Quaternions and particle dynamics in the Euler fluid equations

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JDG & Darryl Holm 06: http://arxiv.org/abs/nlin.CD/0607020


Galanti, JDG & Heritage; Nonlinearity 10, 1675, 1997.

Summary of this talk

**Question:** Do the Euler equations possess some subtle geometric structure that guides the direction of vorticity – see Peter Constantin, *Geometric statistics in turbulence*, SIAM Reviews, 36, 73–98.

1. Quaternions: what are they?

2. **Lagrangian particle dynamics:** We find explicit equations for the Lagrangian derivatives of an ortho-normal co-ordinate system at each point in space. (JDG/Holm 06)

3. For the $3D$-Euler equations; Ertel’s Theorem shows how Euler fits naturally into this framework (JDG, Holm, Kerr & Roulstone 2006).


5. A different direction of vorticity result involving the pressure Hessian.
Lord Kelvin (William Thompson) once said:

Quaternions came from Hamilton after his best work had been done, & though beautifully ingenious, they have been an unmixed evil to those who have touched them in any way.

http://www-groups.dcs.st-and.ac.uk/ history/Mathematicians/Hamilton.html

Kelvin was wrong because quaternions are now used in the computer animation, avionics & robotics industries to track objects undergoing sequences of tumbling rotations.

- **Visualizing quaternions**, by Andrew J. Hanson, MK-Elsevier, 2006.
What are quaternions? (Hamilton (1843))

Quaternions are constructed from a scalar $p$ & a 3-vector $q$ by forming the tetrad

$$p = [p, q] = pI - q \cdot \sigma,$$

$$q \cdot \sigma = \sum_{i=1}^{3} q_i \sigma_i$$

based on the Pauli spin matrices that obey the relations $\sigma_i \sigma_j = -\delta_{ij} - \epsilon_{ijk} \sigma_k$

$$\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$  

Thus quaternions obey the multiplication rule

$$p_1 \otimes p_2 = [p_1p_2 - q_1 \cdot q_2, p_1q_2 + p_2q_1 + q_1 \times q_2].$$  

They are associative but obviously non-commutative.
Quaternions, Rotations and Cayley-Klein parameters

Let \( \hat{p} = [p, q] \) be a unit quaternion with inverse \( \hat{p}^* = [p, -q] \) with \( p^2 + q^2 = 1 \), which guarantees \( \hat{p} \otimes \hat{p}^* = [1, 0] \). For a pure quaternion \( r = [0, r] \) there exists a transformation \( r \rightarrow r' \)

\[
t' = \hat{p} \otimes r \otimes \hat{p}^* = [0, (p^2 - q^2)r + 2p(q \times r) + 2q(r \cdot q)].
\]

Now choose \( p = \pm \cos \frac{1}{2} \theta \) and \( q = \pm \hat{n} \sin \frac{1}{2} \theta \), where \( \hat{n} \) is the unit normal to \( r \)

\[
t' = \hat{p} \otimes r \otimes \hat{p}^* = [0, r \cos \theta + (\hat{n} \times r) \sin \theta],
\]

where

\[
\hat{p} = \pm [\cos \frac{1}{2} \theta, \hat{n} \sin \frac{1}{2} \theta].
\]

This represents a rotation by an angle \( \theta \) of the 3-vector \( r \) about its normal \( \hat{n} \).

The elements of the unit quaternion \( \hat{p} \) are the Cayley-Klein parameters from which the Euler angles can be calculated. All terms are quadratic in \( p \) and \( q \), and thus allow a double covering (\( \pm \)) (see Whittaker 1945).
General Lagrangian evolution equations

Consider the general Lagrangian evolution equation for a 3-vector $w$ such that

\[
\frac{Dw}{Dt} = a(x, t) \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla
\]

transported by a velocity field $u$. Define the scalar $\alpha_a$ the 3-vector $\chi_a$ as

\[
\alpha_a = |w|^{-1}(\hat{w} \cdot a), \quad \chi_a = |w|^{-1}(\hat{w} \times a).
\]

for $|w| \neq 0$. Via the decomposition $a = \alpha_a w + \chi_a \times w$, $|w|$ & $\hat{w}$ satisfy

\[
\frac{D|w|}{Dt} = \alpha_a |w|, \quad \frac{D\hat{w}}{Dt} = \chi_a \times \hat{w}.
\]

