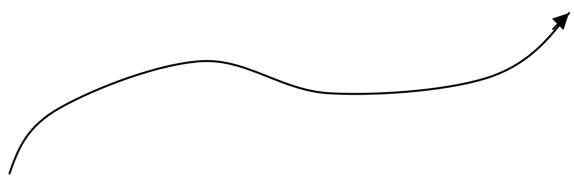
Lagrangian dynamics and statistical geometric structure of turbulence

Charles Meneveau, Yi Li & Laurent Chevillard

2 variations on a theme: $\dot{z} = -z^2$



JOHNS HOPKINS UNIVERSITY

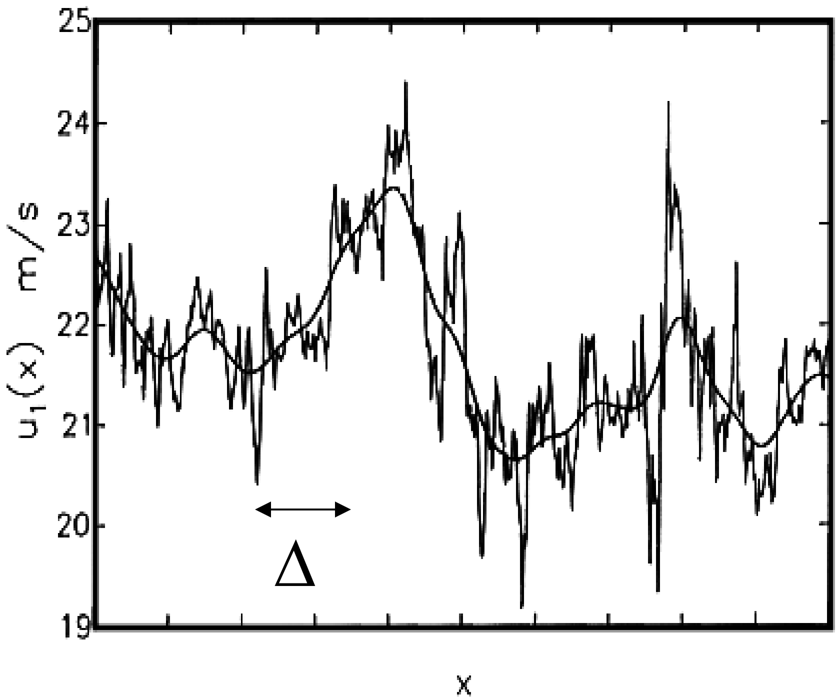
Mechanical Engineering

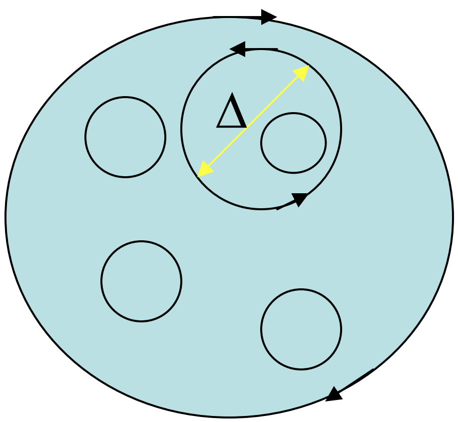
Center for Environmental & Applied Fluid Mechanics

Turbulent flow: multiscale

Characterize velocity field at particular scale
(filter out larger-scale advection): use velocity increments

$$\delta u(\Delta) = u_L(\mathbf{x} + \Delta \mathbf{e}_L) - u_L(\mathbf{x})$$

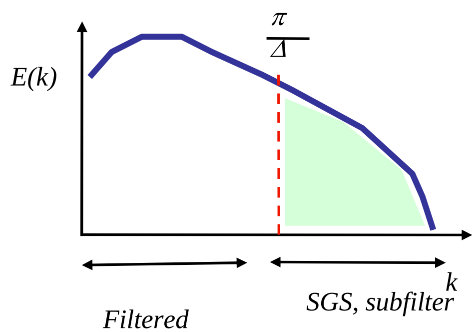




All velocity component increments in all directions, at all scales:

Filtered (coarsened) velocity gradient tensor at scale Δ:

$$\tilde{A}_{ij} = \frac{\partial \tilde{u}_j}{\partial x_i} \qquad \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} \\ \tilde{A}_{31} & \tilde{A}_{23} & \tilde{A}_{33} \end{pmatrix}$$



$$A_{ij} = \frac{\partial u_j}{\partial x_i}$$

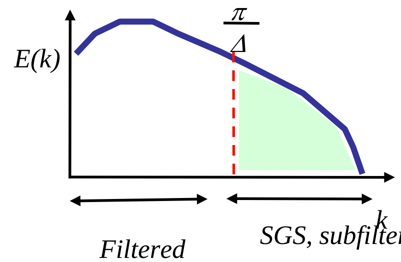
Unfiltered full velocity gradient tensor (Δ=0)

Restricted Euler dynamics in (inertial range of) turbulence:

Restricted Euler: Vieillefosse, Phys. A, **125**, 1985
Cantwell, Phys. Fluids A**4**, 1992
Filtered turbulence: Borue & Orszag, JFM **366**, 1998
Van der Bos *et al.*, Phys Fluids **14**, 2002:

- Filtered Navier-Stokes equations:

$$\frac{\partial \tilde{u}_j}{\partial t} + \tilde{u}_k \frac{\partial \tilde{u}_j}{\partial x_k} = -\frac{\partial \tilde{p}}{\partial x_j} + \nu \nabla^2 \tilde{u}_j - \frac{\partial}{\partial x_k} \tau_{jk}$$



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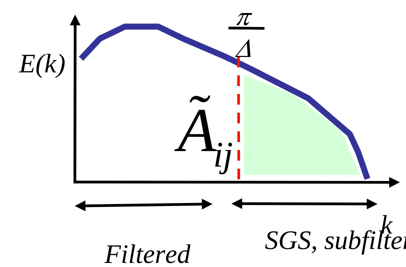

- Take gradient: $\tilde{A}_{ij} = \frac{\partial \tilde{u}_j}{\partial x_i}$

$$\frac{\partial \tilde{A}_{ij}}{\partial t} + \tilde{u}_k \frac{\partial \tilde{A}_{ij}}{\partial x_k} \equiv$$

$$\frac{d\tilde{A}_{ij}}{dt} = -\left(\tilde{A}_{ik}\tilde{A}_{kj} - \frac{\delta_{ij}}{3}\tilde{A}_{mk}\tilde{A}_{km}\right) + \left(-\frac{\partial^2 p}{\partial x_i \partial x_j} + \frac{1}{3}\frac{\partial^2 p}{\partial x_k \partial x_k}\delta_{ij}\right) - \left(\frac{\partial^2 \tau^d_{kj}}{\partial x_i \partial x_k} - \frac{\delta_{ij}}{3}\frac{\partial^2 \tau^d_{kl}}{\partial x_l \partial x_k}\right) + \nu \nabla^2 \tilde{A}_{ij}$$

Self-interaction

Anisotropic Pressure Hessian

Anisotropic subgrid-scale + viscous effects

$$\frac{d\tilde{A}_{ij}}{dt} = -\left(\tilde{A}_{ik}\tilde{A}_{kj} - \frac{\delta_{ij}}{3}\tilde{A}_{mk}\tilde{A}_{km}\right) + H_{ij}$$

Restricted Euler dynamics $H_{ij} = 0$ in (inertial range of) turbulence:

- **Invariants (Cantwell 1992):**

$$Q_\Delta \equiv -\frac{1}{2}\tilde{A}_{ki}\tilde{A}_{ik}$$

$$R_\Delta \equiv -\frac{1}{3}\tilde{A}_{km}\tilde{A}_{mn}\tilde{A}_{nk}$$

$$\tilde{A}_{ji}\frac{d\tilde{A}_{ij}}{dt} = \tilde{A}_{ji}(\tilde{A}_{ik}\tilde{A}_{kj} - \tfrac{1}{3}\tilde{A}_{mk}\tilde{A}_{km}\delta_{ij}) \quad \rightarrow$$

$$\tilde{A}_{jk}\tilde{A}_{ki}\frac{d\tilde{A}_{ij}}{dt} = \tilde{A}_{jk}\underbrace{\tilde{A}_{ki}(\tilde{A}_{ik}\tilde{A}_{kj}}_{\text{Cayley-Hamilton Theorem}} - \tfrac{1}{3}\tilde{A}_{mk}\tilde{A}_{km}\delta_{ij}) \quad \rightarrow$$

$$A_{ik}A_{kn}A_{nj} + PA_{ik}A_{kj} + QA_{ij} + R\delta_{ij} = 0$$

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Cayley-Hamilton Theorem

$$A_{ik}A_{kn}A_{nj} + PA_{ik}A_{kj} + QA_{ij} + R\delta_{ij} = 0$$

Remarkable projection (decoupling)!

More literature:
Equations for all 5 invariants:
Martin, Dopazo & Valiño (Phys. Fluids, 1998)
Equations for eigenvalues,
and higher-dimensional versions:
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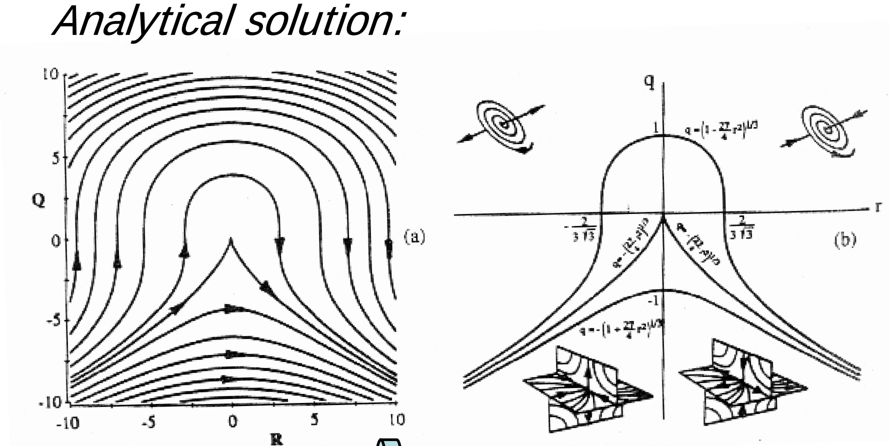
Analytical solution:



From: Cantwell, Phys. Fluids 1992)

- **Singularity in finite time**, but

  - Predicts preference for **axisymmetric extension**

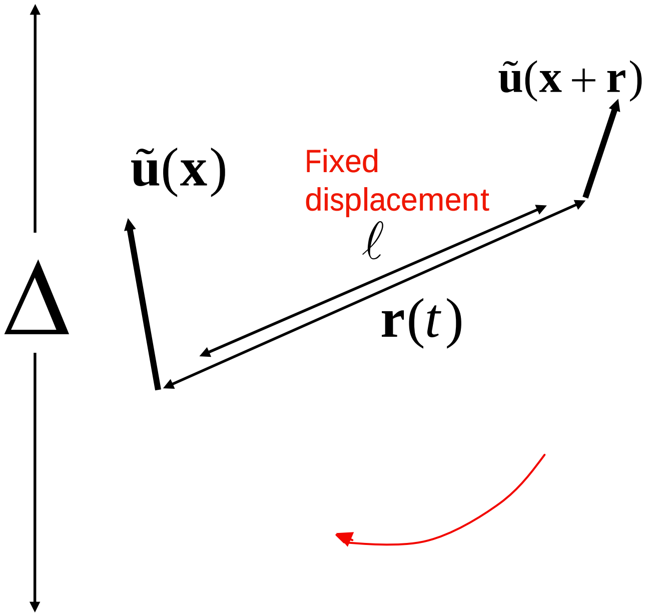
  - Predicts alignment of **vorticity with intermediate eigenvector** of S: β_s

REST OF TALK:

Topic 1: Are there other useful “trace-dynamics” simplifications (other than Q & R)

Topic 2: Stochastic Lagrangian model for evolution of A_{ij}

Another simplifications: Velocity increments



$$\tilde{u}_i(\mathbf{x}+\mathbf{r}) - \tilde{u}_i(\mathbf{x}) = \tilde{A}_{ki} r_k + O(r^2)$$

$$\tilde{A}_{ji} = \frac{\partial \tilde{u}_i}{\partial x_j}$$

Longitudinal

$$\delta u \equiv \tilde{A}_{ki} r_k \frac{r_i}{r} \frac{\ell}{r}$$

$$\delta u(t) \equiv \tilde{\mathbf{A}}(t) : (\hat{\mathbf{r}}(t)\hat{\mathbf{r}}(t))\, \ell = \tilde{A}_{rr} \ell$$

