high precision,
    - needed singularity detection, since singularities are far from reals
- Singularity type
  - $\omega \approx x^{-\beta}$ with $5/6 \leq \beta \leq 1$
- Real singularity? No
  - Vorticity does not grow in 2D → no singularities
  - u=(u,v,0) $\partial_t \omega + u \bullet \nabla \omega = \omega \bullet \nabla u = 0$
  - ω=(0,0, ζ)

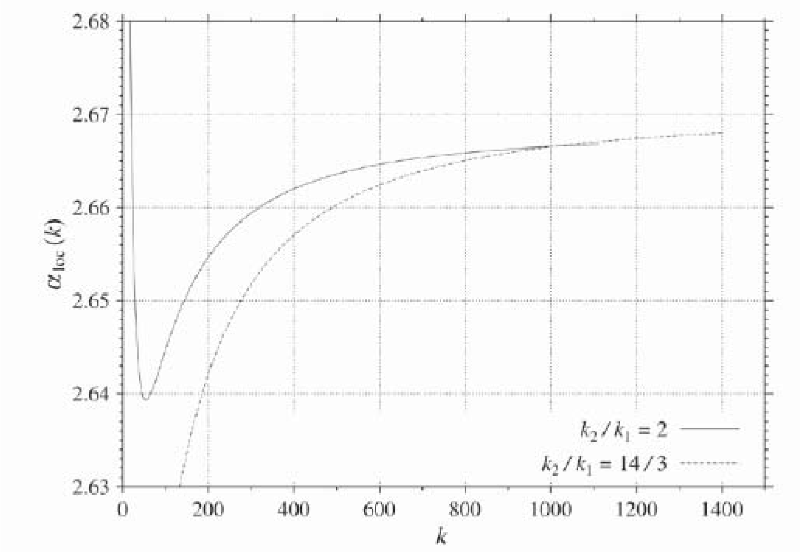


Fig. 3. Local prefactor exponent $\alpha_{loc}(k)$ versus wavenumber for two values of the slope.

Unfolding Singularities

- General method

Unfolding Singularities

- General method
  - Unfolding variable η
  - Mapping $q(x,t, \eta)=0$ defines relation between (x, t) and η
  - Rewrite PDE in terms of (x, t, η)
    - $u = u(x, t, \eta)$
    - $\partial_t = \partial_t - q_\eta^{-1} \eta_t \partial_\eta$
- Special method
  - Include sqrt(ξ) in solution
  - $\xi = \xi(x,t)$ a smooth function
  - Works only for a single sqrt singularity

Boussinesq and Unfolding

Boussinesq eqtns

$$(\partial_t + \mathbf{u}\cdot\nabla)\rho = f$$

$$(\partial_t + \mathbf{u}\cdot\nabla)\zeta = -\partial_z \rho + g$$

$$\mathbf{u} = (u,v) = \nabla^{\perp}\psi = (-\partial_z\psi, \partial_r\psi)$$

$$\zeta = \nabla^2\psi = -\partial_z u + \partial_r v.$$

Unfolding ansatz

$$\mathbf{u} = \mathbf{u}_0 + \xi^{\frac{1}{2}}\mathbf{u}_1$$

$$\rho = \rho_0 + \xi^{\frac{1}{2}}\rho_1$$

$$\zeta = \zeta_0 + \xi^{-\frac{1}{2}}\zeta_1$$

$$\psi = \psi_0 + \xi^{\frac{3}{2}}\psi_1$$

u_i, ρ_i , ψ_i , ζ_i , ξ smooth functions

Unfolded eqtns

- Div

$$\mathbf{u}_0 = \nabla^{\perp}\psi_0$$

$$\mathbf{u}_1 = \frac{3}{2}\psi_1\nabla^{\perp}\xi + \xi\nabla^{\perp}\psi_1$$

- ρ eqtn

$$(\partial_t + \mathbf{u}_0\cdot\nabla)\rho_0 + \frac{1}{2}\alpha\xi\rho_1 + \xi\mathbf{u}_1\cdot\nabla\rho_1 = f$$

$$(\partial_t + \mathbf{u}_0\cdot\nabla)\rho_1 + \mathbf{u}_1\cdot\nabla\rho_0 = 0$$

- ζ definition

$$\zeta_0 = \nabla\times\mathbf{u}_0$$

$$\zeta_1 = \frac{1}{2}\nabla\xi\times\mathbf{u}_1 + \xi\nabla\times\mathbf{u}_1$$