$\alpha_a$ is the growth rate (Constantin 1994) & $\chi_a$ is the ‘swing’ rate. The ‘tetrads’

\[
q_a = [\alpha_a, \chi_a], \quad w = [0, w].
\]

allow us to write this as

\[
\frac{Dw}{Dt} = q_a \odot w.
\]
**Theorem:** (JDG/Holm 06) If $a$ is differentiable in the Lagrangian sense s.t.

$$\frac{Da}{Dt} = b(x, t),$$

(i) For for $|w| \neq 0$, $q_a$ and $q_b$ satisfy the Ricatti equation

$$\frac{Dq_a}{Dt} + q_a \otimes q_a = q_b;$$

(ii) At each point $x$ there exists an orthonormal frame $(\hat{w}, \hat{\chi}_a, \hat{w} \times \hat{\chi}_a) \in SO(3)$

whose Lagrangian time derivative is expressed as

$$\frac{D\hat{w}}{Dt} = D_{ab} \times \hat{w},$$

$$\frac{D(\hat{w} \times \hat{\chi}_a)}{Dt} = D_{ab} \times (\hat{w} \times \hat{\chi}_a),$$

$$\frac{D\hat{\chi}_a}{Dt} = D_{ab} \times \hat{\chi}_a,$$

where the Darboux angular velocity vector $D_{ab}$ is defined as

$$D_{ab} = \chi_a + \frac{c_1}{\chi_a} \hat{w}, \quad c_1 = \hat{w} \cdot (\hat{\chi}_a \times \chi_b).$$
Lagrangian frame dynamics: tracking a particle

The dotted line represents a particle (●) trajectory moving from \((x_1, t_1)\) to \((x_2, t_2)\).

The orientation of the orthonormal unit vectors \(\{\hat{w}, \hat{x}_a, (\hat{w} \times \hat{x}_a)\}\)

is driven by the Darboux vector \(D_{ab} = x_a + c_1 \hat{w}\) where \(c_1 = \hat{w} \cdot (\hat{x}_a \times x_b)\).

Thus we need the ‘quartet’ of vectors to make this process work

\(\{u, w, a, b\}\).
Proof: (i) It is clear that (with \( q_b = [\alpha_b, \chi_b] \))

\[
\frac{D^2 w}{Dt^2} = [0, b] = q_b \otimes w.
\]

Compatibility between this and the \( q \)-equation means that

\[
\left( \frac{Dq_a}{Dt} + q_a \otimes q_a - q_b \right) \otimes w = 0,
\]

(ii) Now consider the ortho-normal frame \((\hat{w}, \hat{\chi}_a, \hat{w} \times \hat{\chi}_a)\) as in the Figure below.

The evolution of \( \chi_a \) comes from

\[
\frac{Dq_a}{Dt} + q_a \otimes q_a = q_b,
\]

and gives

\[
\frac{D\chi_a}{Dt} = -2\alpha_a \chi_a + \chi_b.
\]
\( \mathbf{b} \) can be expressed in this ortho-normal frame as the linear combination

\[
\mathbf{b} = |\mathbf{w}| \left[ \alpha_b \hat{\mathbf{w}} + c_1 \hat{\mathbf{x}}_a + c_2 (\hat{\mathbf{w}} \times \hat{\mathbf{x}}_a) \right],
\]

\[
\chi_b = c_1 (\hat{\mathbf{w}} \times \hat{\mathbf{x}}_a) - c_2 \hat{\mathbf{x}}_a,
\]

where \( c_1 = \hat{\mathbf{w}} \cdot (\hat{\mathbf{x}}_a \times \chi_b) \) and \( c_2 = -(\hat{\mathbf{x}}_a \cdot \chi_b) \). From the Ricatti equation for the tetrad \( \mathbf{q}_a = [\alpha_a, \chi_a] \) (where \( \chi_a = |\chi_a| \))

\[
\frac{D\chi_a}{Dt} = -2\alpha_a \chi_a + \chi_b,
\]

\[
\Rightarrow \quad \frac{D\chi_a}{Dt} = -2\alpha_a \chi_a - c_2,
\]

There follows

\[
\frac{D\hat{\chi}_a}{Dt} = c_1 \chi_a^{-1}(\hat{\mathbf{w}} \times \hat{\mathbf{x}}_a),
\]

\[
\frac{D(\hat{\mathbf{w}} \times \hat{\mathbf{x}}_a)}{Dt} = \chi_a \hat{\mathbf{w}} - c_1 \chi_a^{-1} \hat{\mathbf{x}}_a,
\]

which, together with

\[
\frac{D\hat{\mathbf{w}}}{Dt} = \chi_a \times \hat{\mathbf{w}},
\]

can be re-expressed in terms of the Darboux vector \( \mathbf{D}_a = \chi_a + \frac{c_1}{\chi_a} \hat{\mathbf{w}} \).
Ertel’s Theorem & the 3D Euler equations

\[
\frac{D\omega}{Dt} = \omega \cdot \nabla u = S\omega
\]