Transverse

$$\delta v^2 \equiv \left[\left(\delta_{ij} - \frac{r_i r_j}{r^2} \right) \tilde{A}_{kj} r_k \frac{\ell}{r} \right]^2$$

See: Yi & Meneveau, Phys. Rev. Lett. **95**, 164502, 2005, JFM 2006

Velocity increments: Lagrangian evolution

$$\delta u \equiv \tilde{A}_{ki} r_k \frac{r_i}{r} \frac{\ell}{r}$$

Rate of change of longitudinal velocity increment, following the flow (both end-points in linearized flow):

$$\frac{d}{dt}\delta u = \frac{d}{dt}\left(\tilde{A}_{ki} r_k \frac{r_i}{r} \frac{\ell}{r} \right)$$

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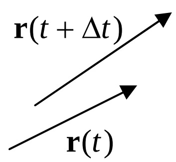

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$$\frac{d\tilde{A}_{ij}}{dt} = -(\tilde{A}_{ik}\tilde{A}_{kj} - \tfrac{1}{D}\tilde{A}_{mk}\tilde{A}_{km}\delta_{ij}) + H_{ij}$$

$$\frac{dr_i}{dt} = \frac{\partial \tilde{u}_i}{\partial x_m} r_m = \tilde{A}_{mi} r_m$$

$$H_{mn} = \left(\frac{\partial^2}{\partial x_m \partial x_k} \left[p\delta_{kn} - \tau_{kn}^{SGS} + 2\nu \tilde{S}_{kn} \right] \right)^{anisotropic}$$

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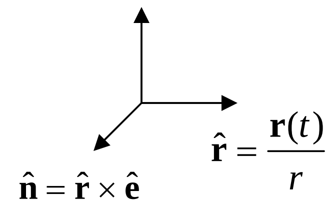
Velocity increments: Lagrangian evolution

$$\frac{d}{dt}\delta u = \frac{1}{l}\left(\delta v^2 - \delta u^2\right) + \frac{1}{D}\underbrace{\tilde{A}_{pq}\tilde{A}_{qp}}\, l + H_{mn}\frac{r_m r_i}{r^2}\, l$$

$$\hat{\mathbf{e}}: \quad \hat{e}_n = \delta u_i\left(\delta_{in} - \frac{r_i r_n}{r^2}\right)\frac{1}{\delta v}$$



Tensor invariant (Q)

Write A in frame formed by:

$$\begin{bmatrix} A_{rr} & A_{re} & A_{rn} \\ A_{er} & A_{ee} & A_{en} \\ A_{nr} & A_{ne} & -(A_{rr}+E_{ee}) \end{bmatrix} = \begin{bmatrix} \delta u & \delta v & 0 \\ ? & ? & ? \\ ? & ? & -(\delta u + ?) \end{bmatrix}\frac{1}{l}$$

$$\tilde{A}_{pq}\tilde{A}_{qp}\, l = \begin{bmatrix} \delta u & \delta v & 0 \\ ? & ? & ? \\ ? & ? & -(\delta u + ?) \end{bmatrix}\begin{bmatrix} \delta u & \delta v & 0 \\ ? & ? & ? \\ ? & ? & -(\delta u + ?) \end{bmatrix}\frac{1}{l} = \left(\delta u^2 + [\delta u + ?]^2 + ? + \ldots\right)\frac{1}{l} = 2\delta u^2\frac{1}{l} + ? + \ldots$$

$$H_{mn} = \left(\frac{\partial^2}{\partial x_m \partial x_k}\left[p\delta_{kn} - \tau_{kn}^{SGS} + 2\nu\tilde{S}_{kn}\right]\right)^{anisotropic} = 0$$

and ? =0

$$\frac{d}{dt}\delta u = -\left(1 - \frac{2}{D}\right)\delta u^2 + \delta v^2$$

Velocity increments: Lagrangian evolution

From a similar derivation for δv:

$$\frac{d}{dt}\delta v = -\frac{2}{\mathrm{l}}\delta u \delta v$$

From a similar derivation for δT, neglecting diffusion and SGS fluxes:

$$\frac{d}{dt}\delta T = -\frac{1}{\mathrm{l}}\delta u \delta T$$

In 2D, vorticity is passive scalar, so for 2D:

$$\frac{d}{dt}\delta \omega_z = -\frac{1}{\mathrm{l}}\delta u \delta \omega_z$$

Velocity increments: Lagrangian evolution

Set of equations:

$$\begin{cases} \frac{d}{dt}\delta u = -\left(1-\frac{2}{D}\right)\delta u^2 + \delta v^2 \\ \frac{d}{dt}\delta v = -2\delta u \delta v \\ \frac{d}{dt}\delta T = -\delta u \delta T \\ \frac{d}{dt}\delta \omega_z = -\delta u \delta \omega_z \quad \text{(only for D=2)} \end{cases}$$

1-D inviscid Burgers:

$$\frac{d}{dt}\delta u = -\delta u^2$$

Angles & vortex stretching (Galati, Gibbon et al, 1997 Gibbon & Holm 2006....):

$$\frac{d}{dt}\alpha = -\alpha^2 + \chi^2$$

$$\frac{d}{dt}\chi = -2\alpha\chi$$

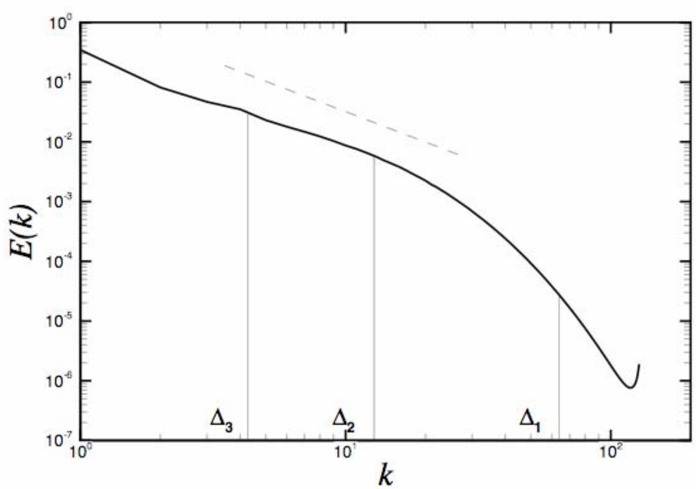
"Advected delta-vee system"

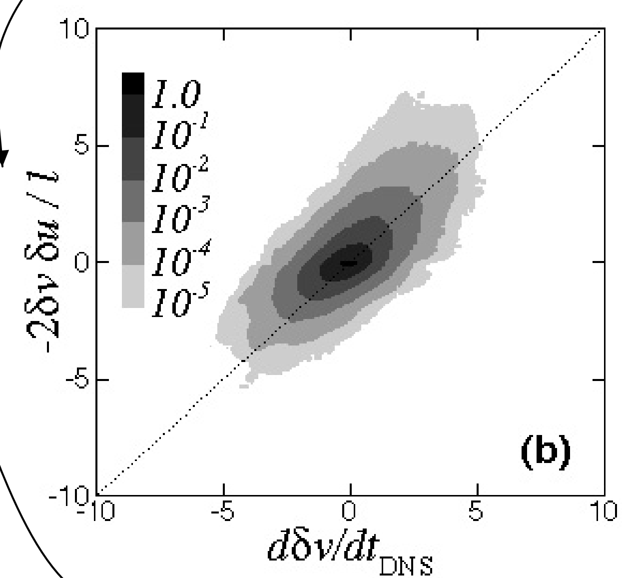
Comparison with DNS,
Lagrangian rate of change of velocity increments:

256^3 DNS, filtered at 40η, Δ=40 η, evaluated δu, δv, and their Lagrangian rate of change of velocity increments numerically

$$\begin{cases} \dfrac{d}{dt}\delta u = \left(-\dfrac{1}{3}\delta u^2 + \delta v^2\right)\dfrac{1}{l} \\[2ex] \dfrac{d}{dt}\delta v = -\dfrac{2}{l}\ \delta u\ \delta v \end{cases}$$

$\rho = 0.51$





$\rho = 0.61$

Evolution from Gaussian initial conditions:

Initial condition:

δu = Gaussian zero mean, unit variance

δv_k = Gaussian zero mean, unit variance, k=1,2

$$\delta v = \sqrt{\delta v_1^2 + \delta v_2^2}$$

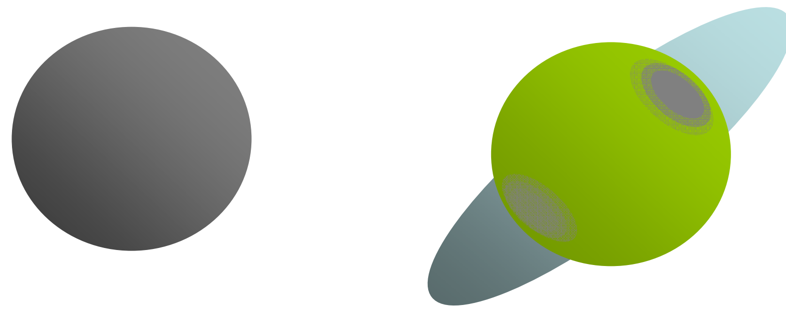
δT = Gaussian zero mean, unit variance

set $\ell = 1$

Technical point #1: Alignment bias correction factor

(thanks to Greg Eyink for pointing out the need for a correction)

$\mathbf{r}(0), \quad |\mathbf{r}(0)| = \ell \qquad\qquad \mathbf{r}(t)$



$$\ell^D d\Omega_0 = r(t)^D d\Omega(t) \qquad \frac{d\Omega(t)}{d\Omega_0} = \left(\frac{\ell}{r(t)}\right)^D$$

$$\frac{dr}{dt} = \delta u(r,t) = \delta u \cdot \left(\frac{r}{\ell}\right)$$

$$\frac{d}{dt}\ln(r/\ell) = \delta u/\ell$$

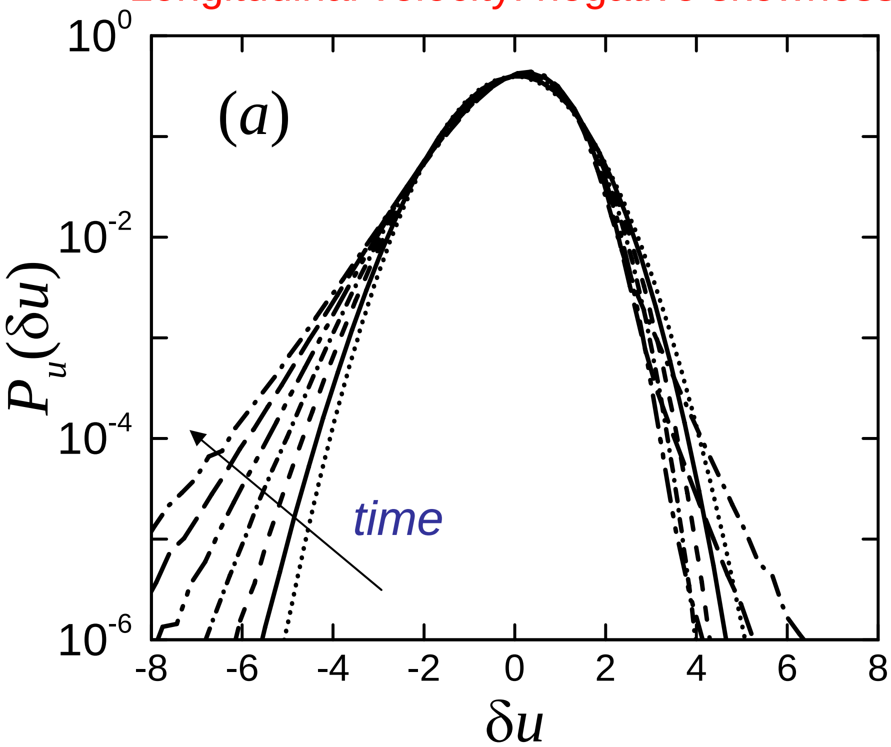
See: Yi & Meneveau, Phys. Rev. Lett. **95**, 164502,

$$\frac{d\Omega(t)}{d\Omega_0} = \exp\left(-D\ell^{-1}\int_0^t \delta u(t')dt'\right) \qquad P \to P\frac{d\Omega(t)}{d\Omega_0}$$

