Incompressible Fluid Turbulence & Generalized Solutions of Euler Equations

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Analytical and Computational Challenges of
Incompressible Flows at High Reynolds Number
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Energy Dissipation at Zero Viscosity

Experiments

H. L. Dryden, Q. Appl. Maths 1, 7 (1943)

K. R. Sreenivasan, Phys. Fluids 27, 1048 (1984)

O. Cadot et al. Phys. Rev. E 56, 427 (1997)

R. B. Pearson et al. Phys. Fluids 14, 1288 (2002)

Simulations

K. R. Sreenivasan, Phys. Fluids 10, 528 (1998)

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Energy dissipation rate in various turbulent flows appears to remain positive as Reynolds number tends to infinity.

Onsager's Theorem

Theorem: Let the velocity

$$\mathbf{u} \in L^3([0,T], B_p^{\alpha}(\mathbf{T}^d)) \cap C([0,T], L^2(\mathbf{T}^d))$$

be a weak (distributional) solution of the incompressible Euler equations with $p \geq 3$ and $\alpha > 1/3$. Then energy is conserved.

The Hölder case ($p = \infty$) was stated by Lars Onsager (1949). For an historical discussion, see Eyink & Sreenivasan, RMP 78, 87 (2006).

The converse: to explain the observed energy dissipation requires $\alpha \leq 1/3$ in the infinite Reynolds number limit. Onsager's prediction of such (near) singularities has been confirmed by experiment and simulation:

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A. Arneodo et al. , Physica A 213, 232 (1995)

P. Kestener and A. Arneodo, Phys. Rev. Lett. 93, 044501 (2004)

Smoothed Equations

The proof of Onsager's theorem (following Constantin et al. (1994)) uses the low-pass filtering operation

$$\overline{\mathbf{u}}_\ell(\mathbf{x}) = \int d^d\mathbf{r}\ G_\ell(\mathbf{r})\mathbf{u}(\mathbf{x}+\mathbf{r})$$

retaining the scales $> \ell$.

The filtered (mollified) equations are

$$\partial_t\overline{\mathbf{u}}_\ell + \boldsymbol{\nabla}\boldsymbol{\cdot}[\overline{\mathbf{u}}_\ell\overline{\mathbf{u}}_\ell + \boldsymbol{\tau}_\ell] = -\boldsymbol{\nabla}\overline{p}_\ell$$

where $\boldsymbol{\tau}_\ell$ is the stress tensor

$$\boldsymbol{\tau}_\ell = \overline{(\mathbf{u}\otimes\mathbf{u})}_\ell - \overline{\mathbf{u}}_\ell \otimes \overline{\mathbf{u}}_\ell,$$

from the eliminated subscale modes.

This is the same approach used in Large-Eddy Simulation (LES) models of turbulent flow. In that case, a closure equation is employed for the stress tensor $\boldsymbol{\tau}_\ell$.

Nonlinear Energy Cascade

Large-scale energy density (per mass):

$$e_\ell = \frac{1}{2}|\overline{\mathbf{u}}_\ell|^2$$

Space transport of large-scale energy:

$$\mathbf{J}_\ell = (e_\ell + \overline{p}_\ell)\overline{\mathbf{u}}_\ell + \overline{\mathbf{u}}_\ell \cdot \boldsymbol{\tau}_\ell$$

Energy flux to length-scales $< \ell$:

$$\Pi_\ell = -\boldsymbol{\nabla}\overline{\mathbf{u}}_\ell : \boldsymbol{\tau}_\ell$$

Large-scale energy balance:

$$\partial_t e_\ell + \boldsymbol{\nabla} \cdot \mathbf{J}_\ell = -\Pi_\ell$$

Turbulent energy cascade is the dynamical transfer of kinetic energy from large-scales to small-scales via Π_ℓ.

