A sufficient condition of regularity for axially symmetric solutions to the Navier-Stokes equations

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Motivation I

$e_1, e_2, e_3 \leftrightarrow x_1, x_2, x_3$ and $e_\varrho, e_\varphi, e_3 \leftrightarrow \varrho, \varphi, x_3$

$$e_\varrho = \cos\varphi e_1 + \sin\varphi e_2, \quad e_\varphi = -\sin\varphi e_1 + \cos\varphi e_2, \quad e_3 = e_3$$

$$v = v_i e_i = v_1 e_1 + v_2 e_2 + v_3 e_3 = v_\varrho e_\varrho + v_\varphi e_\varphi + v_3 e_3$$

$$\partial_t v + v \cdot \nabla v - \Delta v + \nabla p = 0, \qquad \mathrm{div} v = 0$$

$$v_{\varrho,\varphi} = \partial v_\varrho / \partial\varphi = 0, \quad v_{\varphi,\varphi} = 0, \quad v_{3,\varphi} = 0 \quad p_{,\varphi} = 0$$

Motivation II

Ladyzhenskaya (1968), Ukhovskij & Yudovich (1968), Leonardi, Malek, Necas, & Pokorny (1999), Neustupa & Pokorny (2001), Pokorny (2001), Chae & Lee (2002), Zajaczkowski (2004, 2005), Wiegner & Zajaczkowski (2005), and others.

For the Cauchy problem on $]0,T[$, Chae & Lee (2002) proved

$$\int_0^T dt \Big(\int_{\mathbb{R}^3} \frac{1}{\varrho} |v|^\gamma dx \Big)^{\frac{\alpha}{\gamma}} < +\infty \Rightarrow \text{regularity}$$

if $1/\alpha + 1/\gamma \leq 1/2$, $2 < \gamma < +\infty$, $2 < \alpha \leq +\infty$

Questions

The marginal case $\gamma = 2$ and $\alpha = +\infty$? which is an analogue of $L_{3,\infty}$-case in the absence of axial symmetry.

Local Version?

Notation

$$\mathcal{C}(x_0, R) = \{x \in \mathbb{R}^3 \parallel x = (x', x_3),\ x' = (x_1, x_2),$$

$$|x' - x'_0| < R,\ |x_3 - x_{03}| < R\}, \quad \mathcal{C}(R) = \mathcal{C}(0, R), \quad \mathcal{C} = \mathcal{C}(1);$$

$$z = (x, t),\ z_0 = (x_0, t_0), \quad Q(z_0, R) = \mathcal{C}(x_0, R) \times]t_0 - R^2, t_0[,$$

$$Q(R) = Q(0, R), \qquad Q = Q(1).$$

Suitable Weak Solutions

$$v \in L_{2,\infty}(Q) \cap W_2^{1,0}(Q), \qquad p \in L_{\frac{3}{2}}(Q);$$

v and p satisfy NSE's in weak sense;

$$\int\limits_B \varphi(x,t)|v(x,t)|^2 dx + 2\int\limits_{-1}^{t}\int\limits_B \varphi|\nabla v|^2 dxdt'$$

$$\leq \int\limits_{-1}^{t}\int\limits_B \Big(|v|^2(\Delta\varphi + \partial_t\varphi) + v\cdot\varphi(|v|^2 + 2p)\Big)dxdt'$$

for a.a $t \in]-1,0[$ and for all functions $0 \leq \varphi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R}^1)$ vanishing in a neighborhood of the parabolic boundary of Q.

Main Result

Theorem 0.1 *Let v and p be an axially symmetric suitable weak solution to the Navier-Stokes equations in Q. Assume that*

(0.1) $$\mathcal{A}_0 = \operatorname{ess} \sup_{-1 \le t \le 0} \int_{\mathcal{C}} \frac{1}{\varrho} |v(x,t)|^2 dx < +\infty.$$

Then the point $(x,t) = (0,0)$ is a regular point of v, i.e., there exists $r \in]0,1]$ such that v is Hölder continuous in the closure of the cylinder $Q(r)$.