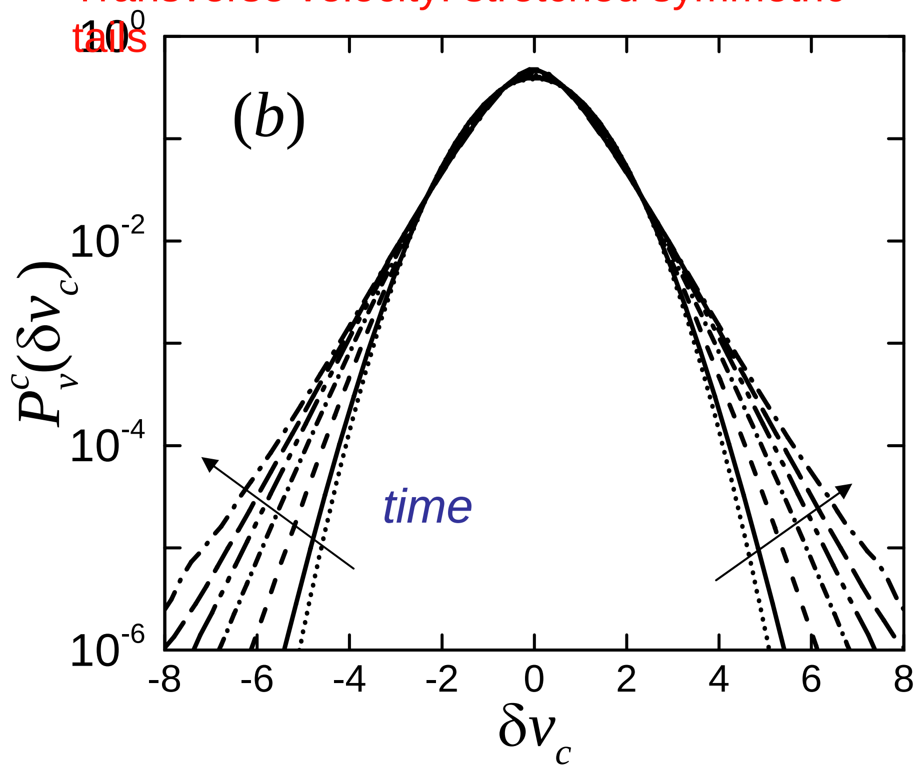
Can be evaluated from advected delta-vee system

Numerical Results: PDFs in 3D

Longitudinal velocity: negative skewness



Transverse velocity: stretched symmetric tails



$$\begin{cases} \frac{d}{dt}\delta u = -\left(1-\frac{2}{D}\right)\delta u^2 + \delta v^2 \\ \frac{d}{dt}\delta v = -2\delta u \delta v \\ \frac{d}{dt}\delta T = -\delta u \delta T \\ \frac{d}{dt}\delta \omega_z = -\delta u \delta \omega_z \quad \text{(only for D=2)} \end{cases}$$

D =3

$$\begin{cases} \frac{d}{dt}\delta u = -\frac{1}{3}\delta u^2 + \delta v^2 \\ \frac{d}{dt}\delta v = -2\delta u \delta v \end{cases}$$

Technical point #2:
Individual component θ is random, uniformly distributed in [0, 2π)

$$\delta v_c = |\delta v| \cos\theta$$

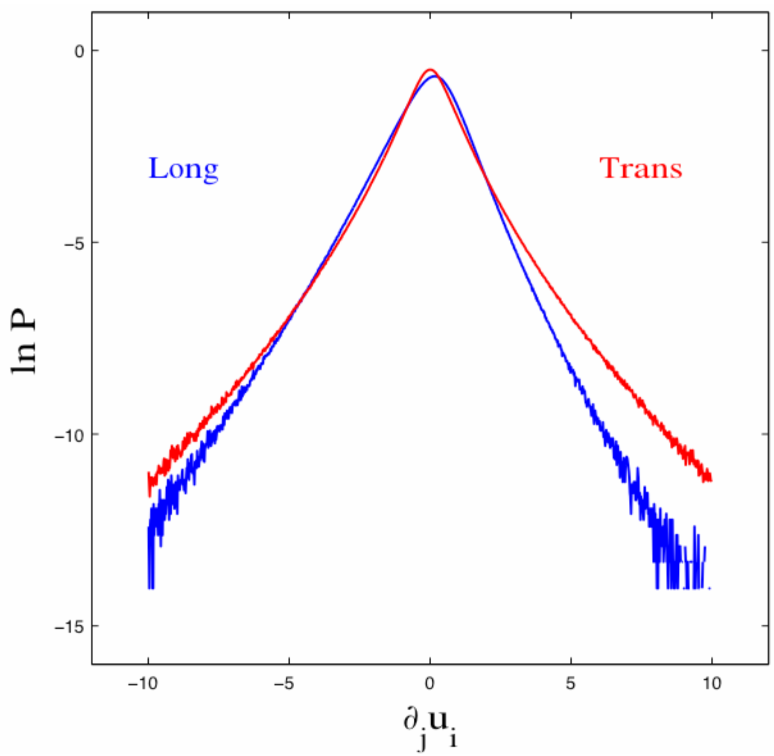
$$P_v^c(\delta v_c) = \frac{1}{\pi}\int_{|\delta v_c|}^{\infty} \frac{P_v(\delta v)}{\sqrt{\delta v^2 - \delta v_c^2}}\, d\delta v$$

Measured intermittency trends:

Longitudinal increment is skewed

Transverse velocity is more intermittent

256^3 DNS



1024^3 DNS
(Gotoh et al. 2002)

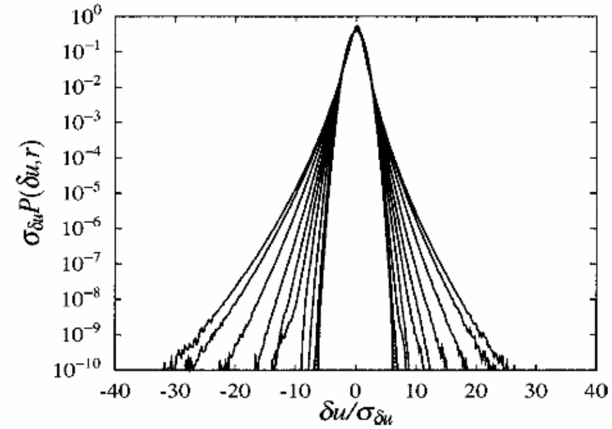


FIG. 15. Variation of the δu_r PDF with r for $R_\lambda = 381$. From the outermost curve, $r_n/\eta = 2^{n-1}dx/\eta = 2.38\times 2^{n-1}$, $n = 1,...,10$, where $dx = 2\pi/1024$. The inertial range corresponds to $n = 6, 7, 8$. Dotted line: Gaussian.

Gotoh, Fukayama, and Nakano

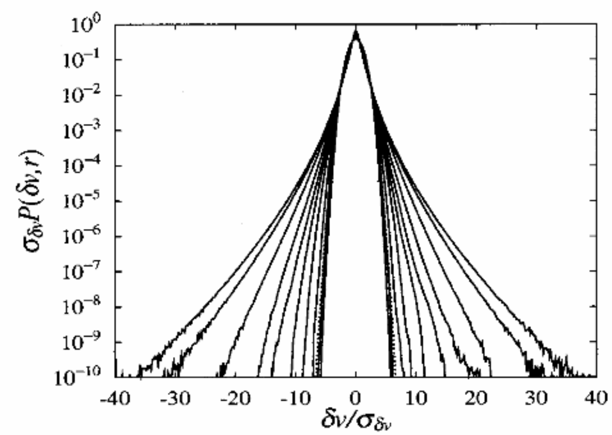
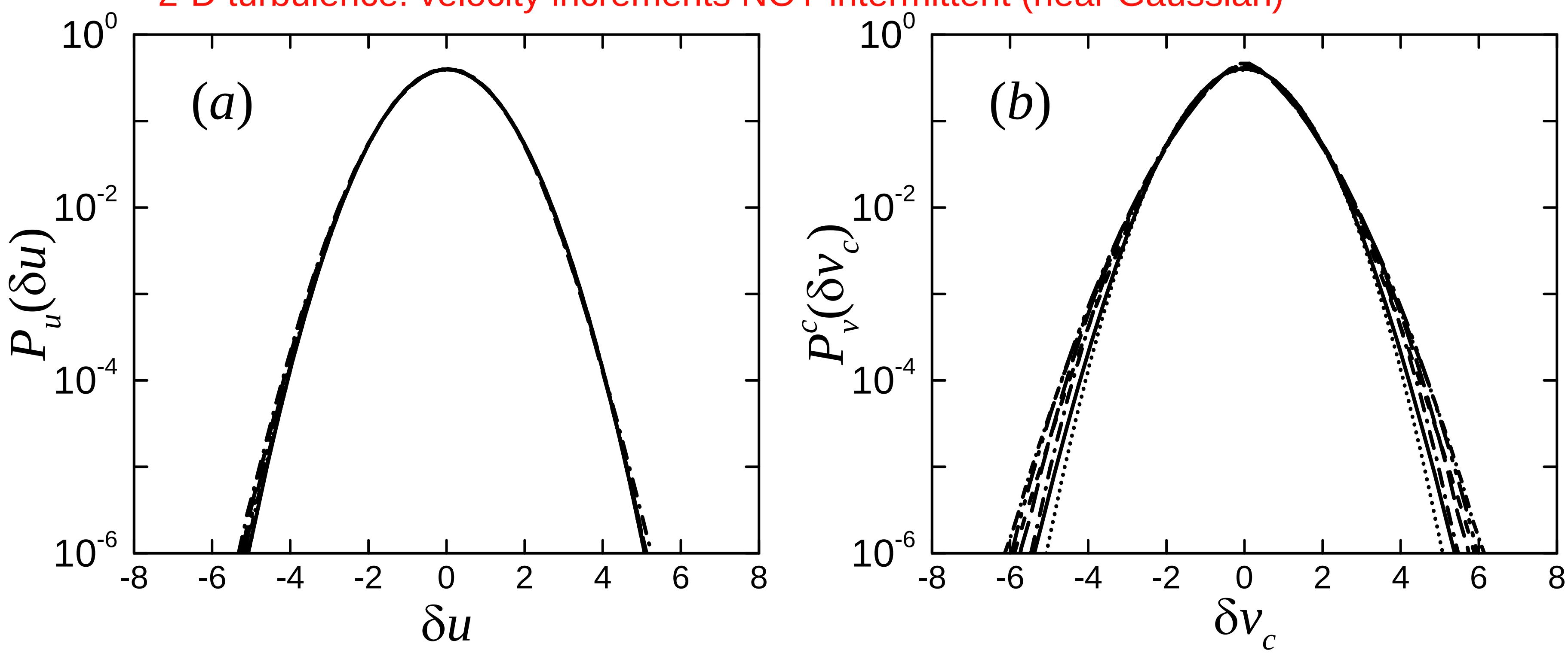


FIG. 16. Variation of PDF for δv_r with r at $R_\lambda = 381$. The classification of curves is the same as in Fig. 17.

Numerical Results: PDFs in 2D

2-D turbulence: velocity increments NOT intermittent (near Gaussian)



$$
\begin{cases}
\frac{d}{dt}\delta u = -\left(1-\frac{2}{D}\right)\delta u^2 + \delta v^2 \\
\frac{d}{dt}\delta v = -2\delta u \delta v \\
\frac{d}{dt}\delta T = -\delta u \delta T \\
\frac{d}{dt}\delta \omega_z = -\delta u \delta \omega_z \quad \text{(only for D=2)}
\end{cases}
$$

D =2

$$
\begin{cases}
\frac{d}{dt}\delta u = \delta v^2 \\
\frac{d}{dt}\delta v = -2\delta u \delta v
\end{cases}
$$

Measured intermittency trends:

No intermittency in 2-D turbulence for velocity increments

J. Paret and P. Tabeling 3127

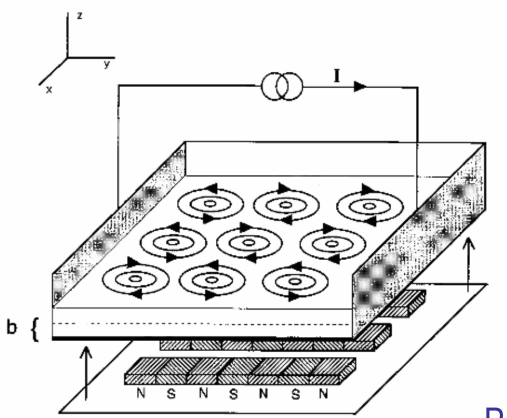


FIG. 1. The experimental set-up.