Proof of Onsager's Theorem

The proof uses crucially that energy flux Π_ℓ depends only upon *velocity-increments*

$$\delta\mathbf{u}(\mathbf{x};\mathbf{r}) \equiv \mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x}).$$

In particular,

$$\boldsymbol{\tau}_\ell = \int d\mathbf{r}\; G_\ell(\mathbf{r})\delta\mathbf{u}(\mathbf{r}) \otimes \delta\mathbf{u}(\mathbf{r})$$

$$-\int d\mathbf{r}\; G_\ell(\mathbf{r})\delta\mathbf{u}(\mathbf{r}) \otimes \int d\mathbf{r}\; G_\ell(\mathbf{r})\delta\mathbf{u}(\mathbf{r})$$

and

$$\boldsymbol{\nabla}\overline{\mathbf{u}}_\ell = -(1/\ell)\int d\mathbf{r}\; (\boldsymbol{\nabla}G)_\ell(\mathbf{r})\delta\mathbf{u}(\mathbf{r})$$

It is then easy to see, for example, that

$$\Pi_\ell(\mathbf{x},t) = O(\ell^{3\alpha-1})$$

if $\mathbf{u}(t) \in C^\alpha(\mathbf{x})$.

Strong Scale-Locality Approximation

It is not hard to show that the energy cascade is *scale-local* when $0 < \alpha < 1$. I.e. most of the large-scale gradient $\boldsymbol{\nabla}\overline{\mathbf{u}}_\ell$ and stress $\boldsymbol{\tau}_\ell$ come from scales $\sim \ell$. See Eyink (2005).

Assuming that energy cascade is strongly UV local leads to the approximate formula

$$\boldsymbol{\tau}^{SL} \approx \overline{\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}} - \overline{\overline{\mathbf{u}}} \otimes \overline{\overline{\mathbf{u}}} \approx C\ell^2 (\boldsymbol{\nabla}\overline{\mathbf{u}})^\top (\boldsymbol{\nabla}\overline{\mathbf{u}}).$$

where $C = (1/d) \int d\mathbf{r}\, |\mathbf{r}|^2 G(\mathbf{r})$ for a spherically symmetric filter kernel G (omitting ℓ subscripts).

Energy flux in the same approximation is

$$\Pi^{SL} = C\ell^2 \left[-\mathrm{Tr}\,(\overline{\mathbf{S}}^3) + \frac{1}{4} \overline{\boldsymbol{\omega}}^\top \overline{\mathbf{S}} \overline{\boldsymbol{\omega}} \right]$$

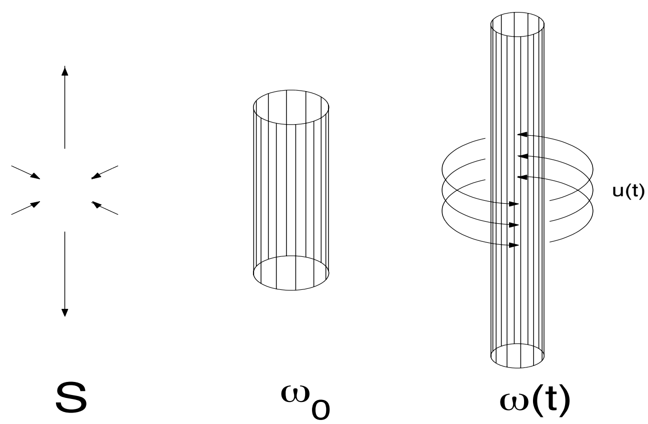
(Borue & Orszag, 1998)

Energy cascade arises from *vortex-stretching* and *strain-skewness*. Note that

$$-\langle \mathrm{Tr}\,(\overline{\mathbf{S}}^3) \rangle = \frac{3}{4} \langle \overline{\boldsymbol{\omega}}^\top \overline{\mathbf{S}} \overline{\boldsymbol{\omega}} \rangle$$

where $\langle \cdot \rangle$ is average over any homogeneous ensemble (Betchov, 1956).

Vortex-Stretching Mechanism



Energy-Cascade by Vortex-Stretching.

A small-scale vortex in a large-scale strain is elongated, on average, along an expanding direction. As the vortex lengthens it also becomes more slender and thus spins up, by Kelvin's Theorem. The spin-up motion produces more stress in the contracting directions and thus opposes the large-scale strain.

Taylor & Green (1937)

"To explain this process is, perhaps, the fundamental problem in turbulent motion. It seems clear that it is intimately associated with diffusion. Suppose that eddying motion of some definite scale is generated in a non-viscous fluid. Consider two particles A, B, situated on the same vortex line in a turbulent fluid of flow and separated initially by small distance d_0.