Euler in vorticity format

**Theorem:** (Ertel 1942) If \( \omega \) satisfies the 3D incompressible Euler equations then any arbitrary differentiable \( \mu \) satisfies

\[
\frac{D}{Dt} (\omega \cdot \nabla \mu) = \omega \cdot \nabla \left( \frac{D\mu}{Dt} \right) \implies \left[ \frac{D}{Dt}, \omega \cdot \nabla \right] = 0.
\]

**Proof:** Consider \( \omega \cdot \nabla \mu \equiv \omega_i \mu, i \)

\[
\frac{D}{Dt}(\omega_i \mu, i) = \frac{D\omega_i}{Dt} \mu, i + \omega_i \left\{ \frac{\partial}{\partial x_i} \left( \frac{D\mu}{Dt} \right) - u_{k,i} \mu, k \right\}
\]

\[
= \left\{ \omega_j u_i, j \mu, i - \omega_i u_{k,i} \mu, k \right\} + \omega_i \frac{\partial}{\partial x_i} \left( \frac{D\mu}{Dt} \right)
\]

zero under summation

In characteristic (Lie-derivative) form, \( \omega \cdot \frac{\partial}{\partial x}(t) = \omega \cdot \frac{\partial}{\partial x}(0) \) is a Lagrangian invariant (Cauchy 1859) and is “frozen in”.
Various references

- Bauer’s thesis 2000 (ETH-Berlin); *Gradient entropy vorticity, potential vorticity and its history*.
Ohkitani’s result & the pressure Hessian

Define the Hessian matrix of the pressure

\[ P = \{p_{ij}\} = \left\{ \frac{\partial^2 p}{\partial x_i \partial x_j} \right\} \]

then Ohkitani took \( \mu = u_i \) (Phys. Fluids, A5, 2576, 1993).

**Result:** The vortex stretching vector \( \omega \cdot \nabla u = S\omega \) obeys

\[ \frac{D(\omega \cdot \nabla u)}{Dt} = \frac{D(S\omega)}{Dt} = \omega \cdot \nabla \left( \frac{Du}{Dt} \right) = -P\omega \]

Thus for Euler, via Ertel’s Theorem, we have the identification:

\[ w \equiv \omega \quad a \equiv \omega \cdot \nabla u = S\omega \quad b \equiv -P\omega \]

with a quartet

\[ (u, w, a, b) \equiv (u, \omega, S\omega, -P\omega). \]
Euler: the variables $\alpha(x, t)$ and $\chi(x, t)$

$$S \hat{\omega} = \alpha \hat{\omega} + \chi \times \hat{\omega}$$


\[\begin{align*}
(\alpha_a) \quad \alpha &= \hat{\omega} \cdot S \hat{\omega} \\
(-\alpha_b) \quad \alpha_p &= \hat{\omega} \cdot P \hat{\omega} \\
\chi &= \hat{\omega} \times S \hat{\omega} \quad (\chi_a) \\
\chi_p &= \hat{\omega} \times P \hat{\omega} \quad (-\chi_b) \\
q &= [\alpha, \chi] \\
q_b &= -q_p = -[\alpha_p, \chi_p]
\end{align*}\]

$$\frac{Dq}{Dt} + q \otimes q + q_p = 0$$

constrained by $TrP = \Delta p = -u_{i,j}u_{j,i} = \frac{1}{2} \omega^2 - TrS^2$. 
Lagrangian frame dynamics: tracking an Euler fluid particle

The dotted line represents the fluid packet (●) trajectory moving from \((x_1, t_1)\) to \((x_2, t_2)\). The orientation of the orthonormal unit vectors 
\[
\{\hat{\omega}, \hat{\chi}, (\hat{\omega} \times \hat{\chi})\}
\]
is driven by the Darboux vector 
\[
\mathcal{D} = \chi + \frac{c_1}{\nabla} \hat{\omega}, \quad c_1 = -\hat{\omega} \cdot (\hat{\chi} \times \chi_p).
\]
Thus the pressure Hessian within \(c_1\) drives the Darboux vector \(\mathcal{D}\).
The $\alpha$ and $\chi$ equations

In terms of $\alpha$ and $\chi$, the Ricatti equation for $q$

$$\frac{Dq_a}{Dt} + q_a \otimes q_a = q_b;$$

becomes

$$\frac{D\alpha}{Dt} = \chi^2 - \alpha^2 - \alpha_p, \quad \frac{D\chi}{Dt} = -2\alpha\chi - \chi_p.$$

(Galanti, JDG & Heritage; Nonlinearity 10, 1675, 1997). Stationary values are

$$\alpha = \gamma_0, \quad \chi = 0, \quad \alpha_p = -\gamma_0^2$$

which correspond to Burgers’-like vortices.