Preliminaries I

Invariant Functionals

$$A(z_0, r; v) = \operatorname{ess} \sup_{t_0 - r^2 < t < t_0} \frac{1}{r} \int\limits_{\mathcal{C}(x_0, r)} |v(x,t)|^2 dx,$$

$$C(z_0, r; v) = \frac{1}{r^2} \int\limits_{Q(z_0, r)} |v|^3 dz,$$

$$E(z_0, r; v) = \frac{1}{r} \int\limits_{Q(z_0, r)} |\nabla v|^2 dz, \quad D(z_0, r; p) = \frac{1}{r^2} \int\limits_{Q(z_0, r)} |p|^{\frac{3}{2}} dz.$$

Preliminaries II

Energy Inequality

$$A(z_0,R/2;v)+E(z_0,R/2;v)\leq c(C^{\frac{2}{3}}(z_0,R;v)+$$
$$+C(z_0,R;v)+D(z_0,R;p))$$

Decay Estimate for Pressure

$$D(z_0,r;p)\leq c\Big[\frac{r}{r_1}D(z_0,r_1;p)+\Big(\frac{r_1}{r}\Big)^2 C(z_0,r_1;v)\Big],$$

which is valid for all $0<r\leq r_1\leq R$

Preliminaries III

Lemma 0.2 *Let* v *and* p *be a suitable weak solution to the Navier-Stokes equations in* Q *and let*

$$A_0 = \sup_{0<r<1} A(0,r;v) < +\infty.$$

Then, for any $r \in]0,1/2[$*, we have*

$$C^{\frac{4}{3}}(0,r;v) + D(0,r;p) + E(0,r;v) \leq c\Big((A_0+1)r^{\frac{1}{2}}(D(0,1;p)+$$

$$+E(0,1;v)) + A_0^4 + A_0^2 + A_0\Big).$$

Preliminaries IV

Lemma 0.3 *Under the conditions of Theorem 0.1, we have*

$$A(z_0, r; v) + C(z_0, r; v) + D(z_0, r; p) + E(z_0, r; v) \leq \mathcal{A} < +\infty$$

for all $z_0 = (x_0, 0)$*,* $x_0 = (0, b)$*,* $|b| \leq 1/4$*, and for all* $0 < r \leq 1/4$*, where* $\mathcal{A}$ *depends on* $D(0, 1; p)$*,* $E(0, 1; v)$*, and* $\mathcal{A}_0$ *only.*

$$\mathcal{P}(R_1, R_2; a) = \{x \in \mathbb{R}^3 \parallel R_1 < |x'| < R_2,\ |x_3| < a\}$$

Preliminaries V

Lemma 0.4 *Let v and p be a suitable weak solution to the NSE's in the set $\widehat{Q} = \mathcal{P}(3/4, 9/4; 3/2)\times] - (3/2)^2, 0[$. Assume that*

$$\int\limits_{\widehat{Q}} |v(z)|^6 dz \le m < +\infty.$$

Then, there exists a function $\Phi_0 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, nondecreasing in each variables, such that

$$|v(z)| + |\nabla v(z)| \le \Phi_0(m, \mathcal{A}_*) < +\infty, \; \mathcal{A}_* = \int\limits_{\widehat{Q}} |p(z)|^{\frac{3}{2}} dz$$

for any $z \in \mathcal{P}(1, 2; 1)\times] - 1, 0[$.

Preliminaries VI

Lemma 0.5 *Assume that all conditions of Theorem 0.1 hold. Then*

$$\int\limits_{\mathcal{C}} \frac{1}{\varrho}|v(x,t)|^2 dx \leq \mathcal{A}_0$$

for all $t \in]-1, 0[$.