If the turbulence is diffusive, in the sense that a concentrated collection of particles spreads into a diffuse cloud (and turbulence is always found to be diffusive), the average distance d between pairs of particles like A and B increases continually.

If the fluid were non-viscous the continual increase in the average value of d^2 would necessarily involve a continual increase in ω^2, ω being the resultant vorticity at any point. In fact, the equation for conservation of circulation in a non-viscous fluid is

$$\frac{\omega}{d} = \frac{\omega_0}{d_0} \quad \text{or} \quad \frac{\omega^2}{d^2} = \frac{\omega_0^2}{d_0^2} \tag{10}$$

where ω_0 is the initial resultant vorticity when $d = d_0$. Hence ω^2 increases continually as d^2 increases.

The mean rate of dissipation of energy in a viscous fluid is

$$\overline{W} = \mu(\overline{\xi^2 + \eta^2 + \zeta^2}) = \mu\overline{\omega^2}, \tag{11}$$

so that if turbulence is set up in a slightly viscous fluid by the formation of large-scale eddies (e.g. as in a wind-tunnel when the wind meets a large-scale obstruction) we may expect first an increase in $\overline{\omega^2}$ in accordance with (10).

When $\overline{\omega^2}$ has increased to some value which depends on the viscosity, it is no longer possible to neglect the effect of viscosity in the equation for the conservation of circulation, so that (10) ceases to be true. Experiment shows, in fact, that in a wind tunnel $\overline{W}$ reaches the definite value indicated by (5) and (6)."

Kelvin Circulation Theorem

For a closed, oriented, rectifiable loop $C \subset \Lambda$ at an initial time t_0, the circulation

$$\Gamma(C,t) = \oint_{C(t)} \mathbf{u}(t)\cdot d\mathbf{x} = \int_{S(t)} \boldsymbol{\omega}(t)\cdot d\mathbf{A}$$

where $C(t)$ is the loop at time t advected by the fluid velocity $\mathbf{u}$, $S(t)$ is any surface spanning that loop, and $\boldsymbol{\omega}(t) = \boldsymbol{\nabla}\times\mathbf{u}(t)$.

Kelvin-Helmholtz Theorem:

$$\frac{d}{dt}\Gamma(C,t) = \nu \oint_{C(t)} \triangle\mathbf{u}(t)\cdot d\mathbf{x}.$$

The Kelvin theorem for all loops C is formally equivalent to the Navier-Stokes equation.

Question:

Does the righthand side vanish as $\nu \to 0$? (See recent work of P. Constantin).

Kelvin Theorem in the Inertial-Range

Large-scale circulation:

$$\overline{\Gamma}_\ell(C,t) = \oint_{\overline{C}_\ell(t)} \overline{\mathbf{u}}_\ell(t){\boldsymbol{\cdot}}d\mathbf{x} = \int_{\overline{S}_\ell(t)} \overline{\boldsymbol{\omega}}_\ell(t){\boldsymbol{\cdot}}d\mathbf{S}$$

where $\overline{C}_\ell(t)$ and $\overline{S}_\ell(t)$ are advected by $\overline{\mathbf{u}}_\ell$, which generates a flow of diffeomorphisms.

Circulation balance:

$$(d/dt)\overline{\Gamma}_\ell(C,t) = \oint_{\overline{C}_\ell(t)} \mathbf{f}_\ell(t){\boldsymbol{\cdot}}\, d\mathbf{x}$$

where

$$\mathbf{f}_\ell = \overline{(\mathbf{u}{\times}\boldsymbol{\omega})}_\ell - \overline{\mathbf{u}}_\ell{\times}\overline{\boldsymbol{\omega}}_\ell = -\boldsymbol{\nabla}{\boldsymbol{\cdot}}\boldsymbol{\tau}_\ell + \boldsymbol{\nabla}k_\ell$$

is the *turbulent vortex-force* and $k_\ell = (1/2)\mathrm{Tr}\,\boldsymbol{\tau}_\ell$ is the subgrid kinetic energy.

