Calculation of complex singular solutions to the 3D incompressible Euler equations

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Numerical Studies

- Axisymmetric flow with swirl and 2D Boussinesq convection
  - -Grauer & Sideris (1991, 1995),Pumir & Siggia (1992)
    Meiron & Shelley (1992), E & Shu (1994)
    Grauer et al (1998), Yin & Tang (2006)

- High symmetry flows
  - -Kida-Pelz flow: Kida (1985), Pelz & coworkers (1994,1997)
  - -Taylor-Green flow: Brachet & coworkers (1983,2005)

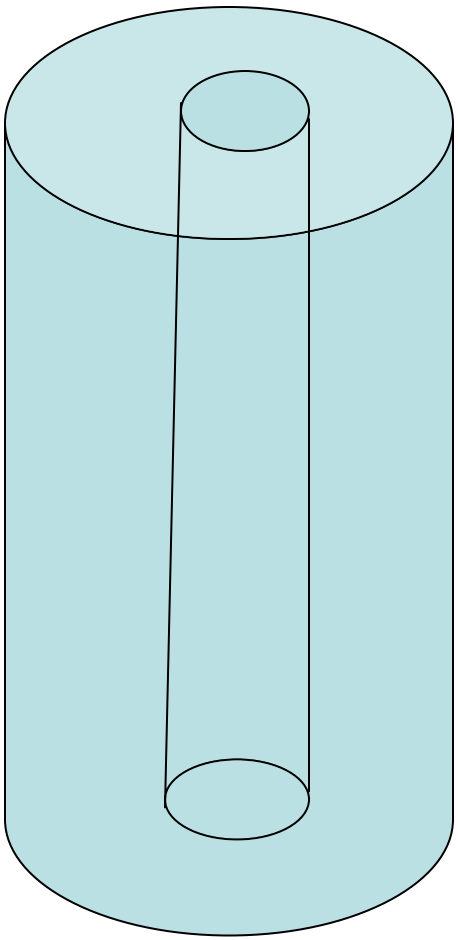
- Antiparallel vortex tubes
  - -Kerr (1993, 2005)
  - -Hou & Li (2006)

- Pauls et al(2006).: Study of complex space singularities for 2D Euler in short time asymptotic regime

Axisymmetric flow with swirl

•Caflisch (1993), Caflisch & Siegel (2004)

$$r^{-1}\partial_r(ru_r) + \partial_z u_z = 0$$
$$\partial_t u_z + \mathbf{u}\cdot\nabla u_z + \partial_z p = 0$$
$$\partial_t u_r + \mathbf{u}\cdot\nabla u_r - r^{-1}u_\theta^2 + \partial_r p = 0$$
$$\partial_t u_\theta + \mathbf{u}\cdot\nabla u_\theta + r^{-1}u_\theta u_r = 0.$$