J. Paret and P. Tabeling 3133

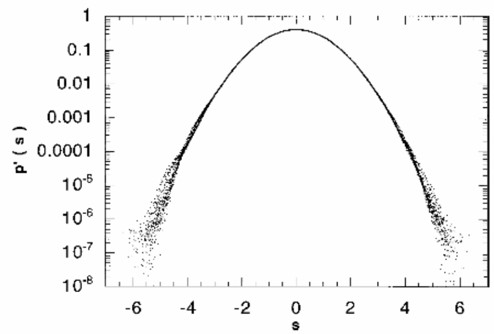


FIG. 12. Rescaled PDF of longitudinal velocity increments for 7 different separations in the inertial range. $s=\delta v/\langle \delta v^2\rangle^{1/2}$.

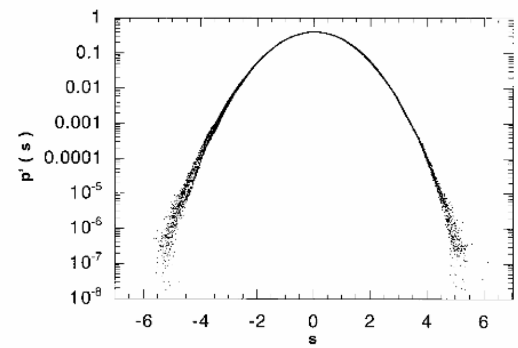


FIG. 15. Rescaled PDF of transverse velocity increments for 7 different separations in the inertial range. $s=\delta v/\langle \delta v^2\rangle^{1/2}$.

Paret & Tabeling, Phys. Fluids, 1998

DNS:

Bofetta et al.
Phys. Rev. E, 2000

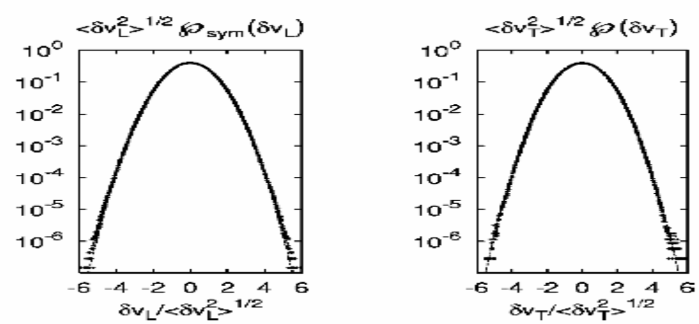


FIG. 6. Left: symmetric part of the longitudinal velocity difference PDF. Right: PDF of transverse velocity differences. The forcing is restricted to a band of wave numbers. Gaussian distributions are shown as solid lines.

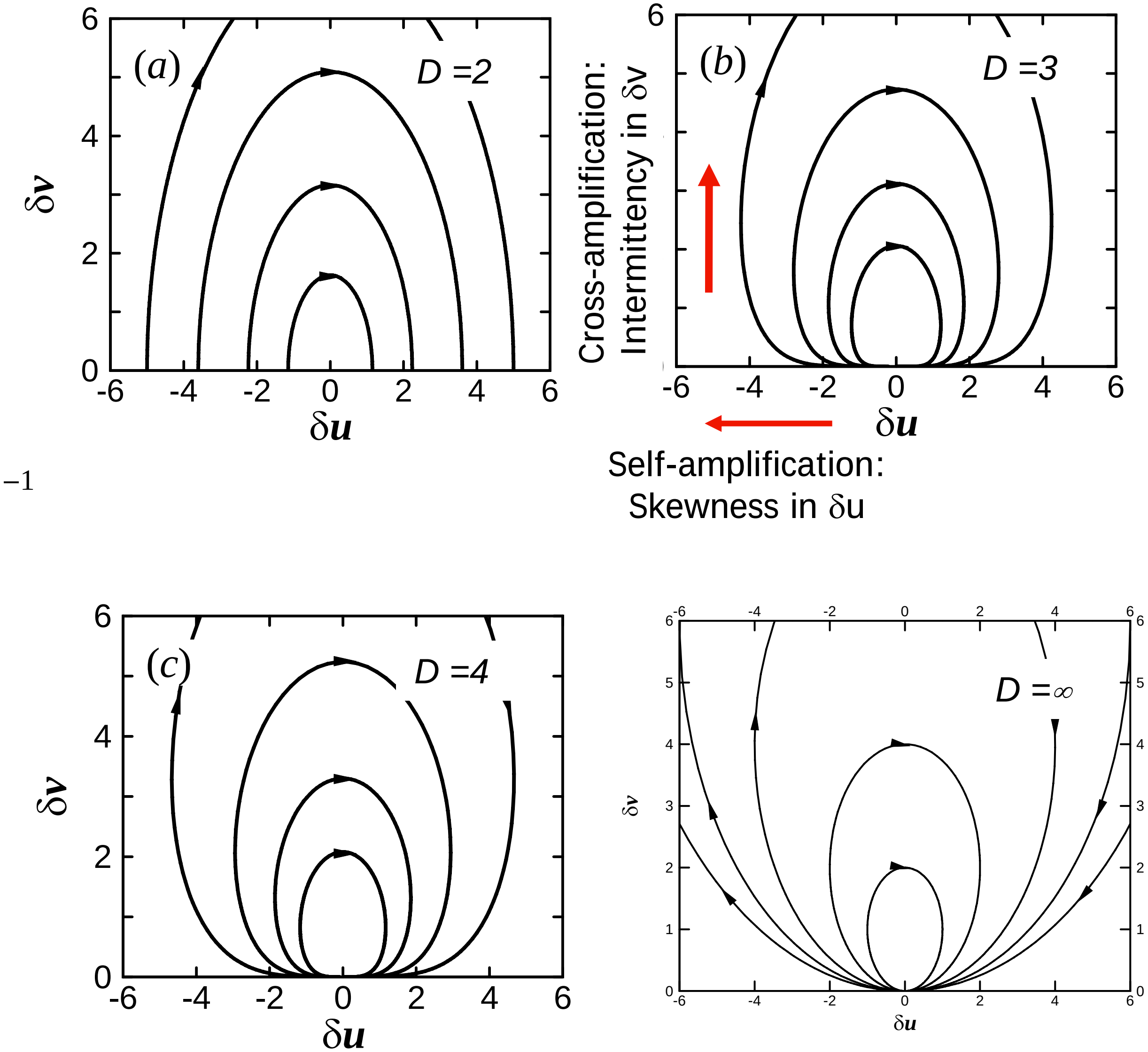
Phase portraits in (δu, δv) phase space:

$$\begin{cases} \dfrac{d}{dt}\delta u = -\left(1 - \dfrac{2}{D}\right)\delta u^2 + \delta v^2 \\ \dfrac{d}{dt}\delta v = -2\delta u \delta v \end{cases}$$

Invariant:

$$U = \left(\delta u^2 + \frac{D}{D+2}\delta v^2\right)\delta v^{2/D-1}$$

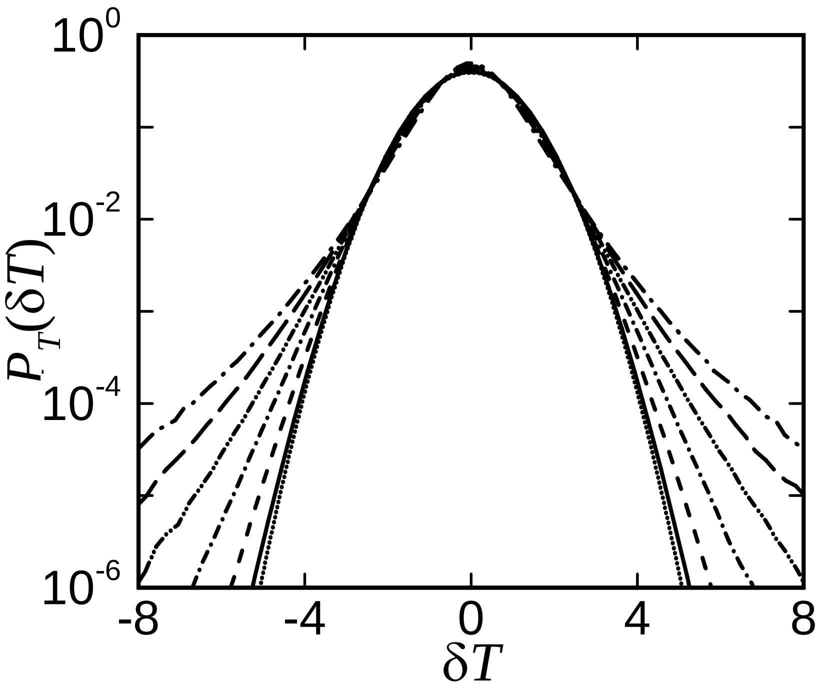
"For small initial δv (particles moving directly towards each other), gradient can become arbitrarily large at later times"

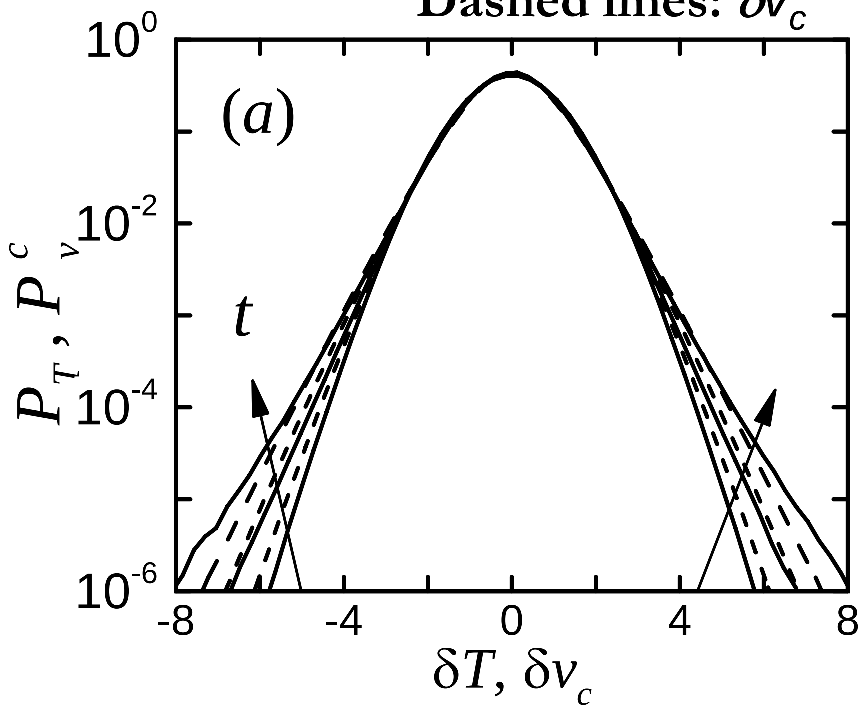


Passive scalar

Solid lines: δT

Dashed lines: δv_c





$$\begin{cases} \frac{d}{dt}\delta u = -\left(1-\frac{2}{D}\right)\delta u^2 + \delta v^2 \\ \frac{d}{dt}\delta v = -2\delta u \delta v \\ \frac{d}{dt}\delta T = -\delta u \delta T \\ \frac{d}{dt}\delta \omega_z = -\delta u \delta \omega_z \quad \text{(only for D=2)} \end{cases}$$

$D = 3$

$$\begin{cases} \frac{d}{dt}\delta u = -\frac{1}{3}\delta u^2 + \delta v^2 \\ \frac{d}{dt}\delta v = -2\delta u \delta v \\ \frac{d}{dt}\delta T = -\delta u \delta T \end{cases}$$