Define *loop-torque* $K_\ell(C) \equiv -\oint_C \mathbf{f}_\ell{\boldsymbol{\cdot}}d\mathbf{x}$. Is the limit

$$\lim_{\ell\to 0} K_\ell(C) = 0?$$

Rigorous Bound

If velocity $\mathbf{u} \in C^{\alpha}$ and $L(C)$ is the length of C

$$|K_\ell(C)| \leq (\text{const.}) L(C) \ell^{2\alpha-1}$$

See G. Eyink, C. R. Physique, 7, 449 (2006).

Thus $K_\ell(C) \to 0$ if C is rectifiable and $h > 1/2$. However, in turbulent flow at infinite Reynolds number the most probable Hölder exponent is $h_* \approx 1/3$ and material curves $C(t)$ are fractal!

A **cascade of circulations** is possible. But note that

$$\langle K_\ell(C) \rangle = 0$$

for any homogeneous average, for all ℓ and C. There is no *mean* cascade of circulations.

Circulation Cascade: Numerical Results

PDF & RMS of subscale torque are nearly independent of $k_c = 2\pi/\ell$ in the turbulent inertial-range: the cascade of circulations is persistent in scale.

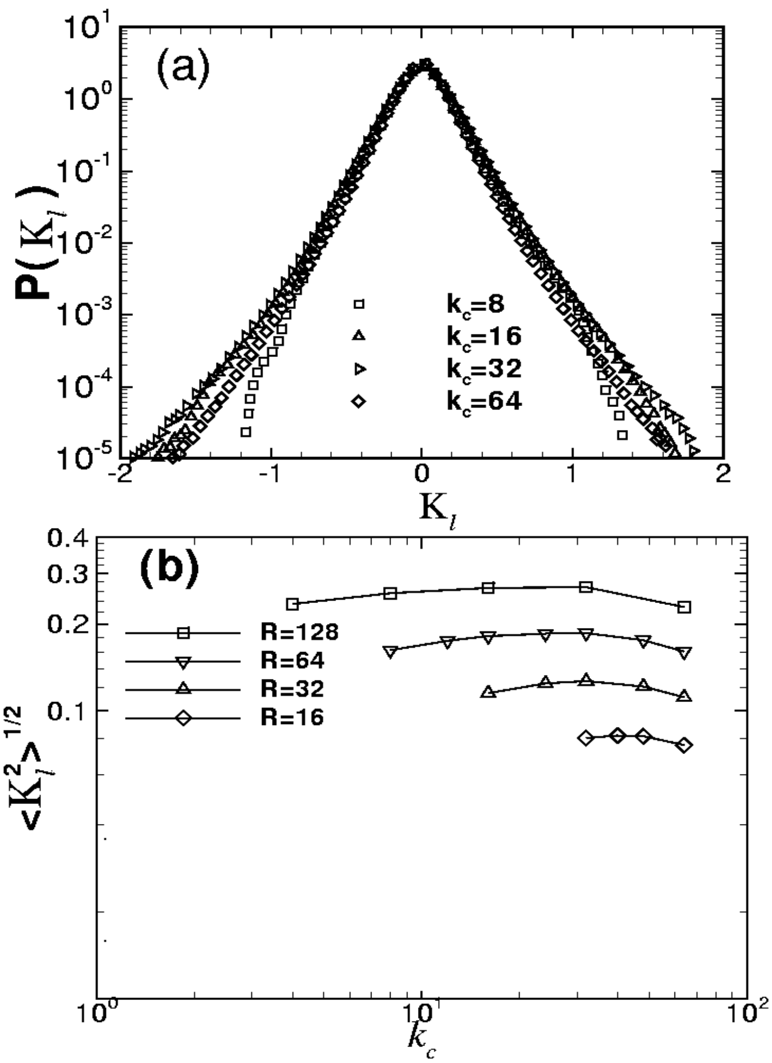


Figure. (a) PDF and (b) RMS of the subscale *loop-torque* $K_\ell(C) = -\oint_C \mathbf{f}_\ell \cdot d\mathbf{x}$, for square loops C of edge-length 64 in 1024^3 DNS of forced 3D hydrodynamic turbulence. (Chen et al., 2006)