•Annular geometry

$$r_1 < r < r_2,\ 0 < z < 2\pi$$

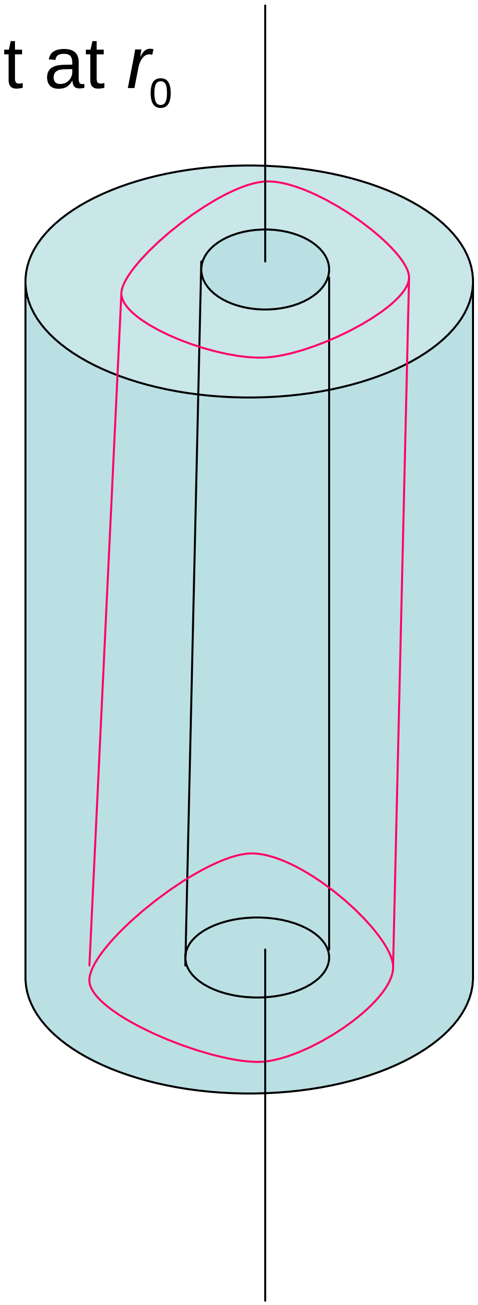
•Steady background flow

$$\mathbf{u} = (0, \bar{u}_\theta, \bar{u}_z)(r)$$

chosen to satisfy Rayleigh's criterion for instability
and an unstable eigenmode $\hat{\mathbf{u}}_1(r)e^{iz+\sigma t}$

Background flow

- Background flow is smoothed vortex sheet at r_0 (motivated by Caflisch, Li, Shelley 1991)



$$\bar{u}_\theta = \begin{cases} \Gamma_1/2\pi r & r_1 < r < r_0 \\ \Gamma_2/2\pi r & r_0 < r < r_2 \end{cases}$$

$$\bar{u}_z = \begin{cases} w_1 & r_1 < r < r_0 \\ w_2 & r_0 < r < r_2 \end{cases}$$

$$\bar{u}_r = 0.$$

- Pure swirling flow is unstable if $|\Gamma_1| > |\Gamma_2|$ (Rayleigh criterion)

Traveling wave solution

- Construct complex, upper-analytic traveling wave solution

  Baker, Caflisch & Siegel (1993)
  Caflisch(1993), Caflisch & Siegel (2004)

$$\mathbf{u} = \bar{\mathbf{u}}(r) + \mathbf{u}_{+}(r, z, t)$$

in which

$$\mathbf{u}_{+} = \sum_{k=1}^{\infty} \hat{\mathbf{u}}_k(r) e^{ik(z - i\sigma t)}$$

- Traveling wave with speed σ in Im(z) direction
- $\hat{\mathbf{u}}_1$ is linearly unstable eigenmode with eigenvalue σ
- Traveling wave speed σ is thus determined from linear eigenvalue problem and is independent of the amplitude

Motivation for traveling wave form

- Construction of solution is greatly simplified
  - Degrees of freedom reduced
- One way coupling among wavenumbers so mode k' depends only on $k < k'$
  - Computational errors minimized since no truncation or aliasing errors in restriction to finite number of Fourier components
- Equation for $\hat{\mathbf{u}}_{\mathbf{k}}$ has form

$$L_k \hat{\mathbf{u}}_{\mathbf{k}} = \mathbf{F}_{\mathbf{k}}(\overline{\mathbf{u}}, \hat{\mathbf{u}}_{\mathbf{1}}, \ldots, \hat{\mathbf{u}}_{\mathbf{k-1}})$$

- L_k is second order ODE operator

Motivation (cont'd)

- Singularities at $z = z_r + i\ z_i$ travel with speed σ in Im z direction, reach real z line in finite time (for $z_i \leq 0$)
- Singularities detected through asymptotics of Fourier coefficients û

(Sulem, Sulem & Frisch 1983)

- Provide information on generic form of singularities

Perturbation construction of real singular solution

Real remainder

- Consider $\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}_+ + \mathbf{u}_- + \tilde{\mathbf{u}}$ where $\mathbf{u}_- = \mathbf{u}_+^*(z^*)$
- $\bar{\mathbf{u}},\ \bar{\mathbf{u}} + \mathbf{u}_+,\ \bar{\mathbf{u}} + \mathbf{u}_-$ are exact solutions of Euler equations
- $\tilde{\mathbf{u}}$ satisfies system of equations in which forcing terms are quadratic, i.e.,

$$\mathbf{u}_+ \cdot \nabla \mathbf{u}_- + \mathbf{u}_- \cdot \nabla \mathbf{u}_+$$

- We want $\mathbf{u}_+,\ \mathbf{u}_- = O(\varepsilon) \Rightarrow \tilde{\mathbf{u}} = O(\varepsilon^2)$

$$\tilde{u}_{reg} \sim O(T),\ \tilde{u}_{\sin g} \sim O(T\varepsilon) + O(\varepsilon^2)$$