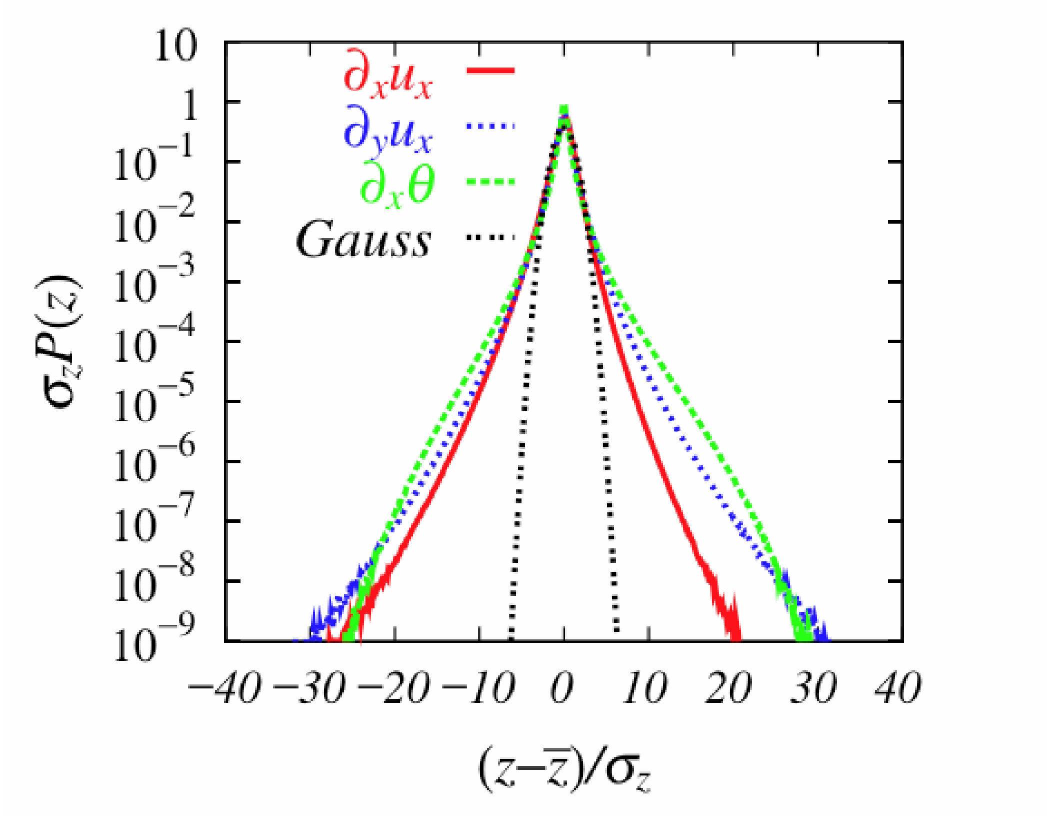
Scalar increments MORE intermittent than transverse velocity, after initial transient

Measured intermittency trends:

Passive scalar transport: $\delta T(\mathbf{l}) = T(\mathbf{x} + l\,\mathbf{e}_L) - T(\mathbf{x})$

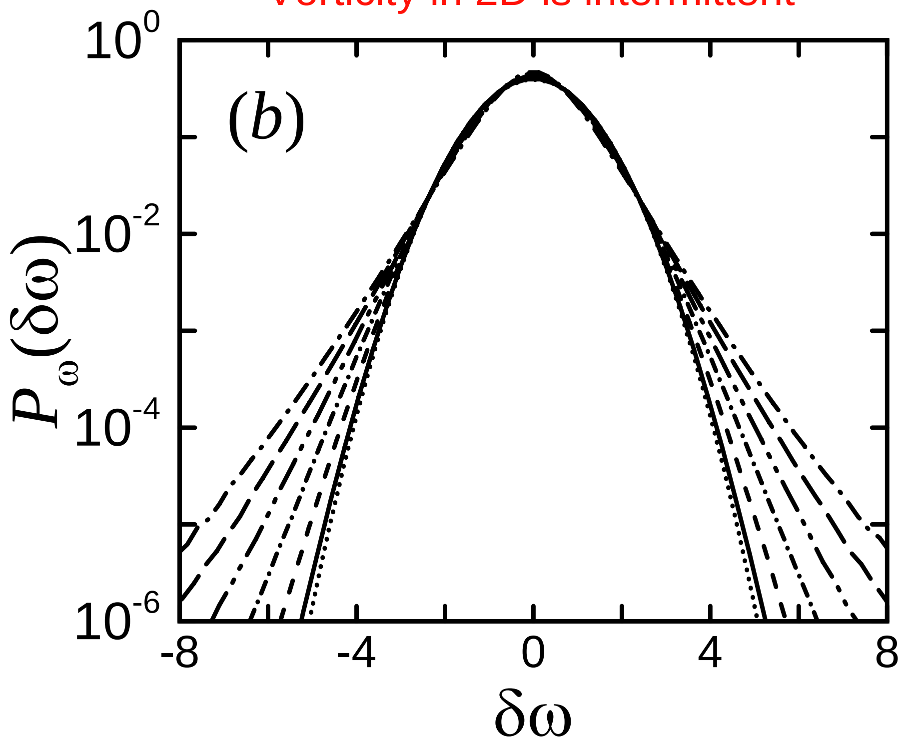
Larger intermittency for scalar increments than for velocity increments

e.g. Antonia et al. Phys. Rev. A 1984, and
Watanabe & Gotoh, NJP, 2004:



Vorticity in 2D

Vorticity in 2D is intermittent

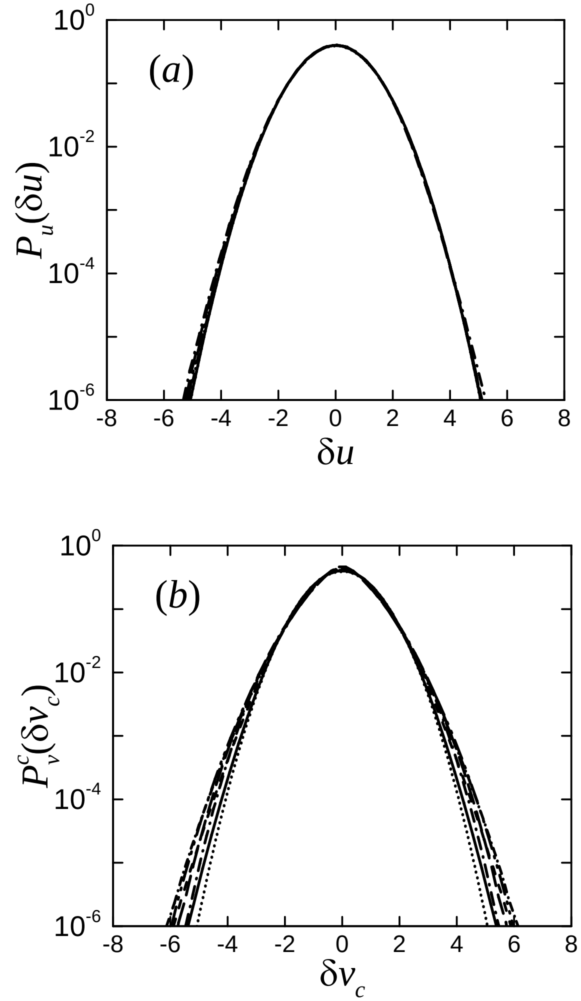


$$\begin{cases} \frac{d}{dt}\delta u = -\left(1-\frac{2}{D}\right)\delta u^2 + \delta v^2 \\ \frac{d}{dt}\delta v = -2\delta u \delta v \\ \frac{d}{dt}\delta T = -\delta u \delta T \\ \frac{d}{dt}\delta \omega_z = -\delta u \delta \omega_z \quad \text{(only for D=2)} \end{cases}$$

D =2

$$\begin{cases} \frac{d}{dt}\delta u = \delta v^2 \\ \frac{d}{dt}\delta v = -2\delta u \delta v \\ \frac{d}{dt}\delta \omega_z = -\delta u \delta \omega_z \end{cases}$$

Recall: velocity in 2D



Measured intermittency trends:

Intermittency in 2-D turbulence for vorticity increments

J. Paret and P. Tabeling 3127

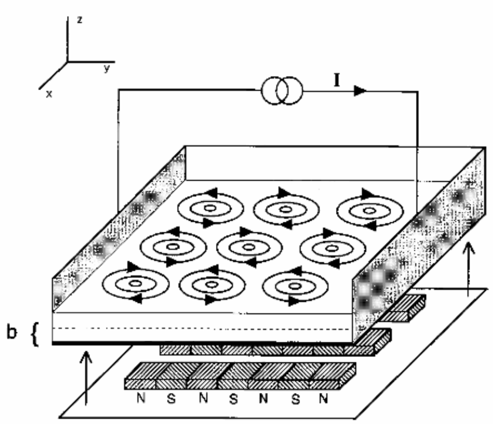


FIG. 1. The experimental set-up.

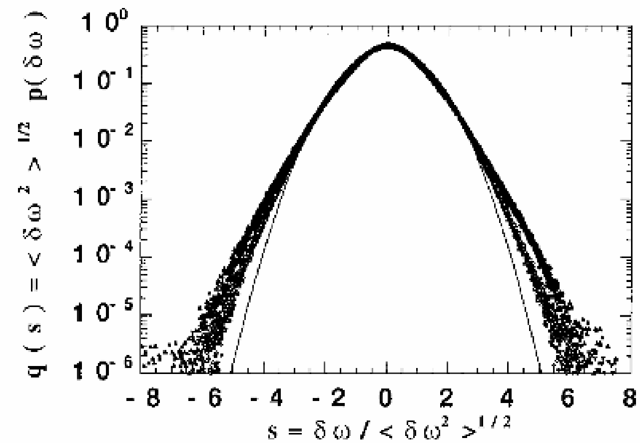


FIG. 5. Normalized distributions of vorticity increments, for five separations of r: 2, 3, 5, 7, and 9 cm.

Paret & Tabeling, Phys. Rev. Lett., 1999

Conclusions so far:

• Non-Gaussian intermittency trends can be explained simply by “Burgers equation-like” dynamics where instead of 1-D we embed a 1-D direction and follow it in a Lagrangian fashion. Non-Gaussian PDFs evolve very quickly (0.3 turn-over time).

• However, for complete local information, we’d need to follow 3 “perpendicular” lines. The we would need

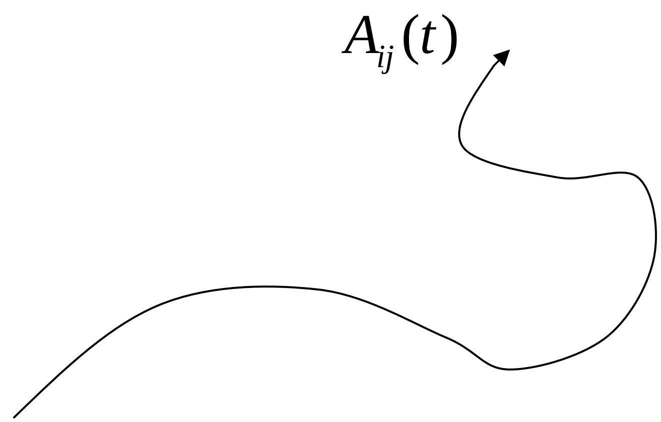
$$3 \times 2 + 3 \text{ angles} = 9$$

variables to be followed.

• Might as well stay with A_{ij} ...