Kraichnan Passive Scalar Model

Definition: For $(\mathbf{x}, t) \in \Lambda \times [0, \infty)$

$$(\partial_t + \mathbf{u}(\mathbf{x}, t) \circ \boldsymbol{\nabla})\theta(\mathbf{x}, t) = 0$$

interpreted in the Stratonovich sense. The advecting velocity $\mathbf{u}(\mathbf{x}, t)$ is a Gaussian (generalized) random field, with mean $\overline{\mathbf{u}}(\mathbf{x}, t)$ and covariance

$$\langle \widetilde{u}^i(\mathbf{x}, t)\widetilde{u}^j(\mathbf{x}', t')\rangle = D^{ij}(\mathbf{x}, \mathbf{x}'; t)\delta(t - t').$$

More precisely,

$$d\theta(\mathbf{x}, t) = -\overline{\mathbf{u}}(\mathbf{x}, t) \cdot \boldsymbol{\nabla}\theta(\mathbf{x}, t)dt - \widetilde{\mathbf{U}}(\mathbf{x}, dt) \circ \boldsymbol{\nabla}\theta(\mathbf{x}, t)$$

and

$$\widetilde{\mathbf{U}}(\mathbf{x}, t) = \sum_{n=0}^{\infty} \sqrt{\lambda_n(t)}\mathbf{e}_n(\mathbf{x}, t)W_n(t),$$

where $\lambda_n(t)$ and $\mathbf{e}_n(\mathbf{x}, t)$ for $n = 0, 1, 2, ...$ are the eigenvalues and eigenfunctions of the positive, trace-class operator with kernel $\mathbf{D}(\mathbf{x}, \mathbf{x}'; t)$ acting on $L^2(\Lambda, \mathbf{R}^d)$ and $W_n(t)$, $n = 0, 1, 2, ...$ independent Brownian motions.

Hölder-continuity in space:

$$E\left(|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')|^2\right) = C|\mathbf{x} - \mathbf{x}'|^{2\alpha}$$

with $0 < \alpha < 1$.

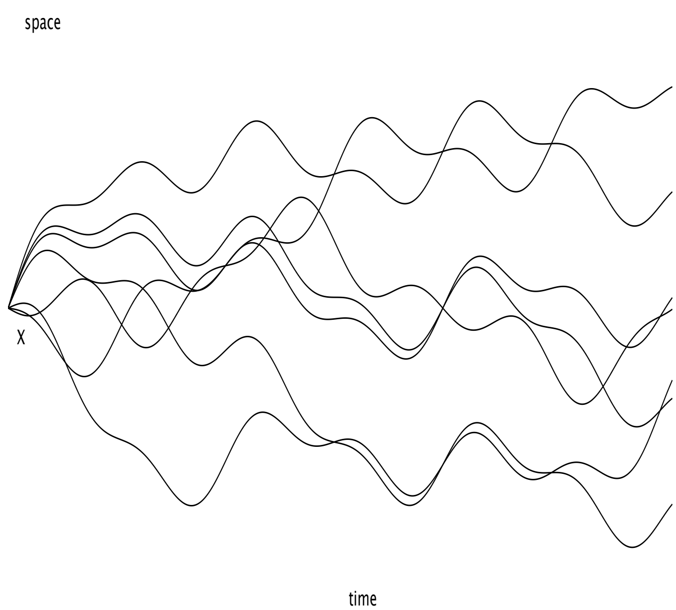
Dissipative Anomaly

For the Hölder velocity case, the scalar equation is *dissipative* and the scalar "energy" is not conserved.

For $t > t'$,

$$\int d\mathbf{x}\,|\theta(\mathbf{x},t)|^2 < \int d\mathbf{x}\,|\theta(\mathbf{x},t')|^2$$

Spontaneous Stochasticity: Non-uniqueness of Lagrangian trajectories for a fixed velocity realization!