- ζ eqtn

$$(\partial_t + \mathbf{u}_0\cdot\nabla)\zeta_1 + \xi\mathbf{u}_1\cdot\nabla\zeta_0 = -\xi\partial_z\rho_1 - \frac{1}{2}\rho_1\partial_z\xi$$

$$(\partial_t + \mathbf{u}_0\cdot\nabla)\zeta_0 + \mathbf{u}_1\cdot\nabla\zeta_1 - \frac{1}{2}\zeta_1\alpha\xi = -\partial_z\rho_0 + g.$$

- desingularization

$$\xi_t + \mathbf{u}_0\cdot\nabla\xi = 0.$$

Unfolded eqtns

- Div

$$\mathbf{u}_0 = \nabla^{\perp}\psi_0$$

$$\mathbf{u}_1 = \frac{3}{2}\psi_1\nabla^{\perp}\xi + \xi\nabla^{\perp}\psi_1$$

- ζ definition

$$\zeta_0 = \nabla\times\mathbf{u}_0$$

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- desingularization

$$\xi_t + \mathbf{u}_0\cdot\nabla\xi = 0.$$

- ρ eqtn

$$(\partial_t + \mathbf{u}_0\cdot\nabla)\rho_0 + \frac{1}{2}\alpha\xi\rho_1 + \xi\mathbf{u}_1\cdot\nabla\rho_1 = f$$

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- ζ eqtn

$$(\partial_t + \mathbf{u}_0\cdot\nabla)\zeta_1 + \xi\mathbf{u}_1\cdot\nabla\zeta_0 = -\xi\partial_z\rho_1 - \frac{1}{2}\rho_1\partial_z\xi$$

$$(\partial_t + \mathbf{u}_0\cdot\nabla)\zeta_0 + \mathbf{u}_1\cdot\nabla\zeta_1 - \frac{1}{2}\zeta_1\alpha\xi = -\partial_z\rho_0 + g.$$

Unfolded eqtns

- Div

$$\mathbf{u}_0 = \nabla^{\perp}\psi_0$$

$$\mathbf{u}_1 = \frac{3}{2}\psi_1\nabla^{\perp}\xi + \xi\nabla^{\perp}\psi_1$$

- ζ definition

$$\zeta_0 = \nabla \times \mathbf{u}_0$$

$$\zeta_1 = \frac{1}{2}\nabla\xi \times \mathbf{u}_1 + \xi\nabla \times \mathbf{u}_1$$

- desingularization

$$\xi_t + \mathbf{u}_0 \cdot \nabla\xi = 0.$$

- ρ eqtn

$$(\partial_t + \mathbf{u}_0 \cdot \nabla)\rho_0 + \frac{1}{2}\alpha\xi\rho_1 + \xi\mathbf{u}_1 \cdot \nabla\rho_1 = f$$

$$(\partial_t + \mathbf{u}_0 \cdot \nabla)\rho_1 + \mathbf{u}_1 \cdot \nabla\rho_0 = 0$$

- ζ eqtn

$$(\partial_t + \mathbf{u}_0 \cdot \nabla)\zeta_1 = -\frac{1}{2}\rho_1\partial_z\xi \qquad \mathbf{u}_1 \cdot \nabla\zeta_0 = -\partial_z\rho_1$$

$$(\partial_t + \mathbf{u}_0 \cdot \nabla)\zeta_0 + \mathbf{u}_1 \cdot \nabla\zeta_1 - \frac{1}{2}\zeta_1\alpha\xi = -\partial_z\rho_0 + g.$$

- This system is well-posed but nonstandard.
- Unfolding through mapping q(x,t, η)=0 leads to a well-posed system that is more complicated but more standard.

Conclusions

- Inviscid singularities may play a role in viscous turbulence.
- Complex variables approach successful for interface problems, including singularity formation and global existence.
- Complex singular solutions for Euler constructed by special methods.
- Unfolding of weak complex singularities and their dynamics.
- Attempting to turn this into a real singular solution for Euler.

Equations for u$^+$

$$r^{-1}\partial_r(ru_r^+)+\partial_z u_z^+=0$$

$$(\overline{u}_z-i\sigma)\partial_z u_z^+ + u_r^+\partial_r\overline{u}_z+\partial_z p^+=a$$

$$(\overline{u}_z-i\sigma)\partial_z u_r^+ - 2r^{-1}\overline{u}_\theta u_\theta^+ + \partial_r p^+=b$$

$$(\overline{u}_z-i\sigma)\partial_z u_\theta^+ + u_r^+\partial_r\overline{u}_\theta + r^{-1}\overline{u}_\theta u_r^+=c$$

