Nonexistence of self-similar singularities for the 3D Euler equations

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0. Introduction

We are concerned on the incompressible fluid equations in $\mathbb{R}^3$:

$$(E)\begin{cases} \dfrac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p \\ \text{div } \ v = 0, \\ v(x,0) = v_0(x), \end{cases}$$

where $v = (v^1, v^2, v^3)$, $v^j = v^j(x,t)$, $j = 1, 2, 3$, is the fluid velocity, and $p = p(x,t)$ is the pressure(L. Euler, 1757).

• Local in time existence of classical solution: For $v_0 \in H^m(\mathbb{R}^3)$, $m > 5/2$ the classical solution exists uniquely at least for 'small time'(Kato, Temam, Brezis,...)

• An outstanding open question:

Is there any local classical solution which evolves into a singularity in a finite time ?

• Beale-Kato-Majda's Blow-up criterion('84):

$$\limsup_{t\to T_*} \|v(t)\|_{H^m} = \infty \Leftrightarrow \int_0^{T_*} \|\omega(t)\|_{L^\infty} dt = \infty,$$

where $m > 5/2$, and ω =curl v is the vorticity.

• Refinements:

(Note the embeddings: $L^\infty \hookrightarrow BMO \hookrightarrow \dot{B}^0_{\infty,\infty}$)

The integrand $\|\omega(t)\|_{L^\infty}$ is replaced by $\|\omega(t)\|_{BMO}$(Kozono-Taniuchi['00]), and later by $\|\omega(t)\|_{\dot{B}^0_{\infty,\infty}}$ (C.['01, '02]; Kozono-Ogawa-Taniuchi['02]; Planchon['03])

• In this talk we are concerned on the possibility of self-similar type of blow-ups of the Euler equations.

• The self-similar singularity is one of the most popular scenarios in search of finite time singularity in nonlinear PDEs.
(e.g nonlinear Schrödinger equations, porous medium equation, ...)

2. Nonexistence of self-similar singularity

• The Euler system (E) has scaling property that if (v, p) is a solution, then for any $\lambda > 0$ and $\alpha \in \mathbb{R}$ the functions

$$v^{\lambda,\alpha}(x,t) = \lambda^{\alpha} v(\lambda x, \lambda^{\alpha+1} t), \quad p^{\lambda,\alpha}(x,t) = \lambda^{2\alpha} p(\lambda x, \lambda^{\alpha+1} t)$$

are also solutions with the initial data $v_0^{\lambda,\alpha}(x) = \lambda^{\alpha} v_0(\lambda x)$.

• In view of this it would be interesting to check if there exists any nontrivial solution $(v(x,t), p(x,t))$ of the form($\alpha \neq -1$),

$$\begin{cases} v(x,t) = \dfrac{1}{(T_* - t)^{\frac{\alpha}{\alpha+1}}} V\left(\dfrac{x}{(T_* - t)^{\frac{1}{\alpha+1}}}\right), \\ p(x,t) = \dfrac{1}{(T_* - t)^{\frac{2\alpha}{\alpha+1}}} P\left(\dfrac{x}{(T_* - t)^{\frac{1}{\alpha+1}}}\right) \end{cases}$$

: self-similar singular solution

• Substituting this into the Euler equation, we find that (V, P) should be a solution of the system

$$(SSE)\begin{cases} \dfrac{\alpha}{\alpha+1}V + \dfrac{1}{\alpha+1}(y\cdot\nabla)V + (V\cdot\nabla)V = -\nabla P \\ \text{div } V = 0, \end{cases}$$

which could be regarded as the Euler version of the Leray equations:

$$(Leray)\begin{cases} \dfrac{1}{2}V + \dfrac{1}{2}(y\cdot\nabla)V + (V\cdot\nabla)V = -\nabla P + \Delta V \\ \text{div } V = 0, \end{cases}$$

• Nonexistence of the self-similar blowing up solutions (in $L^3(\mathbb{R}^3)$) for the 3D Navier-Stokes equations was first proved by Nečas-Ružička -Šverák ('96)
(extended to the case $L^p(\mathbb{R}^3), p > 3$ by Tsai in '98)

• Use of the maximum principle was crucial in the above results for the Navier-Stokes equations.

• To be more specific let us define a scalar function Π and an elliptic operator $\mathcal{L}$ respectively as

$$\Pi = \frac{1}{2}|V|^2 + P + \frac{1}{2}y \cdot V,$$

$$\mathcal{L} = \Delta - (V + \frac{1}{2}y) \cdot \nabla.$$

• If (V, P) is a solution of the Leray equations, then we have the pointwise inequality,

$$\mathcal{L}\, \Pi \geq 0.$$

This provides us the desired maximum principle.