Representation of solution by random characteristics:

$$\theta(\mathbf{x},t) = \int P_{\mathbf{x},t}(d\mathbf{x}'|\mathbf{u})\theta(\mathbf{x}'(t'),t')$$

LeJan-Raimond Weak Solutions

Ito formulation (assume $\overline{\mathbf{u}} = 0$):

$$d\theta(\mathbf{x},t) = \frac{1}{2} D_{ij}(\mathbf{x},\mathbf{x};t)\nabla^i\nabla^j\theta(\mathbf{x},t)dt - \mathbf{U}(\mathbf{x},dt)\boldsymbol{\cdot}\boldsymbol{\nabla}\theta(\mathbf{x},t)$$

or

$$\theta(\mathbf{x},t) = P_{t,t'}\theta(\mathbf{x},t') - \int_{t'}^{t} P_{t,s}\mathbf{U}(\mathbf{x},ds)\boldsymbol{\cdot}\boldsymbol{\nabla}\theta(\mathbf{x},t) = S^{\mathbf{u}}_{t,t'}\theta(\mathbf{x},t')$$

where $P_{t,t'}$ is the Markov semigroup of the diffusion with generator $A(t) = \frac{1}{2}D_{ij}(\mathbf{x},\mathbf{x};t)\nabla^i\nabla^j$ and $S^{\mathbf{u}}_{t,t'}$ is defined by the Wiener chaos (or Krylov-Veretennikov) expansion:

$$S^{\mathbf{u}}_{t,t'} = \sum_{n=0}^{\infty}(-1)^n \int_{t'\le s_1\le s_2\le\cdots\le s_n\le t} P_{t,s_n}\mathbf{U}(ds_n)\boldsymbol{\cdot}\boldsymbol{\nabla}P_{s_n,s_{n-1}}$$

$$\cdots\mathbf{U}(ds_2)\boldsymbol{\cdot}\boldsymbol{\nabla}P_{s_2,s_1}\mathbf{U}(ds_1)\boldsymbol{\cdot}\boldsymbol{\nabla}P_{s_1,t'}.$$

LeJan & Raimond (2002, 2004) prove that this defines a random Markov semigroup (conditioned upon $\mathbf{u}$).

*The resulting weak solutions are *robust*: they are also obtained by smoothing $\mathbf{u} \to \overline{\mathbf{u}}_\ell$ or adding a small scalar diffusivity κ and taking limits $\ell \to 0$ or $\kappa \to 0$.

*Analogous to the "generalized Euler flows" of Brenier & Shnirelman, but for the Cauchy initial-value problem.

Passive Vector Model

Ideal Ohm's Law for vector potential $\mathbf{A}$

$$\partial_t \mathbf{A} + \boldsymbol{\nabla}(\mathbf{u}\cdot\mathbf{A}) - \mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \mathbf{0}$$

or induction equation for magnetic field $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$

$$\partial_t \mathbf{B} + \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B}) = \mathbf{0}$$

Alfvén's Theorem: Conservation of magnetic flux

$$\Phi(C,t) = \oint_{C(t)} \mathbf{A}(t)\cdot d\mathbf{x} = \int_{S(t)} \mathbf{B}(t)\cdot d\mathbf{S}.$$

More generally, a *passive k-form* satisfies

$$\partial_t \omega^k + L_{\mathbf{u}}\omega^k = 0$$

with $L_{\mathbf{u}}$ the Lie-derivative along the vector field $\mathbf{u}$. This is equivalent to conservation of the integral invariants

$$I^k(t) = \int_{C^k(t)} \omega^k(t)$$

for any k-dimensional volume $C^k(t)$ comoving with $\mathbf{u}$. Then $k = 0$ is the passive scalar, $k = 1$ the passive vector and $k = 2$ the passive magnetic field.

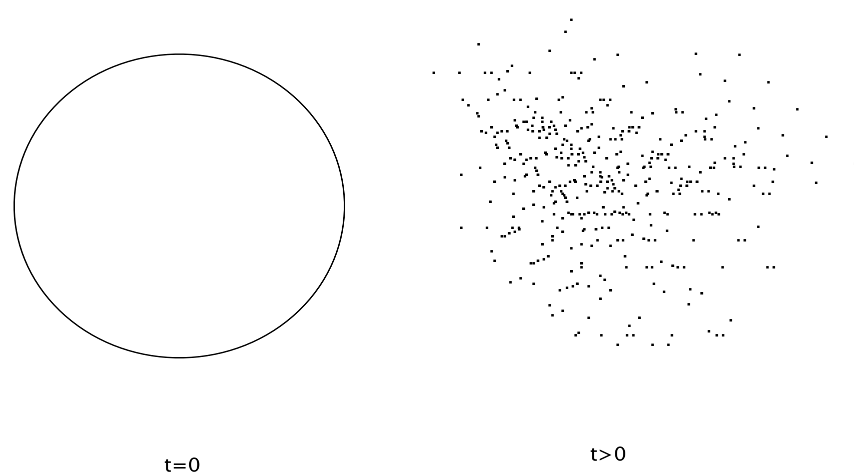
The *Kazantsev model* of kinematic dynamo uses a white-in-time velocity field $\mathbf{u}(t)$ in these equations, interpreted in Stratonovich sense.