Scaling and Blow up I

Ad absurdum. Let $z=0$ is a singular point of v. Then $\exists$ $\{R_k\}_{k=1}^{\infty}$ such that $R_k\to 0$ as $k\to+\infty$ and

$$\frac{1}{R_k^2}\int\limits_{Q(R_k)}|v|^3dz\geq\varepsilon>0$$

for all $k\in\mathbb{N}$.

$$u^k(y,s)=R_kv(R_ky,R_k^2s),\quad q^k(y,s)=R_k^2p(R_ky,R_k^2s),$$

where $e=(y,s)\in Q(1/R_k)$.

Scaling and Blow up II

$$x_k^b = (0, bR_k), \quad y^b = (0, b), \quad z_k^b = (x_k^b, 0), \quad e^b = (y^b, 0)$$

$$|b| R_k < 1/4, \qquad aR_k < 1/4$$

$$C(z_k^b, aR_k; v) = C(e^b, a; u^k) \leq \mathcal{A},$$

$$E(z_k^b, aR_k; v) = E(e^b, a; u^k) \leq \mathcal{A},$$

$$A(z_k^b, aR_k; v) = A(e^b, a; u^k) \leq \mathcal{A},$$

$$D(z_k^b, aR_k; p) = D(e^b, a; q^k) \leq \mathcal{A}$$

for all $k \geq k_0(a, b)$.

Scaling and Blow up III

Limit functions u and q satisfy the NSE's on $\mathbb{R}^3 \times \mathbb{R}_-$, $\mathbb{R}_- = \{s \in \mathbb{R} \,\|\, s \leq 0\}$. They are called an ancient solution to the NSE's. For any $a > 0$,

$$u^k \rightharpoonup u \qquad \text{in } W_2^{1,0}(Q(a)), \qquad u^k \overset{\star}{\rightharpoonup} u \qquad \text{in } L_{2,\infty}(Q(a)),$$

$$u^k \to u \qquad \text{in } L_3(Q(a)), \qquad q^k \rightharpoonup q \qquad \text{in } L_{\frac{3}{2}}(Q(a))$$

and

$$C(e^b, a; u) \leq \mathcal{A}, \qquad A(e^b, a; u) \leq \mathcal{A},$$

$$E(e^b, a; u) \leq \mathcal{A}, \qquad D(e^b, a; q) \leq \mathcal{A}$$

Scaling and Blow up IV

$$\operatorname{ess}\sup_{-\infty<s\leq 0}\int\limits_{\mathbb{R}^3}\frac{|u(y,t)|^2}{|y'|}dy\leq \mathcal{A}_0,$$

$$\frac{1}{R_k^2}\int\limits_{Q(R_k)}|v|^3dz=\int\limits_{Q}|u^k|^3de\to\int\limits_{Q}|u|^3de\geq\varepsilon,$$

and, for any $a>1$,

$$u^k\to u\qquad \text{in } C([-1,0];L_{\frac{9}{8}}(\mathcal{C}(a))).$$

Scaling and Blow up V

$$\Big(\int\limits_{\mathcal{P}(r_1,r_2;h)} |u(y,0)|^{\frac{9}{8}} dy\Big)^{\frac{8}{9}} \le \Big(\int\limits_{\mathcal{P}(r_1,r_2;h)} |u^k(y,0)-u(y,0)|^{\frac{9}{8}} dy\Big)^{\frac{8}{9}} +$$

$$+\Big(\int\limits_{\mathcal{P}(r_1,r_2;h)} |u^k(y,0)|^{\frac{9}{8}} dy\Big)^{\frac{8}{9}} = \alpha_k + \beta_k,$$

$$\alpha_k \longrightarrow 0$$

Scaling and Blow up VI

$$\beta_k = \Big(R_k^{-\frac{15}{8}} \int\limits_{\mathcal{P}(R_k r_1, R_k r_2; R_k h)} |v(x,0)|^{\frac{9}{8}} dx \Big)^{\frac{8}{9}} \le$$

$$\le c(r_1, r_2, h) \Big(\frac{1}{R_k} \int\limits_{\mathcal{P}(R_k r_1, R_k r_2; R_k h)} |v(x,0)|^2 dx \Big)^{\frac{1}{2}} \le$$