• In the derivation of the above inequality the existence of the laplacian(dissipation) term in the Leray equations is essential.

• Since the laplacian term is absent in the self-similar Euler equations, we cannot expect to have similar maximum principle.

• Therefore, we need different argument from Nečas-Ružička -Šverák's or Tsai's to exclude the self-similar singularity.

• Previous results for self-similar Euler system(SSE):

Theorem 1 (C. '04) *If* $V \in H^1(\mathbb{R}^3)$ *is a nontrivial(nonzero) classical solution of (SSE) in* $\mathbb{R}^3$ *, then the helicity of* V *is equal to zero, namely* $\int_{\mathbb{R}^3} V \cdot \Omega dx = 0,$ *where* $\Omega = \mathrm{curl} V$.

Main Results:

• Given a smooth velocity field $v(x,t)$, the particle trajectory map $a \mapsto X(a,t)$ is defined by the solution of the ODE system,

$$\frac{\partial X(a,t)}{\partial t} = v(X(a,t),t) \quad ; \quad X(a,0) = a \in \mathbb{R}^3.$$

Theorem 2 *There exists no finite time blowing up self-similar solution (v,p) to the 3D Euler equations represented by (V,P) above under the following assumptions:*

(i) Before singular time T_* the smooth solution v generates a particle trajectory map $a \mapsto X(a,t)$, which is an $C^1(\mathbb{R}^3 : \mathbb{R}^3)$ diffeomorphism.

(ii) The vorticity $\Omega =$curl V is nonzero, and there exists $p_1 > 0$ such that the $\Omega \in L^p(\mathbb{R}^3;\mathbb{R}^3)$ for all $p \in (0,p_1)$.

Remarks

• The condition (i), which is equivalent to the existence of the 'back-to-label map', $A(\cdot, t) = X^{-1}(\cdot, t)$, is guaranteed by a decay condition(regardless of its rate) for the velocity V(*P. Constantin, private communication*).

• For example, if $\Omega \in L^1_{loc}(\mathbb{R}^3; \mathbb{R}^3)$ and there exist constants R, K and $\varepsilon_1, \varepsilon_2 > 0$ such that $|\Omega(x)| \leq K e^{-\varepsilon_1 |x|^{\varepsilon_2}}$ for $|x| > R$, then we have $\Omega \in L^p(\mathbb{R}^3; \mathbb{R}^3)$ for all $p \in (0, 1)$.

• In the zero vorticity case $\Omega = 0$, since $\text{div } V = 0$ and $\text{curl } V = 0$, we have $V = \nabla h$, where $h(x)$ is a harmonic function. Hence, we have an easy example of self-similar blow-up,

$$v(x,t) = \frac{1}{(T_* - t)^{\frac{\alpha}{\alpha+1}}} \nabla h \left(\frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}} \right),$$

in $\mathbb{R}^3$, which is also the case of the 3D Navier-Stokes($\alpha = 1$). We do not consider this case in the theorem.

The previous theorem is a corollary of the following more general theorem.

Theorem 3 *Let v be a $C([0,T);C^1(\mathbb{R}^3))$ solution to (E), which satisfies the condition (i) of previous theorem. Suppose we have a representation of the vorticity of the solution v to the 3D Euler equations by*

$$\omega(x,t) = \Psi(t)\Omega(\Phi(t)x) \qquad \forall t \in [0,T)$$

where $\Omega = \mathrm{curl}\, V$ for some V, and there exists $p_1 > 0$ such that $\Omega \in L^p(\mathbb{R}^3)$ for all $p \in (0, p_1)$. Then, necessarily either $\det(\Phi(t)) \equiv \det(\Phi(0))$ on $[0,T)$, or $\Omega = 0$.

Proof of Theorem 2 from Theorem 3.

We apply Theorem 3 with

$$\Phi(t) = (T_* - t)^{-\frac{1}{\alpha+1}} I, \quad \text{and} \quad \Psi(t) = (T_* - t)^{-1},$$

where I is the unit matrix in $\mathbb{R}^{3\times 3}$. If $\alpha \neq -1$ and $t \neq 0$, then

$$\det(\Phi(t)) = (T_* - t)^{-\frac{3}{\alpha+1}} \neq T_*^{-\frac{3}{\alpha+1}} = \det(\Phi(0)).$$

Hence, we conclude that $\Omega = 0$. $\square$

Proof of Theorem 3.