Is Alfvén's Theorem Valid for a Hölder Velocity?

Consider the Kraichnan ensemble of velocities $\mathbf{u}(t)$, with $0 < \alpha < 1$. Then is

$$(d/dt)\Phi(C,t) = 0 \ ?$$

A material loop C(t) may not even exist! Because of stochastic splitting of Lagrangian particles in a non-smooth velocity, an initial loop C may "explode" at any $t > 0$ into a cloud of disconnected particles:



For smooth velocity ($\alpha = 1$), the white-noise velocity generates a random flow of diffeomorphisms. See Kunita (1990). Here Alfvén's Theorem holds in the standard sense, with the usual proof.

Generalized Stochastic Flow of Maps

Lagrangian flow maps $\boldsymbol{\xi}_t$ satisfy

$$(d/dt)\boldsymbol{\xi}_t(\mathbf{a}) = \mathbf{u}(\boldsymbol{\xi}_t(\mathbf{a}) \circ, t).$$

The conditional distribution of flow configurations $\boldsymbol{\xi}$ at time t given the velocity realization $\mathbf{u}$, or $P_{\mathbf{u}}[\boldsymbol{\xi}, t]$, then satisfies the *stochastic Liouville equation*:

$$(d/dt)P_{\mathbf{u}}[\boldsymbol{\xi}, t] = -\int d\mathbf{a} \frac{\delta}{\delta \boldsymbol{\xi}(\mathbf{a})} \cdot \left(\mathbf{u}(\boldsymbol{\xi}(\mathbf{a}), t) \circ P_{\mathbf{u}}[\boldsymbol{\xi}, t] \right).$$

It is formally equivalent to the Ito equation:

$$(d/dt)P_{\mathbf{u}}[\boldsymbol{\xi}, t] =$$

$$\frac{1}{2} \int d\mathbf{a} \int d\mathbf{a}' \frac{\delta^2}{\delta \xi^i(\mathbf{a}) \delta \xi^j(\mathbf{a}')} \left(D^{ij}(\boldsymbol{\xi}(\mathbf{a}), \boldsymbol{\xi}(\mathbf{a}'); t) P_{\mathbf{u}}[\boldsymbol{\xi}, t] \right)$$

$$- \int d\mathbf{a} \frac{\delta}{\delta \boldsymbol{\xi}(\mathbf{a})} \cdot \left(\widetilde{\mathbf{u}}(\boldsymbol{\xi}(\mathbf{a}), t) P_{\mathbf{u}}[\boldsymbol{\xi}, t] \right)$$

$$= (A^*(t)P_{\mathbf{u}})[\boldsymbol{\xi}, t] - \int d\mathbf{a} \frac{\delta}{\delta \boldsymbol{\xi}(\mathbf{a})} \cdot \left(\widetilde{\mathbf{u}}(\boldsymbol{\xi}(\mathbf{a}), t) P_{\mathbf{u}}[\boldsymbol{\xi}, t] \right).$$

Averaging over $\mathbf{u}$ gives the forward Kolmogorov equation $\partial_t P[\boldsymbol{\xi}, t] = A^*(t)P[\boldsymbol{\xi}, t]$ with

$$A^*(t) = \frac{1}{2} \int d\mathbf{a} \int d\mathbf{a}' \frac{\delta^2}{\delta \xi^i(\mathbf{a}) \delta \xi^j(\mathbf{a}')} D^{ij}(\boldsymbol{\xi}(\mathbf{a}), \boldsymbol{\xi}(\mathbf{a}'); t).$$

Random Family of Diffusions in Hilbert Space

$G(\Lambda)$ = group of volume-preserving diffeomorphisms
$S(\Lambda)$ = semigroup of Borel volume-preserving maps

$$G(\Lambda) \subset_{\text{dense}} S(\Lambda) \subset_{\text{closed}} L^2(\Lambda, \mathbf{R}^d)$$

See Y. Brenier (2003).