When tubes & sheets bend & tangle then $\chi \neq 0$ and $q$ becomes a full tetrad driven by $q_p$ which is coupled back through the elliptic pressure condition.

**Note:** Off-diagonal elements of $P$ change rapidly near intense vortical regions across which $\chi_p$ and $\alpha_p$ change rapidly.
Phase plane

On Lagrangian trajectories, the $\alpha - \chi$ equations become

\[
\frac{\partial \alpha}{\partial t} = \chi^2 - \alpha^2 - \alpha_p, \quad \frac{\partial \chi}{\partial t} = -2\alpha\chi + C_p.
\]

where $C_p = -\hat{\chi} \cdot \chi_p$.

In regions of the $\alpha - \chi$ phase plane where $\alpha_p = \text{const}$, $C_p = \text{const}$ there are 2 critical points:

\[(\alpha, \chi) = (\pm \alpha_0, \chi_0) \quad 2\alpha_0^2 = \alpha_p + [\alpha_p^2 + C_p^2]^{1/2}\]

- The critical point in the LH-half-plane $(-\alpha_0, \chi_0)$ is an unstable spiral;
- The critical point in the RH-half-plane is $(\alpha_0, \chi_0)$ is a stable spiral.
The next few slides: remarks on the “direction of vorticity” in Euler

1. The BKM theorem

2. • The work of Constantin, Fefferman & Majda 1996 and Constantin 1994
   • The work of Deng, Hou & Yu 2005/6
   • Can our quaternionic Ricatti equation give anything in terms of $P$?
The Beale-Kato-Majda Theorem (CMP 94, 61-6, 1984)

Theorem: There exists a global solution of the Euler equations $u \in C([0, \infty]; H^s) \cap C^1([0, \infty]; H^{s-1})$ for $s \geq 3$ if, for every $t^* > 0$,

$$\int_0^{t^*} \|\omega(\tau)\|_{L^\infty(\Omega)} d\tau < \infty.$$ 

The proof is based on $\|\nabla u\|_\infty \leq c \|\omega\|_\infty [1 + \log H_3]$.

Thus one needs to numerically monitor only $\int_0^{t^*} \|\omega(\tau)\|_\infty d\tau$.

Corollary: If a singularity is observed in a numerical experiment of the form $\|\omega\|_\infty \sim (t^* - t)^{-\beta}$ then $\beta$ must lie in the range $\beta \geq 1$ for the singularity to be genuine & not an artefact of the numerical calculation.
Constantin, Fefferman & Majda; Comm PDEs, 21, 559-571, 1996

The image $W_t$ of a set $W_0$ is given by $W_t = X(t, W_0)$. $W_0$ is said to be **smoothly directed** if there exists a length $\rho > 0$ and a ball $0 < r < \frac{1}{2}\rho$ such that:

1. $\hat{\omega}(\cdot, t)$ has a Lipschitz extn to the ball of radius $4\rho$ centred at $X(q, t)$ &

$$M = \lim_{t \to T} \sup_{q \in W_0^*} \int_0^t \| \nabla \hat{\omega}(\cdot, t) \|^2_{L^\infty(B_{4\rho})} dt < \infty.$$ 

i.e. the direction of vorticity is well-behaved in the nbhd of a set of trajectories.

2. The condition $\sup_{B_{3r}(W_t)} |\omega(x, t)| \leq m \sup_{B_r(W_t)} |\omega(x, t)|$ holds for all $t \in [0, T)$ with $m = \text{const} > 0$; i.e. this nbhd captures large & growing vorticity but not so that it overlaps with another similar region & $\sup_{B_4r(W_t)} |u(x, t)| \leq U(t) := \sup_x |u(x, t)| < \infty$ (Cordoba & Fefferman 2001; for tubes).