Lagrangian Stochastic model for full velocity gradient tensor: $A_{ij} = \frac{\partial u_j}{\partial x_i}$

$$\frac{dA_{ij}}{dt} = \underbrace{-A_{iq}A_{qj}}_{\text{Self-stretching}} \underbrace{- \frac{\partial^2 p}{\partial x_i \partial x_j} + \nu \frac{\partial^2 A_{ij}}{\partial x_q \partial x_q}}_{\text{unclosed}} + \underbrace{W_{ij}}_{\text{forcing}}$$



L. Chevillard & CM, PRL 2006 (in press

Review of various models

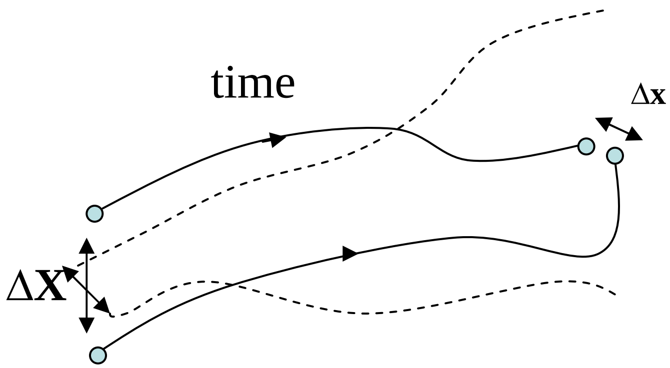
$$\frac{d}{dt}A_{ij} = -A_{iq}A_{qj} - \frac{\partial^2 p}{\partial x_i \partial x_j} + \nu \frac{\partial^2 A_{ij}}{\partial x_q \partial x_q}$$

- Restricted Euler Dynamics (Vieillefosse 84-Cantwell 92)
  $\frac{\partial^2 p}{\partial x_i \partial x_j} = -\frac{\delta_{ij}}{3}\mathrm{Tr}(\mathbf{A}^2)$ and $\nu = 0 \rightarrow$ Finite time singularity
- Lognormality of Pseudo-dissipation $\varphi = \mathrm{Tr}(\mathbf{A}\mathbf{A}^T)$ (Girimaji-Pope 90)
  $\rightarrow$ Strong *a-priori* assumption
- Linear damping term (Martin *et al.* 98)
  $\frac{\partial^2 p}{\partial x_i \partial x_j} = -\frac{\delta_{ij}}{3}\mathrm{Tr}(\mathbf{A}^2)$ and $\nu \frac{\partial^2 \mathbf{A}}{\partial x_q \partial x_q} = -\frac{1}{\tau}\mathbf{A} \rightarrow$ Finite time singularity

Using the material **Deformation** (**Cauchy-Green** Tensor **C**)

- Tetrad's model (Chertkov-Pumir-Shraiman 99)
  $\frac{\partial^2 p}{\partial x_i \partial x_j} = -\frac{\mathrm{Tr}(\mathbf{A}^2)}{\mathrm{Tr}(\mathbf{C}^{-1})} C^{-1}_{ij}$ and $\nu = 0 \rightarrow$ Non stationary
- Differential damping term (Jeong-Girimaji 03)
  $\frac{\partial^2 p}{\partial x_i \partial x_j} = -\frac{\delta_{ij}}{3}\mathrm{Tr}(\mathbf{A}^2)$ and $\nu \frac{\partial^2 \mathbf{A}}{\partial x_q \partial x_q} = -\frac{\mathrm{Tr}(\mathbf{C}^{-1})}{3\tau}\mathbf{A}$
  $\rightarrow$ Non stationary

Focus on Lagrangian pressure field: *p(X,t)*



Change of variables:

$$\frac{\partial^2 p}{\partial x_i \partial x_j} = \frac{\partial X_p}{\partial x_i}\frac{\partial X_q}{\partial x_j}\frac{\partial^2 p}{\partial X_p \partial X_q} + \left(\frac{\partial}{\partial x_i}\frac{\partial X_q}{\partial x_j}\right)\frac{\partial p}{\partial X_q}$$

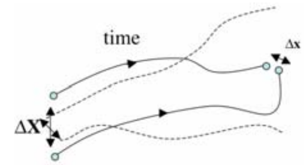
$$\frac{\partial^2 p}{\partial x_i \partial x_j} \approx \frac{\partial X_p}{\partial x_i}\frac{\partial X_q}{\partial x_j}\frac{\partial^2 p}{\partial X_p \partial X_q}$$

L. Chevillard & CM, PRL 2006 (in press):

3 main ingredients

1. Proposed Pressure Hessian model:

Assume that Lagrangian pressure Hessian is isotropic if t-τ is long enough for "memory loss" of dispersion process



$$\frac{\partial^2 p}{\partial x_i \partial x_j} \approx \frac{\partial X_p}{\partial x_i} \frac{\partial X_q}{\partial x_j} \quad \frac{\partial^2 p}{\partial X_p \partial X_q}$$

Deformation tensor: $D_{ij} = \dfrac{\partial x_j}{\partial X_i}$

Cauchy-Green tensor: $C_{ij} = D_{ik} D_{jk}$

Inverse: $(\mathbf{C}^{-1})_{ij} = \dfrac{\partial X_k}{\partial x_i} \dfrac{\partial X_k}{\partial x_j}$

Poisson constraint: $\dfrac{dA_{ii}}{dt} = -A_{iq}A_{qi} - \dfrac{\partial^2 p}{\partial x_i \partial x_i} = 0$

$$(\mathbf{C}^{-1})_{ii} \quad \frac{1}{3} \frac{\partial^2 p}{\partial X_k \partial X_k} = -A_{iq}A_{qi}$$

$$\frac{\partial^2 p}{\partial X_p \partial X_q} \approx \frac{1}{3} \frac{\partial^2 p}{\partial X_k \partial X_k} \delta_{pq}$$

$$\frac{\partial^2 p}{\partial x_i \partial x_j} \approx (\mathbf{C}^{-1})_{ij} \frac{1}{3} \frac{\partial^2 p}{\partial X_k \partial X_k}$$

???

$$\frac{\partial^2 p}{\partial x_i \partial x_j} = -\frac{(\mathbf{C}^{-1})_{ij}}{(\mathbf{C}^{-1})_{nn}} A_{pq} A_{qp}$$

Equivalent to "tetrad model"
(more formally derived)

2. Proposed viscous Hessian model:

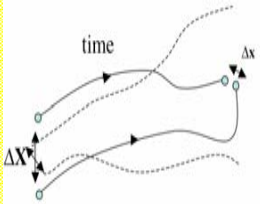
Similar approach (Jeong & Girimaji, 2003)

$$\nu \frac{\partial^2 \mathbf{A}}{\partial x_i \partial x_j} \approx \frac{\partial X_p}{\partial x_i} \frac{\partial X_q}{\partial x_j} \quad \nu \frac{\partial^2 \mathbf{A}}{\partial X_p \partial X_q}$$

Characteristic Lagrangian displacement after **A**'s Lagrangian correlation time scale (Kolm time):

$$\delta X \sim (disp\ veloc) \times (correl\ \ time) \sim u' \tau_K \sim \lambda$$

δX ~ Taylor microscale !



$$(\nu / \delta X^2) \sim \nu / \lambda^2 \sim T^{-1}$$

$$\nu \frac{\partial^2 \mathbf{A}}{\partial X_p \partial X_q} \approx \frac{\nu}{(\delta X)^2} \mathbf{A} \frac{1}{3} \delta_{pq}$$

$$\nu \frac{\partial^2 \mathbf{A}}{\partial X_p \partial X_q} \approx \frac{1}{T} \mathbf{A} \frac{1}{3} \delta_{pq}$$

$$\nu \frac{\partial^2 A_{ij}}{\partial x_m \partial x_m} \approx -\frac{(\mathbf{C}^{-1})_{mm}}{3T} A_{ij}$$

3. Short-time memory material deformation:
(Markovianization)

Equation for deformation tensor:

$$\frac{d\mathbf{D}}{dt} = \mathbf{DA}$$

Formal Solution in terms of time-ordered exponential function:

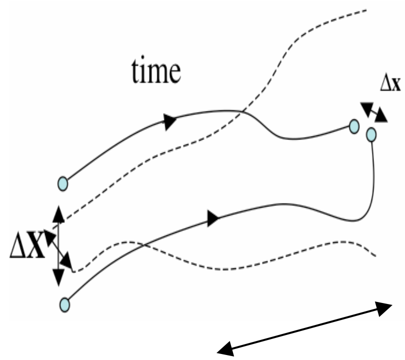
$$\mathbf{D}(t) = \mathbf{D}(0)\prod_{t0}^{t} \exp[\mathbf{A}(t')dt'] = \mathbf{D}(t-\tau)\mathbf{d}_\tau(t)$$

where:

$$\mathbf{d}_\tau(t) = \prod_{t-\tau}^{t} \exp[\mathbf{A}(t')dt'] \approx \exp[\mathbf{A}(t)\tau]$$

Short-time (Markovian) Cauchy-Green:

$$\mathbf{c}_\tau(t) = \exp[\mathbf{A}(t)\tau]\exp[\mathbf{A}^T(t)\tau]$$



$\Delta t = \tau$

Two time-scales tested:

- Mean Kolmogorov time

$$\tau = \tau_K = cT\,\mathrm{Re}^{-1/2}$$

- Local time: strain-rate from **A**:

$$\tau = \Gamma(2S_{ij}S_{ij})^{-1/2}$$

Lagrangian stochastic model for A:

Set of 9 (8) coupled nonlinear stochastic ODE's:

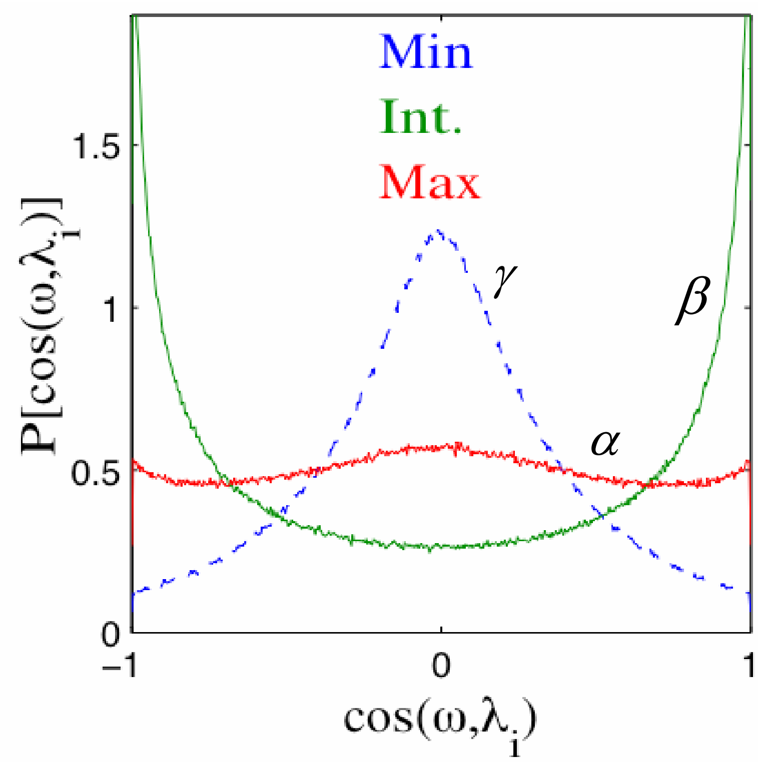
$$d\mathbf{A} = \left(-\mathbf{A}^2 + \frac{Tr(\mathbf{A}^2)}{Tr(\mathbf{c}_\tau^{-1})}\mathbf{c}_\tau^{-1} - \frac{Tr(\mathbf{c}_\tau^{-1})}{3T}\mathbf{A} \right) dt + d\mathbf{W}$$

dW: white-in-time Gaussian forcing (trace-free-isotropic-covariance structure - unit variance (in units of T)

L. Chevillard & CM, PRL 2006 (in press):

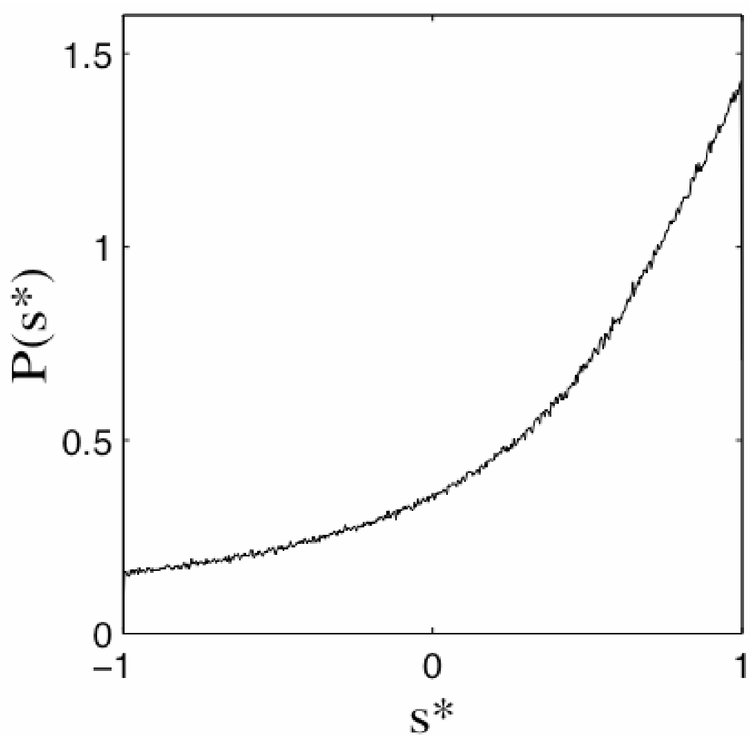
Recall Phenomenology:

Preferential voricity alignment
(Tsinober, Ashurst et al..):



$$\alpha \geq \beta \geq \gamma$$

Preferred strain-state
(Lund & Rogers, 1994)



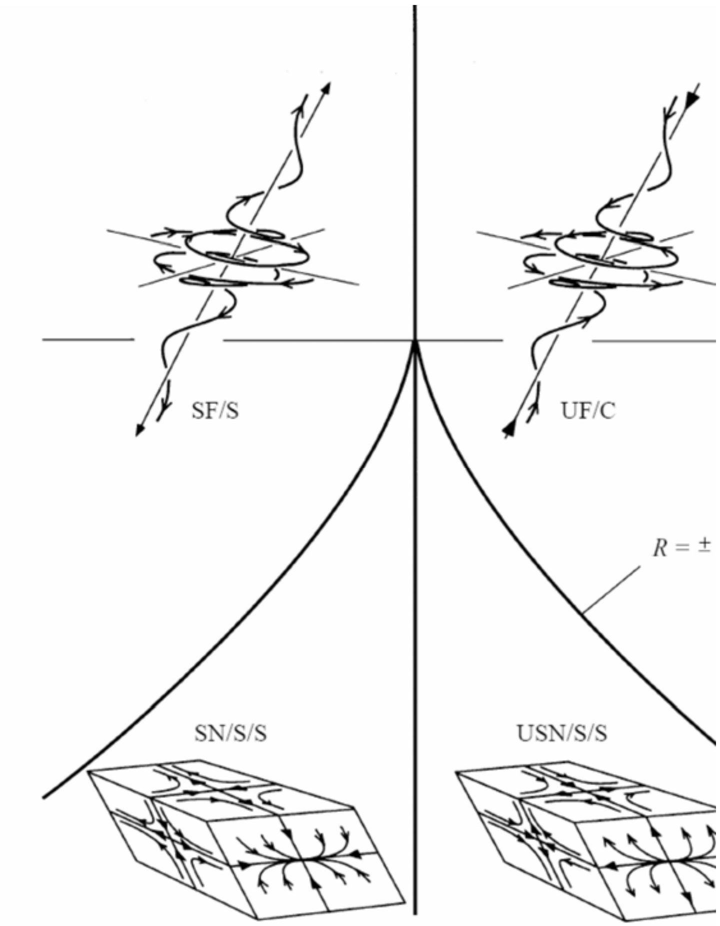
$$s^* = \frac{-3\sqrt{6}\alpha\beta\gamma}{(\alpha^2 + \beta^2 + \gamma^2)^{3/2}}$$

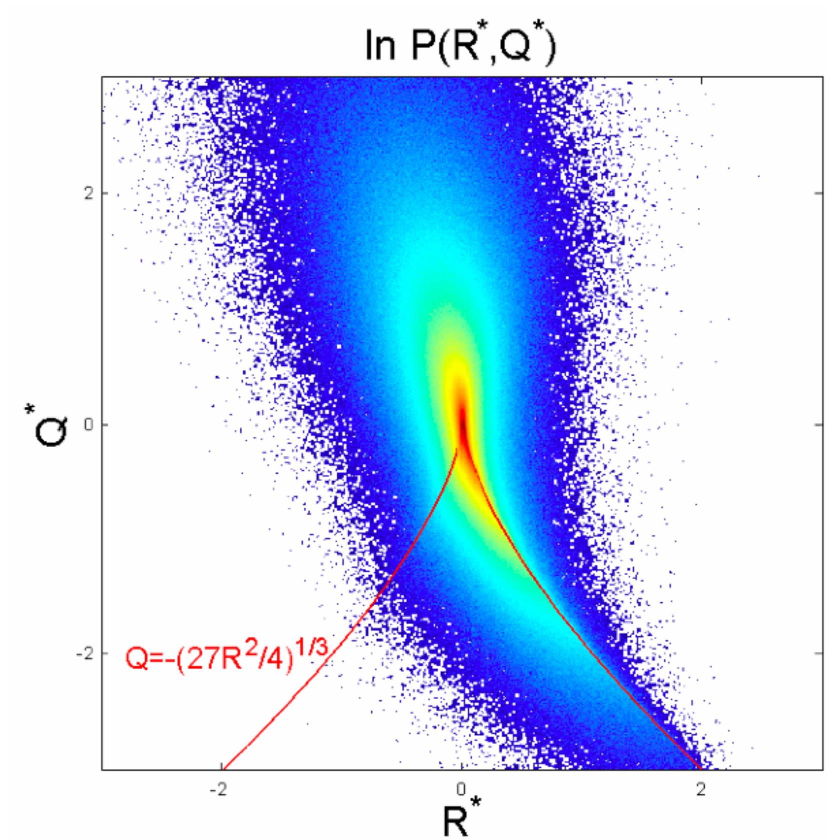
Recall Phenomenology:

Local flow topology (Cantwell, 1992):

$$Q = -\frac{1}{2} A_{pq} A_{qp} \qquad R = -\frac{1}{3} A_{pq} A_{qm} A_{mp}$$

DNS data: pearl-shape R-Q plane:





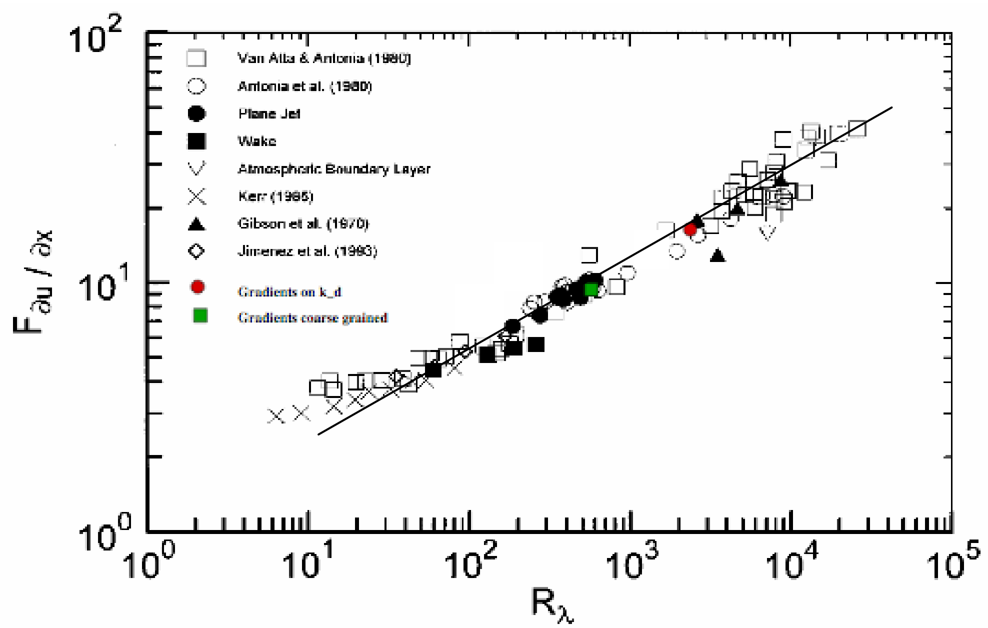
Recall Phenomenology:

Statistical Intermittency (stretched PDFs) and anomalous scaling of moments

$$\left\langle A_{11}^{p}\right\rangle : \ \mathrm{Re}^{F_L(p)} \Rightarrow \left\langle A_{11}^{p}\right\rangle \sim \left\langle A_{11}^{2}\right\rangle^{F_L(p)/F_L(2)}$$

$$\left\langle A_{12}^{p}\right\rangle : \ \mathrm{Re}^{F_T(p)}$$

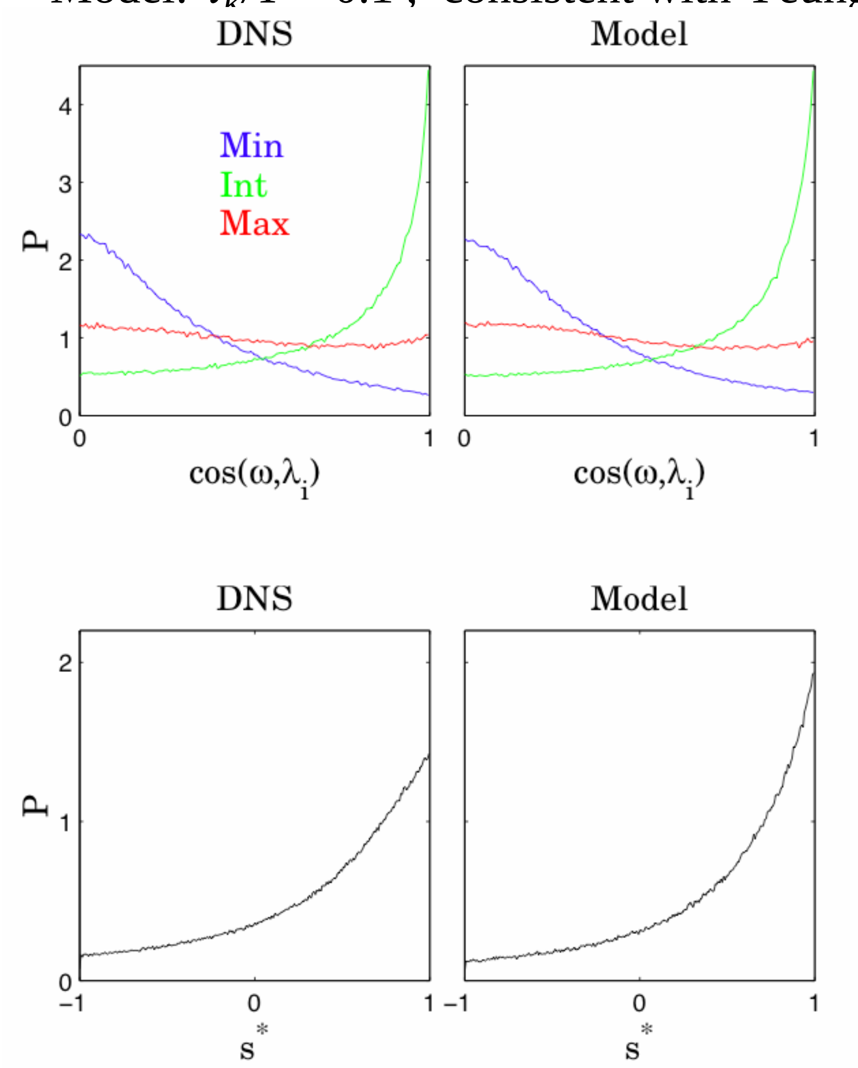
$$F = \frac{\left\langle A_{11}^{4}\right\rangle}{\left\langle A_{11}^{2}\right\rangle^{2}}$$



(Sreenivasan & Antonia)

Results & Comparison with DNS:

- DNS: 256^3: $R_\lambda = 150$
- Model: $\tau_\kappa/T = 0.1$, consistent with Yeung et al. (JoT 2006) at same R_λ