$$a=-u^+\cdot\nabla u_z^+$$

$$b=-u^+\cdot\nabla u_r^+ + r^{-1}u_\theta'^2$$

$$c=-u^+\cdot\nabla u_\theta^+ - r^{-1}u_\theta^+ u_r^+$$

$$u_r^+\overline{\omega}_z = r^{-1}\partial_r(r\overline{u}_\theta)u_r^+$$

Simplified eqtn for u^{+}_{r}

$$\partial_r (r^{-1}\partial_r (r u_r^{+})) + \partial_z^2 u_r^{+} - \eta u_r^{+} = d$$

$$\eta = (\bar{u}_z - i\sigma)^{-1}\{\partial_r^2 \bar{u}_z - r^{-1}\partial_r \bar{u}_z - 2r^{-1}(\bar{u}_z - i\sigma)^{-1}\bar{u}_\theta \bar{\omega}_z\}$$

$$d = (\bar{u}_z - i\sigma)^{-1}\{-\partial_r a + \partial_z b + 2r^{-1}\bar{u}_\theta (\bar{u}_z - i\sigma)^{-1} c\}$$

Instability of u_k equations

- Solution of k eqtn depends on k' with k'<k
- Roundoff error grows as k increases
- Controlled through use of ultra high precision
  - MPFUN by David Bailey
- Limitation on size of computation

Hele-Shaw

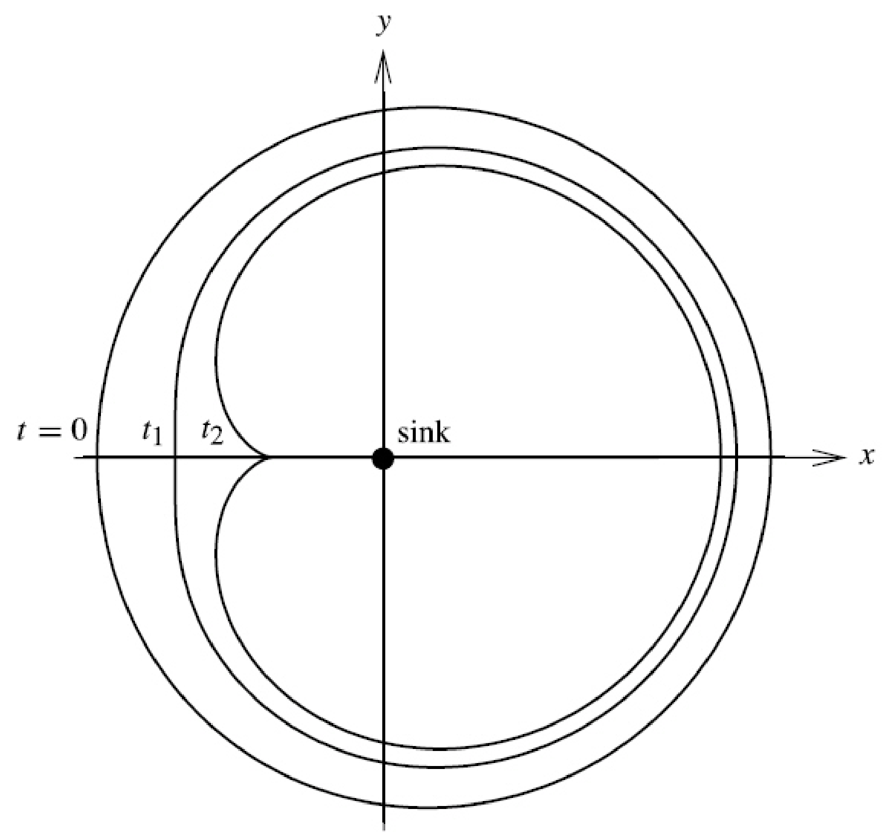
- Flow through porous media with a free boundary
  - Darcy's law and incompressibility