Krylov-Veretennikov expansion:

$$S^{\mathbf{u}}_{t,t'} = P_{t,t'} + \sum_{n=1}^{\infty} \int_{t'}^{t} dt_1 \int_\Lambda d\mathbf{a}_1 \int_{t'}^{t_1} dt_2 \int_\Lambda d\mathbf{a}_2 \cdots \int_{t'}^{t_{n-1}} dt_n \int_\Lambda d\mathbf{a}_n$$

$$P_{t,t_1}\left(\widetilde{\mathbf{u}}(\boldsymbol{\xi}(\mathbf{a}_1), t_1)\cdot\frac{\delta}{\delta\boldsymbol{\xi}(\mathbf{a}_1)}\right) P_{t_1,t_2}\left(\widetilde{\mathbf{u}}(\boldsymbol{\xi}(\mathbf{a}_2), t_2)\cdot\frac{\delta}{\delta\boldsymbol{\xi}(\mathbf{a}_2)}\right) P_{t_2,t_3}$$

$$\cdots P_{t_{n-1},t_n}\left(\widetilde{\mathbf{u}}(\boldsymbol{\xi}(\mathbf{a}_n), t_n)\cdot\frac{\delta}{\delta\boldsymbol{\xi}(\mathbf{a}_n)}\right) P_{t_n,t'},$$

$P_{t,t'}$ is the Markov semigroup of the diffusion on $L^2(\Lambda, \mathbf{R}^d)$ with generator

$$A(t) = \frac{1}{2}\int_\Lambda d\mathbf{a} \int_\Lambda d\mathbf{a}' D^{ij}(\boldsymbol{\xi}(\mathbf{a}), \boldsymbol{\xi}(\mathbf{a}'); t)\frac{\delta^2}{\delta\xi^i(\mathbf{a})\delta\xi^j(\mathbf{a}')}$$

These formal arguments suggest a picture of random splitting of Lagrangian flows in the Kraichnan model.

Loop Equation for Passive Vector

Eulerian equation for magnetic flux:

$$\partial_t \Phi(C,t) + \int_0^{2\pi} d\theta \ u_i(C(\theta),t) \circ \frac{\delta}{\delta C^i(\theta)} \Phi(C,t) = 0$$

Cf. A. A. Migdal (1993).

Ito formulation:

$$\partial_t \Phi(C,t) + \int_0^{2\pi} d\theta \ u_i(C(\theta),t) \frac{\delta}{\delta C^i(\theta)} \Phi(C,t)$$

$$= \frac{1}{2} \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' \ D_{ij}(C(\theta), C(\theta'), t) \frac{\delta^2}{\delta C^i(\theta) \delta C^j(\theta')} \Phi(C,t)$$

The righthand side is a diffusion in *free loop space* on the manifold Λ. See Driver & Röckner (1992). The LeJan-Raimond approach can be applied again!

Finally, for $t > t'$ (?)

$$\Phi(C,t) = \int P_{C,t}(dC'|\mathbf{u}) \Phi(C'(t'),t').$$

The magnetic flux is *not* conserved, except on average.

Conjecture for Circulations of Euler Solutions

Eyink (2006) conjectured the following *martingale property* for circulations of generalized Euler solutions:

$$E[\Gamma(C,t')|\{\Gamma(C,s), s < t\}] = \Gamma(C,t) \quad \text{for} \quad t' > t$$

The average is over random Lagrangian paths, given the past history of the circulation.

The average circulation in the future is given by the last known value.

This martingale property can be formally derived from generalized Least-Action Principle of Brenier-Shnirelman, $\min_P S[P]$ with

$$S[P] = \frac{1}{2}\int P(d\boldsymbol{\xi}) \int_{t_0}^{t_f} dt \int_\Lambda d\mathbf{a}|\dot{\boldsymbol{\xi}}(\mathbf{a},t)|^2.$$

But:

*What sets the arrow of time in Hamilton's Principle? Are circulations martingales backward in time?

*Is the statistical version of Kelvin Theorem (martingale property) sufficient for G. I. Taylor's argument?

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