$$\le c(r_1, r_2, h) \Big(\int\limits_{\mathcal{P}(R_k r_1, R_k r_2; R_k h)} \frac{|v(x,0)|^2}{|x'|} dx \Big)^{\frac{1}{2}} \to 0$$

$$\Rightarrow u(\cdot, 0) = 0 \qquad \text{in} \quad \mathbb{R}^3.$$

Estimates of Axially Sym Solutions I

Proposition 0.6 *Let* V *and* P *be a sufficiently smooth axially symmetric solution to the Navier-Stokes equations in* $\widetilde{Q} = \widetilde{\mathcal{P}} \times]-2^2, 0[$*, where* $\widetilde{\mathcal{P}} = \mathcal{P}(1/4, 3; 2)$*. Then, there exists a non-decreasing function* $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\sup_{z \in \mathcal{P}(1,2;1) \times]-1,0[} \Big(|V(z)| + |\nabla V(z)| \Big) \leq \Phi(\mathcal{A}_2),$$

where

$$\mathcal{A}_2 = \sup_{-2^2 < t < 0} \int_{\widetilde{\mathcal{P}}} |V(x,t)|^2 dx + \int_{\widetilde{Q}} \Big(|\nabla V|^2 + |V|^3 + |P|^{\frac{3}{2}} \Big) dz.$$

Estimates of Axially Sym Solutions II

Lemma 0.7 *Under assumptions of Proposition 0.6, there exists a function* $\Phi_1 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$*, non-decreasing in each variable, such that*

$$\text{(0.2)} \quad \sup_{-(7/4)^2 < t < 0} \int_{\widetilde{\mathcal{P}}_1} |V^a(x,t)|^q dx \le \Phi_1(q, \mathcal{A}_2), \qquad 1 \le q + \infty.$$

Here, $V^a = (V_\varrho, V_3)$*,* $|V^a| = \sqrt{|V_\varrho|^2 + |V_3|^2}$*,*
$\widetilde{\mathcal{P}}_1 = \mathcal{P}(5/16, 11/4; 7/4)$*, and* $\widetilde{Q}_1 = \widetilde{\mathcal{P}}_1 \times] - (7/4)^2, 0[$*.*

Estimates of Axially Sym Solutions III

Lemma 0.8 *Under assumptions of Proposition 0.6, there exists a non decreasing function* $\Phi_5 : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\int\limits_{\widetilde{Q}_2} |V_\varphi|^6 dz \leq \Phi_5(\mathcal{A}_2),$$

where $\widetilde{Q}_2 = \widetilde{\mathcal{P}}_2 \times]-(3/2)^2, 0[$ *and* $\widetilde{\mathcal{P}}_2 = \mathcal{P}(3/8, 5/2; 3/2)$*.*

Corollary 0.9 *Under assumptions of Proposition 0.6, there exists a non-decreasing function* $\Phi_6 : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\int\limits_{\widetilde{Q}_2} |V|^6 dz \leq \Phi_6(\mathcal{A}_2).$$

Decay I

Given $R > 1$,

$$\widetilde{Q}^b_R = \widetilde{\mathcal{P}}^b_R \times]-(2R)^2, 0[,$$

where $b \in \mathbb{R}$ and

$$\widetilde{\mathcal{P}}^b_R = \widetilde{\mathcal{P}}_R + be_3, \qquad \widetilde{\mathcal{P}}_R = \mathcal{P}(R/4, 3R; 2R).$$

Scale ancient solution u and q in the following way

$$u^R(x,t) = Ru(Rx + be_3, R^2 t), \qquad q^R(x,t) = R^2 q(Rx + be_3, R^2 t)$$

for $z = (x,t) \in \widetilde{Q}$.