• By consistency with the initial condition, $\omega_0(x) = \Psi(0)\Omega(\Phi(0)x)$, and hence $\Omega(x) = \Psi(0)^{-1}\omega_0([\Phi(0)]^{-1}x)$.

• Using this fact, we can rewrite the representation of self-similar solution in the form,

$$\omega(x,t) = G(t)\omega_0(F(t)x) \qquad \forall t \in [0,T),$$

where $G(t) = \Psi(t)/\Psi(0)$, $F(t) = [\Phi(0)]^{-1}\Phi(t)$.

• In order to prove the theorem it suffices to show that either $\det(F(t)) = 1$ for all $t \in [0,T)$, or $\omega_0 = 0$

• We set $A(x,t) := X^{-1}(x,t)$, which is the back-to-label map. Taking curl of the first equation of (E), we obtain the vorticity evolution equation,

$$\frac{\partial \omega}{\partial t} + (v \cdot \nabla)\omega = (\omega \cdot \nabla)v.$$

• This, taking dot product with ω, leads to

$$\frac{\partial |\omega|}{\partial t} + (v \cdot \nabla)|\omega| = \alpha|\omega|,$$

where $\alpha(x,t)$ is defined as

$$\alpha(x,t) = \begin{cases} \displaystyle\sum_{i,j=1}^{3} S_{ij}(x,t)\xi_i(x,t)\xi_j(x,t) & \text{if} \quad \omega(x,t) \neq 0 \\ 0 & \text{if} \quad \omega(x,t) = 0 \end{cases}$$

with

$$S_{ij} = \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right), \quad \text{and} \quad \xi(x,t) = \frac{\omega(x,t)}{|\omega(x,t)|}.$$

• In terms of the particle trajectory mapping we can rewrite the equation for $|\omega(x,t)$ as

$$\frac{\partial}{\partial t}|\omega(X(a,t),t)| = \alpha(X(a,t),t)|\omega(X(a,t),t)|.$$

• Integrating this along the particle trajectories $\{X(a,t)\}$, we have

$$|\omega(X(a,t),t)| = |\omega_0(a)| \exp\left[\int_0^t \alpha(X(a,s),s)ds\right].$$

• Taking into account the simple estimates

$$-\|\nabla v(\cdot,t)\|_{L^\infty} \le \alpha(x,t) \le \|\nabla v(\cdot,t)\|_{L^\infty} \quad \forall x \in \mathbb{R}^3,$$

we obtain that

$$\begin{aligned}|\omega_0(a)| \exp\left[-\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty} ds\right] &\le |\omega(X(a,t),t)| \\ &\le |\omega_0(a)| \exp\left[\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty} ds\right],\end{aligned}$$

which, using the back to label map, can be rewritten as

$$\begin{aligned}|\omega_0(A(x,t))| \exp\left[-\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty} ds\right] &\le |\omega(x,t)| \\ &\le |\omega_0(A(x,t))| \exp\left[\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty} ds\right].\end{aligned}$$

• Combining this with the self-similar representation formula, we have

$$
\begin{aligned}
|\omega_0(A(x,t))| \exp\left[-\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty} ds\right] &\leq G(t)|\omega_0(F(t)x)| \\
&\leq |\omega_0(A(x,t))| \exp\left[\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty} ds\right].
\end{aligned}
$$

• Given $p \in (0, p_1)$, computing $L^p(\mathbb{R}^3)$ norm of the each side of (1), we derive

$$
\begin{aligned}
\|\omega_0\|_{L^p} \exp\left[-\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty} ds\right] &\leq G(t)[\det(F(t))]^{-\frac{1}{p}} \|\omega_0\|_{L^p} \\
&\leq \|\omega_0\|_{L^p} \exp\left[\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty} ds\right],
\end{aligned}
$$

where we used the fact $\det(\nabla A(x,t)) \equiv 1$.

• Now, suppose $\Omega \neq 0$, which is equivalent to assuming that $\omega_0 \neq 0$, then we divide the above inequalities by $\|\omega_0\|_{L^p}$ to obtain

$$\exp\left[-\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty} ds\right] \leq G(t)[\det(F(t))]^{-\frac{1}{p}}$$
$$\leq \exp\left[\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty} ds\right].$$

• If there exists $t_1 \in (0,T)$ such that $\det(F(t_1)) \neq 1$, then either $\det(F(t_1)) > 1$ or $\det(F(t_1)) < 1$.