**Theorem: (CFM 1996)** Assume that $W_0$ is smoothly directed as in (i)–(ii).

Then $\exists$ a time $\tau > 0$ & a constant $\Gamma$ s.t. for any $0 \leq t_0 < T$ and $0 \leq t - t_0 \leq \tau$

$$\sup_{B_r(W_t)} |\omega(x, t)| \leq \Gamma \sup_{B_\rho(W_t)} |\omega(x, t_0)|.$$
The work of Deng, Hou & Yu; Comm PDEs, 31, 293–306, 2006

Consider a family of vortex line segments $L_t$ in a region of max-vorticity. Denote by $L(t)$ the arc length of $L_t$, $\hat{n}$ the unit normal & $\kappa$ the curvature. DHY define

$$U_{\hat{\omega}}(t) \equiv \max_{x,y \in L_t} |(u \cdot \hat{\omega})(x,t) - (u \cdot \hat{\omega})(y,t)|,$$

$$U_n(t) \equiv \max_{L_t} |u \cdot \hat{n}|, \text{ and } M(t) \equiv \max \left( \| \nabla \cdot \hat{\omega} \|_{L^\infty(L_t)}, \| \kappa \|_{L^\infty(L_t)} \right).$$

**Theorem: (Deng, Hou & Yu 06):** Let $A, B \in (0, 1)$ with $B = 1 - A$, and $C_0$ be a positive constant. If

1. $U_{\hat{\omega}}(t) + U_n(t) \lesssim (T - t)^{-A},$
2. $M(t)L(t) \leq C_0,$
3. $L(t) \gtrsim (T - t)^B,$

then there will be no blow-up up to time $T$.

Also J. Deng, T. Y. Hou & X. Yu; Comm. PDEs, 30, 225-243, 2005.
Using the pressure Hessian

(see also Chae: $\int_0^T \|S\hat{\omega} \cdot P\hat{\omega}\|_\infty \, d\tau < \infty$; Comm. P&A-M., 109, 1–21, 2006).

**Theorem:** (JDG, Holm, Kerr & Roulstone 06): $\exists$ a global solution of the Euler equations, $u \in C([0, \infty]; H^s) \cap C^1([0, \infty]; H^{s-1})$ for $s \geq 3$ if

$$\int_0^T \|\chi_p\|_{L^\infty(D)} \, d\tau < \infty,$$

with the exception of when $\hat{\omega}$ becomes collinear with an e-vec of $P$ at $t = T$.

**Proof:** With $|S\hat{\omega}|^2 = \alpha^2 + \chi^2$,

$$\frac{D|S\hat{\omega}|}{Dt} \leq -\alpha|S\hat{\omega}| + \frac{\alpha|\alpha_p| + |\chi||\chi_p|}{(\alpha^2 + \chi^2)^{1/2}}.$$

Because $D|\omega|/Dt = \alpha|\omega|$, our concern is with $\alpha \geq 0$

$$\frac{D|S\hat{\omega}|}{Dt} \leq |\alpha_p| + |\chi_p|.$$

Possible that $|P\hat{\omega}|$ blows up simultaneously as the angle between $\hat{\omega}$ and $P\hat{\omega} \rightarrow 0$ thus keeping $\chi_p$ finite; i.e. $\int_0^t \|\chi_p\|_{L^\infty(D)} \, d\tau < \infty$ but $\int_0^t \|\alpha_p\|_{L^\infty(D)} \, d\tau \rightarrow \infty$. 
Frame dynamics & the Frenet-Serret equations

With \( \hat{w} \) as the unit tangent vector, \( \hat{\chi} \) as the unit bi-normal and \( \hat{w} \times \hat{\chi} \) as the unit principal normal, the matrix \( N \) can be formed

\[
N = \left( \hat{w}^T, (\hat{w} \times \hat{\chi})^T, \hat{\chi}^T \right),
\]

with

\[
\frac{DN}{Dt} = GN, \quad G = \begin{pmatrix}
0 & -\chi_a & 0 \\
\chi_a & 0 & -c_1\chi_a^{-1} \\
0 & c_1\chi_a^{-1} & 0
\end{pmatrix}.
\]