Decay II

$$\sup_{z\in\widetilde{Q}_0}\Big\{|u^R(z)|+|\nabla u^R(z)|\Big\}\leq\Phi(\mathcal{A}_2),$$

where $\widetilde{Q}_0=\mathcal{P}(1,2;1)$ and

$$\mathcal{A}_2=\sup_{-2^2\leq t\leq 0}\int\limits_{\widetilde{\mathcal{P}}}|u^R(x,t)|^2dx+\int\limits_{\widetilde{Q}}\Big(|\nabla u^R|^2+|u^R|^3+|q^R|^{\frac{3}{2}}\Big)dz.$$

Decay III

$$\sup_{(y,s)\in Q^b_R} \Big\{ R|u(y,s)| + R^2|\nabla u(y,s)| \Big\} \leq \Phi(c\mathcal{A}),$$

where $Q^b_R = \widetilde{\mathcal{P}}^b_{0R} \times]-R^2, 0[$, $\widetilde{\mathcal{P}}^b_{0R} = be_3 + \mathcal{P}_{0R}$,
$\mathcal{P}_{0R} = \mathcal{P}(R, 2R; R)$

$$\Rightarrow \quad |y'||u(y', b, s)| + |y'|^2|\nabla u(y', b, s)| \leq \Phi(c\mathcal{A})$$

for any $b \in \mathbb{R}$, for any $|y'| > 20$, and for any $s \in [-20, 0]$

$$\Rightarrow \quad |u(y,s)| + |\nabla u(y,s)| \leq c\Phi(c\mathcal{A}) = c(\mathcal{A})$$

for any $|y'| > 20$ and for any $s \in [-20, 0]$

Backward Uniqueness

$$\omega(u) = \nabla \wedge u \quad \Rightarrow \quad \partial_t \omega - \Delta\omega = \omega \cdot \nabla u - u \cdot \nabla\omega$$

So,

$$|\partial_t \omega - \Delta\omega| \le c(\mathcal{A})(|\omega| + |\nabla\omega|), \qquad |\omega| \le c(\mathcal{A})$$

for any $|y'| > 20$ and for any $s \in [-20, 0]$ and

$$\omega(\cdot, 0) = 0 \qquad \text{in} \quad \mathbb{R}^3$$

$$\Rightarrow \quad \omega(y, s) = 0$$

for any $|y'| > 20$ and for any $s \in [-20, 0]$.

Unique Continuation I

$$\nabla \wedge u \equiv 0 \qquad \text{in} \quad \mathbb{R}^3 \setminus \{y' \neq 0\} \times [-8, 0]$$

Unique Continuation II

$$\mathcal{A}_0 \geq \operatorname{ess} \sup_{-20 \leq s \leq 0} \int_{|y'| \leq 40} \frac{|u(y,s)|^2}{|y'|} dy$$

$$\Rightarrow \quad \int_{-\infty}^{+\infty} dy_3 \int_{|y'| \leq 40} \frac{|u(y,s)|^2}{|y'|} dy' \leq \mathcal{A}_0 < +\infty,$$

for any $s \in S \subset [-20, 0]$ and $|S| = 20$.

$$\Rightarrow \quad \nabla \wedge u(\cdot, s) \equiv 0 \qquad \text{in} \quad \mathbb{R}^3$$

for any $s \in S$.

Unique Continuation III

For any $y_0 \in \{|y'| \leq 30,\ y_3 \in \mathbb{R}\}$,

$$B(y_0, 1) \subset \{|y'| \leq 40,\ y_3 \in \mathbb{R}\}$$

and, since u is harmonic,

$$|u(y_0, s)| \leq c\Big(\int_{B(y_0,1)} |u(y,s)|^2 dy\Big)^{\frac{1}{2}} \leq c\Big(\int_{|y'|\leq 40} |u(y,s)|^2 dy\Big)^{\frac{1}{2}} \leq$$

$$\leq c\sqrt{40\mathcal{A}_0}$$

for any $s \in S \cap [-8, 0] \Rightarrow u(\cdot, s)$ is bounded in $\mathbb{R}^3$ for any $s \in S \cap [-8, 0] \Rightarrow u(\cdot, s) = 0$ in $\mathbb{R}^3$ for any $s \in S \cap [-8, 0]$.