• In either case, setting $t = t_1$ and passing $p \searrow 0$ in the above inequalities, we deduce that

$$\int_0^{t_1} \|\nabla v(\cdot,s)\|_{L^\infty} ds = \infty.$$

• This contradicts with the assumption that the flow is smooth on $(0,T)$, i.e $v \in C([0,T);C^1(\mathbb{R}^3;\mathbb{R}^3))$. $\square$

Divergence-free transport equation

• The previous argument in the proof of main theorem can also be applied to the following transport equations by a divergence-free vector field in $\mathbb{R}^n$, $n \geq 2$.

$$(T) \begin{cases} \dfrac{\partial \theta}{\partial t} + (v \cdot \nabla)\theta = 0, \\ \quad \operatorname{div} v = 0, \\ \theta(x, 0) = \theta_0(x), \end{cases}$$

where $v = (v_1, \cdots, v_n) = v(x, t)$, and $\theta = \theta(x, t)$.

• In view of the invariance of the transport equation under the scaling transform,

$$\begin{aligned} v(x,t) &\mapsto v^{\lambda,\alpha}(x,t) = \lambda^{\alpha} v(\lambda x, \lambda^{\alpha+1} t), \\ \theta(x,t) &\mapsto \theta^{\lambda,\alpha,\beta}(x,t) = \lambda^{\beta} \theta(\lambda x, \lambda^{\alpha+1} t) \end{aligned}$$

for all $\alpha, \beta \in \mathbb{R}$ and $\lambda > 0$, the self-similar blowing up solution is of the form,

$$\begin{aligned} v(x,t) &= \frac{1}{(T_* - t)^{\frac{\alpha}{\alpha+1}}} V\left(\frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}} \right), \\ \theta(x,t) &= \frac{1}{(T_* - t)^{\beta}} \Theta\left(\frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}} \right) \end{aligned}$$

for $\alpha \neq -1$ and t sufficiently close to T_*.

• We have following theorem.

Theorem 4 *Suppose there exist* $\alpha \neq -1$, $\beta \in \mathbb{R}$ *and solution* (V, Θ) *to the system (ST) with* $\Theta \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)$ *for some* p_1, p_2 *such that* $0 < p_1 < p_2 \leq \infty$. *Then,* $\Theta = 0$.

Corollary 1 *There exist no self-similar blow-ups for the density dependent Euler equations, the 2D (inviscid) Boussinesq system, and the 2D quasi-geostrophic equations under the appropriate integrability conditions.*

• Similarly to the case of the Euler equations the above theorem is a corollary of the following one.

Theorem 5 *Suppose there exists $T > 0$ such that there exists a representation of the solution $\theta(x,t)$ to the system (T) by*

$$\theta(x,t) = \Psi(t)\Theta(\Phi(t)x) \qquad \forall t \in [0,T).$$

Assume there exist $p_1 < p_2$ with $p_1, p_2 \in (0,\infty]$ such that $\Theta \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)$. Then, necessarily either $\det(\Phi(t)) \equiv \det(\Phi(0))$ and $\Psi(t) \equiv \Psi(0)$ on $[0,T)$, or $\Theta = 0$.

- Density-dependent Euler equations in $\mathbb{R}^n$, $n \geq 2$.

$$\begin{cases} \dfrac{\partial \rho v}{\partial t} + \operatorname{div}(\rho v \otimes v) = -\nabla p, \\ \dfrac{\partial \rho}{\partial t} + (v \cdot \nabla)\rho = 0, \\ \quad \operatorname{div} \ v = 0, \\ v(x,0) = v_0(x), \quad \rho(x,0) = \rho_0(x), \end{cases}$$

where $v = (v_1, \cdots, v_n) = v(x,t)$ is the velocity, $\rho = \rho(x,t) \geq 0$ is the scalar density of the fluid, and $p = p(x,t)$ is the pressure.

- The Boussinesq system in $\mathbb{R}^2$.

$$\begin{cases} \dfrac{\partial v}{\partial t} + (v \cdot \nabla) v = -\nabla p + \theta e_1, \\ \dfrac{\partial \theta}{\partial t} + (v \cdot \nabla)\theta = 0, \\ \quad \text{div } v = 0, \\ v(x,0) = v_0(x), \qquad \theta(x,0) = \theta_0(x) \end{cases}$$

where $v = (v_1, v_2) = v(x,t)$ is the velocity, $e_1 = (1,0)$, and $p = p(x,t)$ is the pressure, while $\theta = \theta(x,t)$ is the temperature function.