    $$\mathbf{u} = V\mathbf{j} - k\nabla p \qquad \nabla \cdot \mathbf{u} = 0$$

  - Boundary conditions

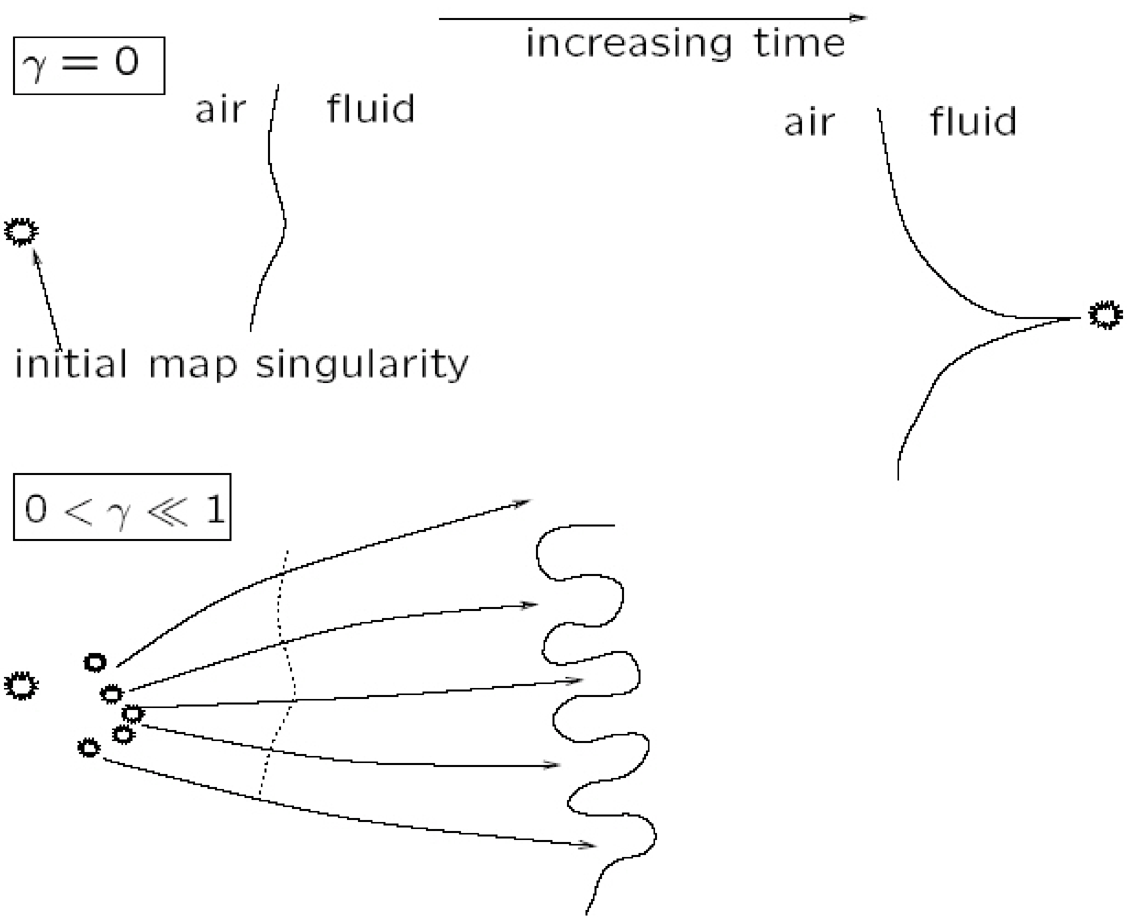
    $$p = 0 \qquad \mathbf{u} \cdot \mathbf{n} = V_n$$

- Exact solution with cusp singularities in the boundary



Hele-Shaw

- The zero surface tension limit $\gamma \to 0$ is singular. Singularities in the complex plane move toward the real boundary, but they can be preceded by daughter singularities (Tanveer, Siegel, …).



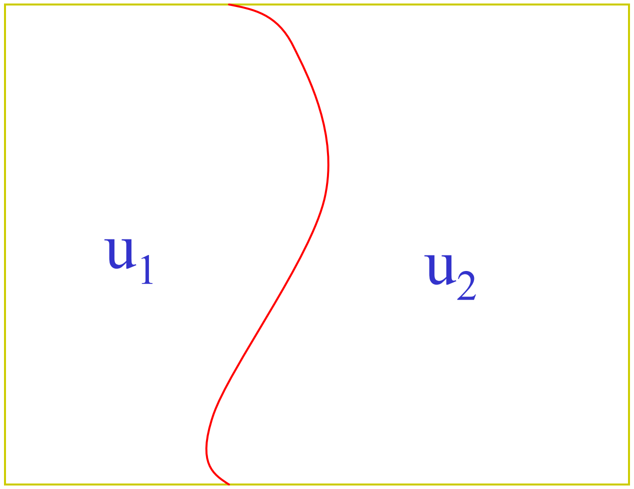
Muskat Problem

- Two sided Hele-Shaw
  - Darcy's law and incompressibility (i=1,2)

$$\mathbf{u}_i = V\mathbf{j} - k_i \nabla p_i \qquad \nabla \cdot \mathbf{u}_i = 0$$

  - Boundary conditions

$$p_1 = p_2, \qquad \mathbf{u}_1 \cdot \mathbf{n} = \mathbf{u}_2 \cdot \mathbf{n} = V_n$$