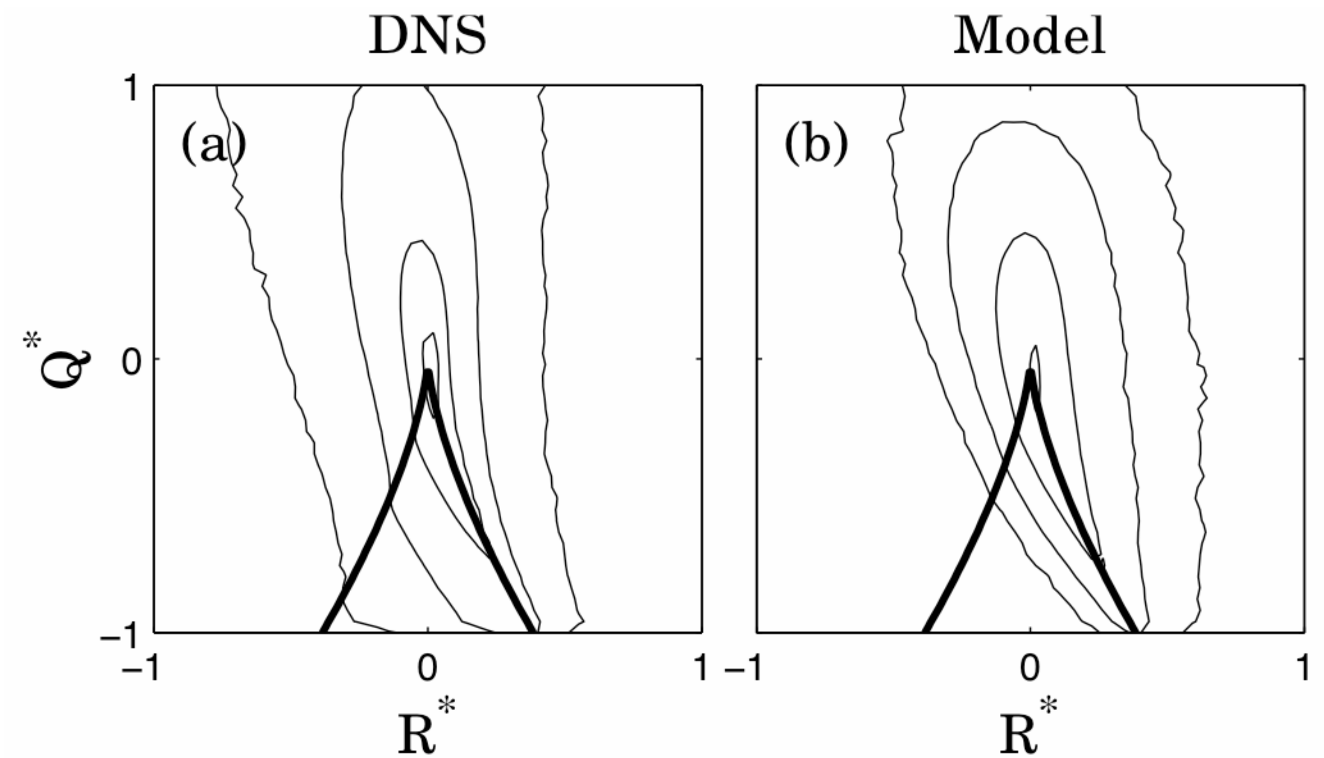


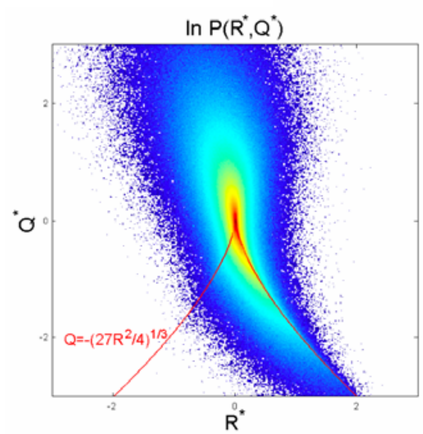
- Alignment of vorticity
  w=ε:(**A**-**A**^{T})/2 with
  S=(**A**+**A**^{T})/2 eigenvectors

  Best alignment with **intermediate**

- PDF of strain-state parameter s*:
  prevalence of **axisymmetric extension**

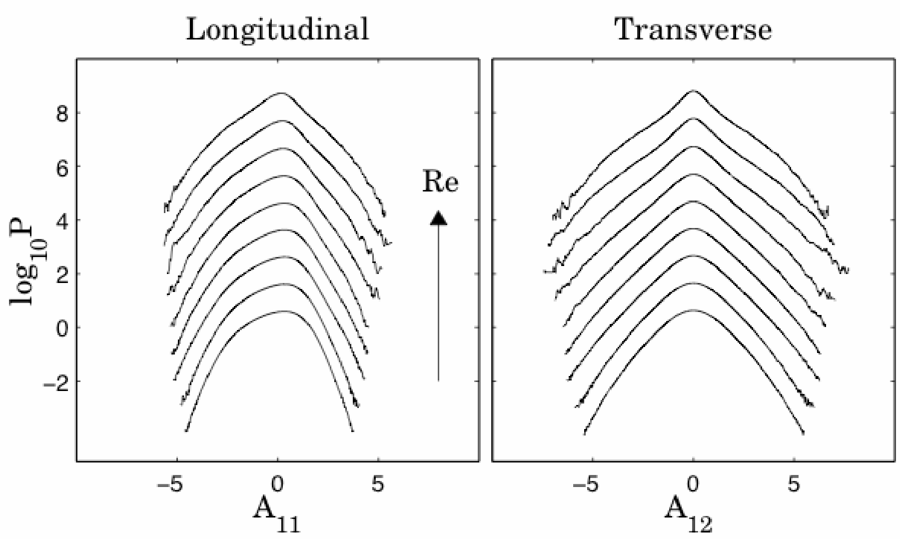
Results & Comparison with DNS: R-Q diagram





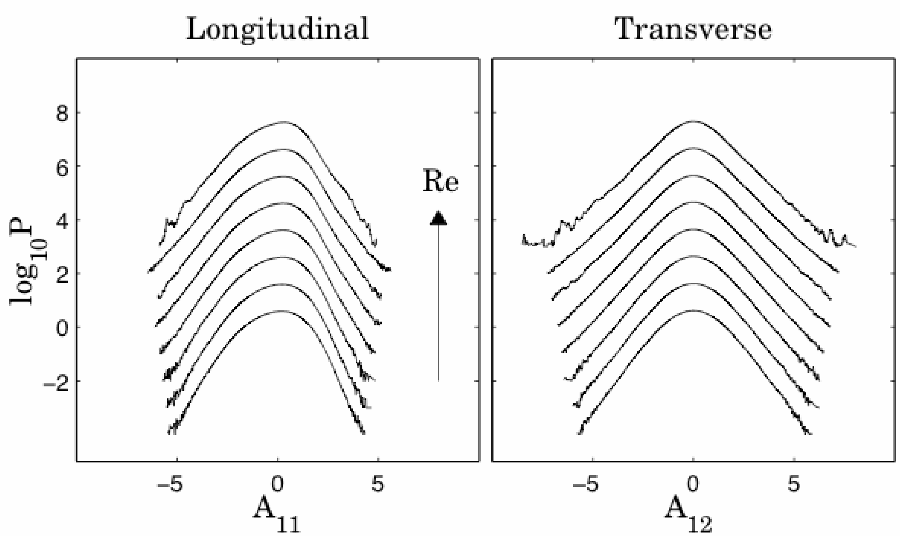
- Shows "Viellefosse tail" also seen in Van der Bos et al (2002) experimental data
- Good but not "perfect" agreement (better than previous tetrad model)

Intermittency (PDFs as function or Re or parameter Γ)



$\tau=\tau_K$:

- Deforms as funcion of Re (realistic),
- Not very realistic at large Re ("R_λ" > 200)

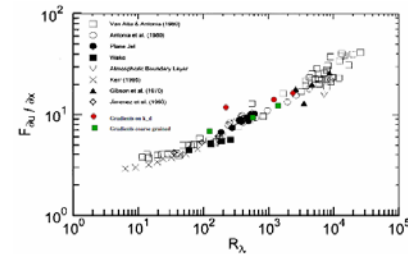


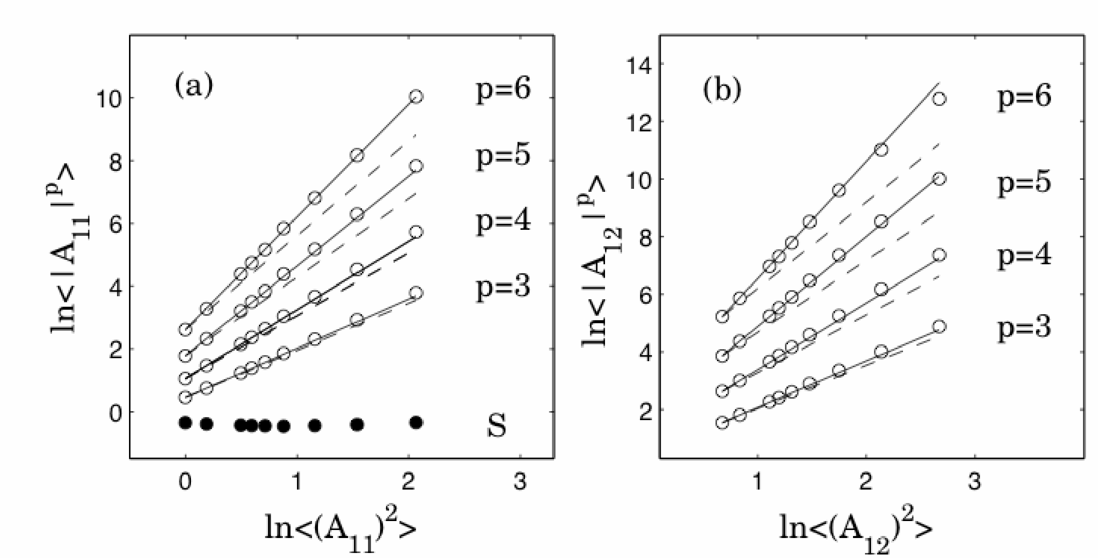
$\tau=\Gamma/(2\mathbf{S}:\mathbf{S})^{1/2}$:

- Quite realistic,
- But model diverges if $\Gamma < \Gamma_{crit}$

Intermittency (Relative anomalous scaling exponents:)

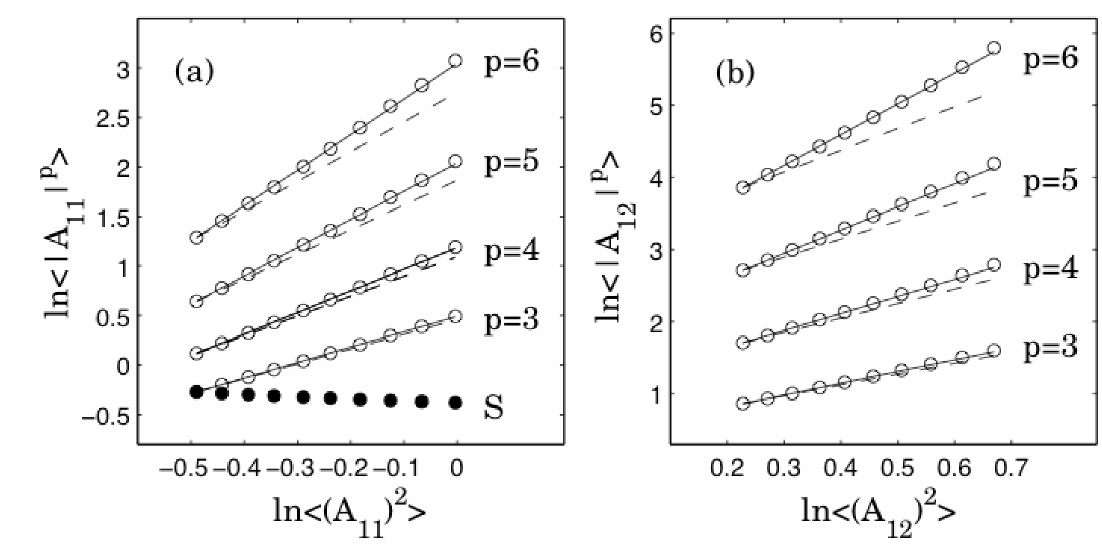
- - - Kolmogorov (1941), ——— Multifractal scaling (Nelkin 1991)





$\tau = \tau_K$:

- $\mu_L = 0.25$
- $\mu_T = 0.36$
- Skewness S ~ -0.35 to -0.5



$\tau = \Gamma/(2\mathbf{S:S})^{1/2}$:

- $\mu_L = 0.25$
- $\mu_T = 0.40$
- Skewness S ~ -0.35 to -0.5

Conclusions:

• It appears that quite a bit (more than previously thought) about turbulence phenomenology may be understood from "local" - "self-stretching" terms "-z^2" or "-A^2" - here we have:

- **(i) "projected" into special directions that simplify things**
- **(ii) "modeled" regularizing terms to get stationary behavior for entire A**

• Simple advected delta-vee system "explains" many qualitative intermittency trends from a very low-dimensional system of ODEs - long time behavior wrong….

• Statistically stationary system of 8 forced ODE's has been proposed - derived from grad(Navier-Stokes) and using physically motivated models for pressure Hessian and viscous Hessian

• Model reproduces structural geometric features of turbulence (RQ, s*, alignments) **and** statistical intermitency measures such as long tails in PDFs, stronger intermittency in transverse directions, and anomalous relative scaling exponents (in a small range of Re).