The Frenet-Serret equations for a space-curve are

\[
\frac{dN}{ds} = FN \quad \text{where} \quad F = \begin{pmatrix}
0 & \kappa & 0 \\
-\kappa & 0 & \tau \\
0 & -\tau & 0
\end{pmatrix},
\]

where \( \kappa \) is the curvature and \( \tau \) is the torsion.
The arc-length derivative $d/ds$ is defined by

$$\frac{d}{ds} = \hat{\omega} \cdot \nabla.$$ 

The evolution of the curvature $\kappa$ and torsion $\tau$ may be obtained from Ertel’s theorem expressed as the commutation of operators $\left[ \frac{D}{Dt}, \omega \cdot \nabla \right] = 0$

$$\alpha_a \frac{d}{ds} + \left[ \frac{D}{Dt}, \frac{d}{ds} \right] = 0.$$ 

This commutation relation immediately gives

$$\alpha_a F + \frac{DF}{Dt} = \frac{dG}{ds} + [G, F].$$ 

Thus Ertel’s Theorem gives explicit evolution equations for the curvature $\kappa$ and torsion $\tau$ that lie within the matrix $F$ and relates them to $c_1, \chi_a$ and $\alpha_a$. 
Mixing

Consider a passive vector line-element $\delta \ell$ in a flow transported by an independent velocity field $u$. For small $\delta \ell$ we have the same equations as Euler for $\omega$

$$\frac{D\delta \ell}{Dt} = \delta \ell \cdot \nabla u$$
$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla$$

Following the analogy with Euler, Ertel’s Theorem holds so there is a $b$-field:

$$\frac{D(\delta \ell \cdot \nabla u)}{Dt} = \delta \ell \cdot \nabla \left( \frac{Du}{Dt} \right)$$

$Du/Dt$ represents any dynamics one wishes to impose on the problem. Thus all the conditions hold for Theorem 1:

1. $w = \delta \ell$
2. $a = \delta \ell \cdot \nabla u$
3. $b = \delta \ell \cdot \nabla \left( \frac{Du}{Dt} \right) \rightarrow$ a Ricatti equation plus an ortho-normal frame ...
Ideal MHD

Consider a magnetic field $B$ coupled to a fluid ($\text{div} \ u = 0 = \text{div} B$)

$$\frac{D u}{D t} = B \cdot \nabla B - \nabla p \quad \frac{D B}{D t} = B \cdot \nabla u$$

Defining Elsasser variables with $\pm$-material derivatives (two time-clocks)

$$v^\pm = u \pm B \quad \frac{D^\pm}{D t} = \frac{\partial}{\partial t} + v^\pm \cdot \nabla$$

the magnetic field $B$ and $v^\pm$ satisfy with $\text{div} v^\pm = 0$

$$\frac{D^\pm v^\mp}{D t} = -\nabla p \quad \frac{D^\pm B}{D t} = B \cdot \nabla v^\pm$$

Moffatt (1985) suggested that $B$ takes the place of $\omega$ in ideal MHD.
Ertel’s Theorem (proof omitted) for this system is

\[
\frac{D^\mp (B \cdot \nabla v^\pm)}{D t} = -PB.
\]

With two time-clocks, we have the correspondence

\[
w \equiv B \quad a^\pm \equiv B \cdot \nabla v^\pm \quad b \equiv -PB
\]

\[
\alpha_{pb} = \hat{B} \cdot PB \quad \chi_{pb} = \hat{B} \times PB
\]

Define tetrads \(q^\pm\) and \(q_{pb}\) as follows

\[
q^\pm = [\alpha^\pm, \chi^\pm] \quad q_{pb} = [\alpha_{pb}, \chi_{pb}].
\]

The tetrads \(q^\pm\) satisfy the compatibility relation

\[
\frac{D^\mp q^\pm}{D t} + q^\pm \otimes q^\mp + q_{pb} = 0
\]
MHD-Lagrangian frame dynamics

We have 2 sets of orthonormal vectors $\hat{B}$, $(\hat{B} \times \hat{\chi}^{\pm})$, $\hat{\chi}^{\pm}$ acted on by their opposite Lagrangian time derivatives.

$$
\frac{D^\mp \hat{B}}{Dt} = D^\mp \times \hat{B},
$$

$$
\frac{D^\mp (\hat{B} \times \hat{\chi}^{\pm})}{Dt} = D^\mp \times (\hat{B} \times \hat{\chi}^{\pm}),
$$

$$
\frac{D^\mp \hat{\chi}^{\pm}}{Dt} = D^\mp \times \hat{\chi}^{\pm}
$$

where the pair of Darboux vectors $D^\mp$ are defined as

$$
D^\mp = \chi^\mp - \frac{c_1^\mp}{\chi^\mp} \hat{B}, \quad c_1^\mp = \hat{B} \cdot [\hat{\chi}^{\pm} \times (\chi_{pb} + \alpha^{\pm} \chi^\mp)].
$$