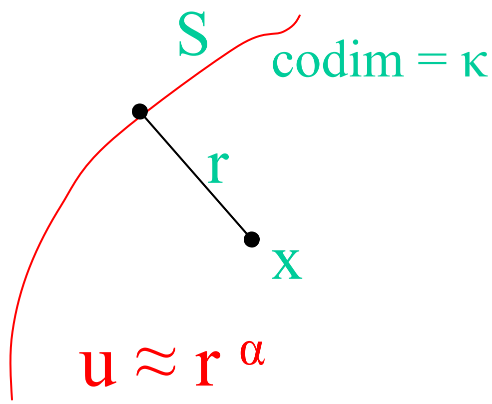
- Singularities
  - No exact solutions
- Analysis by Siegel, Howison & REC
  - Global existence in stable case (more viscous fluid moving into less viscous)
    - Initial data in Sobolev space, then becomes analytic for t>0
  - Analytic construction of singularities in unstable case
    - Curvature singularities, cusps not analyzed

Derivation of Singularity Requirements for Inviscid Energy Dissipation

- For singularity set S of codimension κ, singularity order α

$$dx = r^{\kappa-1} dr dx_S \qquad r = \mathrm{dist}(S)$$

$$u \approx r^{\alpha}$$



- Time derivative of energy $u_t + u \cdot \nabla u + \nabla p = 0$

$$(d/dt)\int |u|^2\, dx = \int u \cdot (u \cdot \nabla) u + u \cdot \nabla p dx$$

$$= \int_S \int r^{3\alpha-1} r^{\kappa-1} dr dx_S$$

- The convective integral is nonzero, only if it isn't absolutely integrable; i.e. $3\alpha - 1 + \kappa - 1 < -1$

$$3\alpha + \kappa < 1$$

Upper analytic solutions

- Look for upper analytic solution (k≥0)

$$u = \bar{u} + u^{+}$$

$$\bar{u} = (0, \bar{u}_z, \bar{u}_\theta)(r)$$

$$u^{+} = (u_r^{+}, u_z^{+}, u_\theta^{+})(r, z)$$

$$u^{+}(r,z) = \sum_{k\geq 1} \hat{u}_k(r) e^{ikz}$$

$$u^{+}(r,z,t) = \sum_{k\geq 1} \hat{u}_k(r) e^{ikz+\sigma kt}$$

- Because wavenumbers add, the coupling is one way (Siegel)

$$M_k \hat{u}_k = A_k(\sigma, \hat{u}_0, \ldots, \hat{u}_{k-1})$$

$$M_k = M_k(\sigma, \hat{u}_0)$$

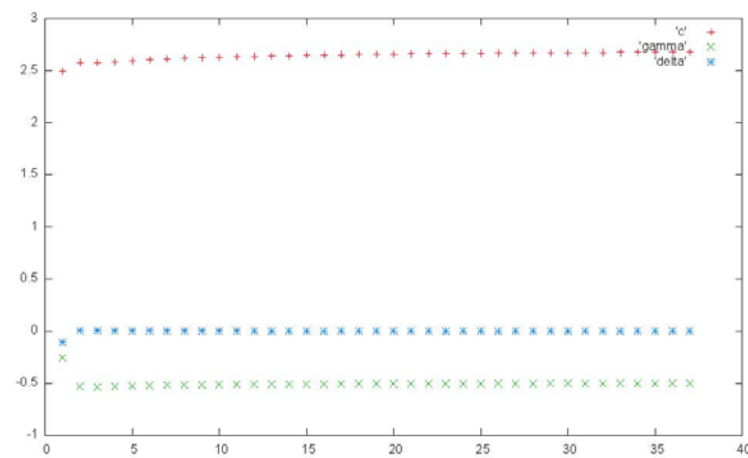
3D Euler (Pelz initial data)

Siegel & REC (2006)

- Solution method
  - Moore's approximation: $u = u_+ + u_-$
    - u_+ upper analytic in x,y,z, $u_- = u_+^*$ lower analytic, no interaction between them
  - Traveling wave ansatz

$$u_+(x, y, z, t) = u_+(x, y, z - i\sigma t)$$

  - No need for ultra-high precision
  - Highly symmetric (Kida)



- Singularity type
  - $u_+ \approx \varepsilon\, x^{-1/2}$
  - $\omega_+ \approx \varepsilon\, x^{-3/2}$
- Real singularity? ?
  - Satisfies known singularity criteria
  - Attempting to construct real singular solution as $u = u_+ + u_- + \varepsilon^2 u_c$