- The 2D quasi-geostrophic equation

$$(QG)\begin{cases} \dfrac{\partial \theta}{\partial t} + (v \cdot \nabla)\theta = 0, \\ v = -\nabla^{\perp}(-\Delta)^{-\frac{1}{2}}\theta \left(= \nabla^{\perp} \displaystyle\int_{\mathbb{R}^2} \frac{\theta(y,t)}{|x-y|} dy \right), \\ \theta(x,0) = \theta_0(x), \end{cases}$$

where $\nabla^{\perp} = (-\partial_2, \partial_1)$.

3. Nonexistence of ‘asymptotically’ self-similar singularity

• We now consider the possibility of ‘asymptotic’ evolution of the local smooth solution toward a self-similar singularity as $t \to T$(the possible singular time).

Theorem 6 *Let $v \in C([0,T); B^1_{\infty,1}(\mathbb{R}^3))$ be a classical solution to the 3D Euler equations. Suppose there exist $p_1 > 0$, $\alpha > -1$, $\bar{V} \in C^1(\mathbb{R}^3)$ such that $\bar{\Omega} =$curl $\bar{V} \in L^q(\mathbb{R}^3)$ for all $q \in (0, p_1)$, and*

$$\lim_{t \nearrow T} (T-t) \left\| \omega(\cdot, t) - \frac{1}{T-t} \bar{\Omega} \left(\frac{\cdot}{(T-t)^{\frac{1}{\alpha+1}}} \right) \right\|_{\dot{B}^0_{\infty,1}} = 0.$$

Then, $\bar{\Omega} = 0$, and $v \in C([0, T+\delta); B^1_{\infty,1}(\mathbb{R}^3))$ for some $\delta > 0$.

• The proof uses the following continuation principle for local solution.

Proposition 1 *Let* $v \in C([0,T); B^1_{\infty,1}(\mathbb{R}^3))$ *be a classical solution to the 3D Euler equations. There exists an absolute constant* $\eta > 0$ *such that if*

$$\inf_{0 \le t < T} (T-t)\|\omega(t)\|_{\dot{B}^0_{\infty,1}} < \eta,$$

then, $v \in C([0,T+\delta); B^1_{\infty,1}(\mathbb{R}^3))$ *for some* $\delta > 0$*.*

• The proof of this proposition is a slight variation of local a priori estimate in the Besov space.

Outline of the Proof:

• We change from physical variables $(x, t) \in \mathbb{R}^3 \times [0, T)$ into 'self-similar variable' $(y, s) \in \mathbb{R}^3 \times [0, \infty)$ as follows:

$$y = \frac{x}{(T-t)^{\frac{1}{\alpha+1}}}, \quad s = \frac{\alpha}{\alpha+1} \log\left(\frac{T}{T-t}\right).$$

• Based on this change of variables, we transform $(v, p) \mapsto (V, P)$ according to

$$v(x,t) = \frac{1}{(T-t)^{\frac{\alpha}{\alpha+1}}} V(y,s), \quad p(x,t) = \frac{1}{(T-t)^{\frac{2\alpha}{\alpha+1}}} P(y,s).$$

• Substituting (v, p) into the Euler system we obtain the

$$(E_1)\begin{cases} \dfrac{\alpha}{\alpha+1}V_s + \dfrac{\alpha}{\alpha+1}V + \dfrac{1}{\alpha+1}(y\cdot\nabla)V + (V\cdot\nabla)V = -\nabla P, \\ \operatorname{div} V = 0, \\ V(y,0) = V_0(y) = T^{\frac{\alpha}{\alpha+1}} v_0(T^{\frac{1}{\alpha+1}} y). \end{cases}$$

• In terms of V our convergence condition is translated into

$$\lim_{s\to\infty} \|\Omega(\cdot, s) - \bar{\Omega}(\cdot)\|_{\dot{B}^0_{\infty,1}} = 0,$$

where we set $\Omega = \operatorname{curl} V$.

• From this we show easily that $\bar{V}$ is a stationary solution of (E_1).

• Using the previous nonexistence result we have $\bar{\Omega} = 0$.

• Hence the convergence hypothesis of the theorem reduces to

$$\lim_{t \nearrow T} (T - t) \|\omega(\cdot, t)\|_{\dot{B}^0_{\infty,1}} = 0.$$

• Applying our continuation principle, we can continue our local solution beyond T. □

4. Nonexistence of asymptotically self-similar solutions for the 3D Navier-Stokes equations

Here we are concerned on the following 3D Navier-Stokes equations.

$$
(\mathrm{NS})\begin{cases}
\dfrac{\partial v}{\partial t} + (v\cdot\nabla)v = -\nabla p + \Delta v, \quad (x,t)\in\mathbb{R}^3\times(0,\infty)\\
\mathrm{div}\ v = 0, \quad (x,t)\in\mathbb{R}^3\times(0,\infty)\\
v(x,0) = v_0(x), \quad x\in\mathbb{R}^3
\end{cases}
$$

Theorem 7 *Let* $p \in [3, \infty)$, *and* $v \in C([0, T); L^p(\mathbb{R}^3))$ *be a classical solution to (NS). Suppose there exists* $\bar{V} \in L^p(\mathbb{R}^3)$ *such that*

$$\lim_{t \nearrow T} (T - t)^{\frac{p-3}{2p}} \left\| v(\cdot, t) - \frac{1}{\sqrt{T-t}} \bar{V}\left(\frac{\cdot}{\sqrt{T-t}}\right) \right\|_{L^p} = 0.$$

Then, $\bar{V} = 0$, *and* $v \in C([0, T + \delta); L^p(\mathbb{R}^3))$ *for some* $\delta > 0$.

• Hou and Li obtained previously this result for $p \in (3, \infty)$ in '06, and the proof can be substantially simplified if we use the following continuation principle.

Proposition 2 *Let* $p \in [3, \infty)$*, and* $v \in C([0,T); L^p(\mathbb{R}^3))$ *be a classical solution to (NS). There exists a constant* $\eta > 0$ *depending on* p *such that if*

$$\inf_{0 \le t < T} (T-t)^{\frac{p-3}{2p}} \|v(t)\|_{L^p} < \eta,$$

then, $v \in C([0, T+\delta); L^p(\mathbb{R}^3))$ *for some* $\delta > 0$*.*

- For $p = 3$ this reduces to the small data global regularity result in $L^3(\mathbb{R}^3)$ due to Kato('84)

- For $p > 3$ proof is immediate from Leray's result('34) on the blow-up rate estimate,

$$\|v(t)\|_{L^p} \ge \frac{C}{(T_* - t)^{\frac{p-3}{2p}}},$$

where T_* is the assumed first blow-up time.

Localization

- We denote $B(z,r) = \{x \in \mathbb{R}^3 \,|\, |x - z| < r\}$ below.

Theorem 8 *Let $p \in [3, \infty)$, and $v \in C([0,T); L^p(\mathbb{R}^3))$ be a classical solution to (NS). Suppose either one of the followings hold.*

(i) Let $q \in [3, \infty)$. Suppose there exists $\bar{V} \in L^p(\mathbb{R}^3)$ and $R \in (0, \infty)$ such that we have

$$\lim_{t \nearrow T} (T-t)^{\frac{q-3}{2q}} \sup_{t<\tau<T} \left\| v(\cdot, \tau) - \frac{1}{\sqrt{T-\tau}} \bar{V}\left(\frac{\cdot - z}{\sqrt{T-\tau}} \right) \right\|_{L^q(B(z, R\sqrt{T-t}))} = 0$$

(ii) Let $q \in [2, 3)$. Suppose there exists $\bar{V} \in L^p(\mathbb{R}^3)$ such that the above holds for all $R \in (0, \infty)$.

Then, $\bar{V} = 0$, and (z, T) is the regular point of $v(x,t)$.

• We note that, in contrast to Theorem 7, besides the localization in space the range of $q \in [2,3)$ is also allowed for the possible convergence of the local classical solution to the self-similar profile.

• For the proof we use the following regularity criterion for the (NS) due to Gustafson-Kang-Tsai('06):

Theorem 9 *Let $q \in (3/2, \infty)$. Suppose v is a suitable weak solution of (NS) in a cylinder, $Q = B(z, r_1) \times (t - r_1^2, T)$ in the sense of Caffarelli-Kohn-Nirenberg. Then, there exists a constant $\eta = \eta(q) >$ such that if*

$$\limsup_{r \searrow 0} \left\{ r^{\frac{q-3}{q}} \operatorname{ess} \sup_{t-r^2<\tau<t} \|v(\cdot,\tau)\|_{L^q(B(z,r))} \right\} \leq \eta,$$

then (z, T) is the regular point of $v(x,t)$.