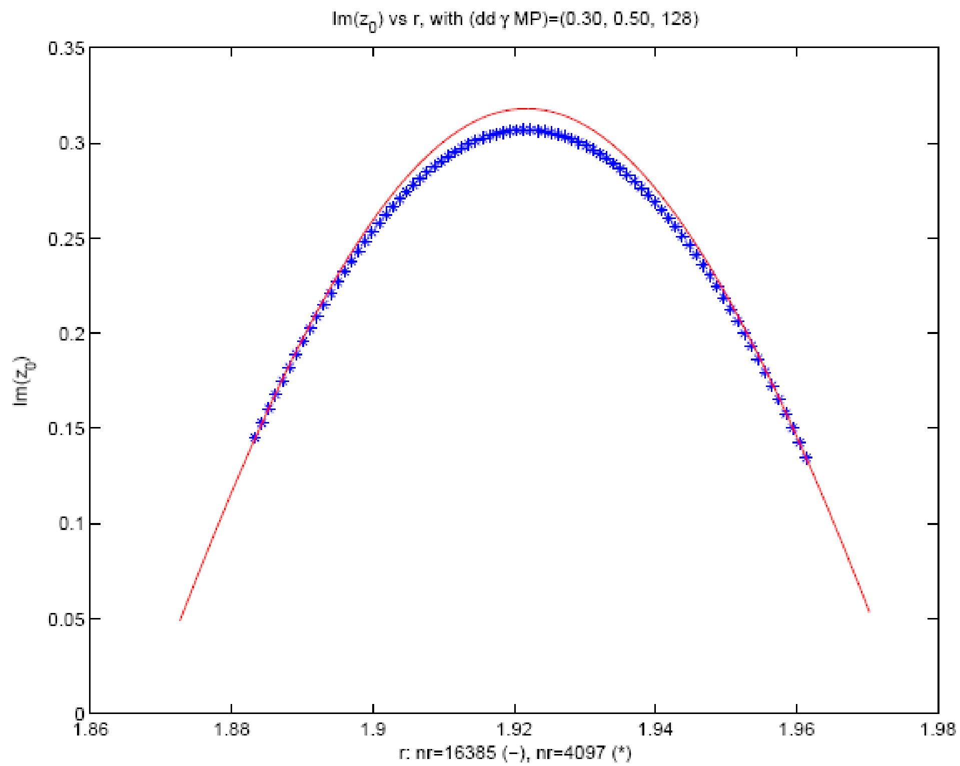
- Full construction requires analysis showing that singularity of $\tilde{\mathbf{u}}$ is same or weaker than that of $\mathbf{u}_+,\ \mathbf{u}_-$

- Similar approach used in studies of singularity formation on vortex sheets
  - Caflisch & Orellana (1989), Duchon & Robert (1988)
  - Siegel, Caflisch, Howison (2004)
  - Cordoba(2006)

- For vortex sheets, singularity formation is associated with ill-posedness

- For Euler equations, traveling wave solution comes from balance between instability and nonlinearity

- Numerical construction in Caflisch (1993) was for $\bar{\mathbf{u}}_{\mathbf{z}} = 0$
- Singularity position depends on r, i.e., $-\mathrm{Im}\ z = \rho(r)$
- Result: $u_{r+} \sim c(z - i\sigma t - i\, r^2)^{\alpha}$ where $\alpha = -1/3$
- Amplitude of u_{r+} (i.e., c) is $O(1)$



Vortex sheet analogue

- Vortex sheets in Boussinesq approximation
  Siegel (1992), (1995)
  γ - vortex sheet strength
  A - density difference
- Pure Boussinesq ($A=1,\ \gamma=0$) $\Rightarrow$ traveling waves of O(1) amplitude
- Pure vortex sheet ($A=0,\ \gamma=1$) $\Rightarrow$ no traveling waves due to conservation of vorticity on sheet
- For $A<<1,\ \gamma=1$ small amplitude ε traveling waves
  $\varepsilon \to 0$ as A $\to 0$.

Numerical method

- Pseudospectral in *z*, 4th order discretization (in r) for L_k
- Background velocities

$$\bar{\mathbf{u}}_{\mathbf{z}} = \sin\left(\frac{\theta_\gamma \pi}{2}\right)\bar{\mathbf{u}}_{\mathbf{z}0},\ \bar{\mathbf{u}}_\theta = \cos\left(\frac{\theta_\gamma \pi}{2}\right)\bar{\mathbf{u}}_{\theta 0}$$

$$0 < \theta_\gamma < 1$$

- Numerical method is accurate but unstable

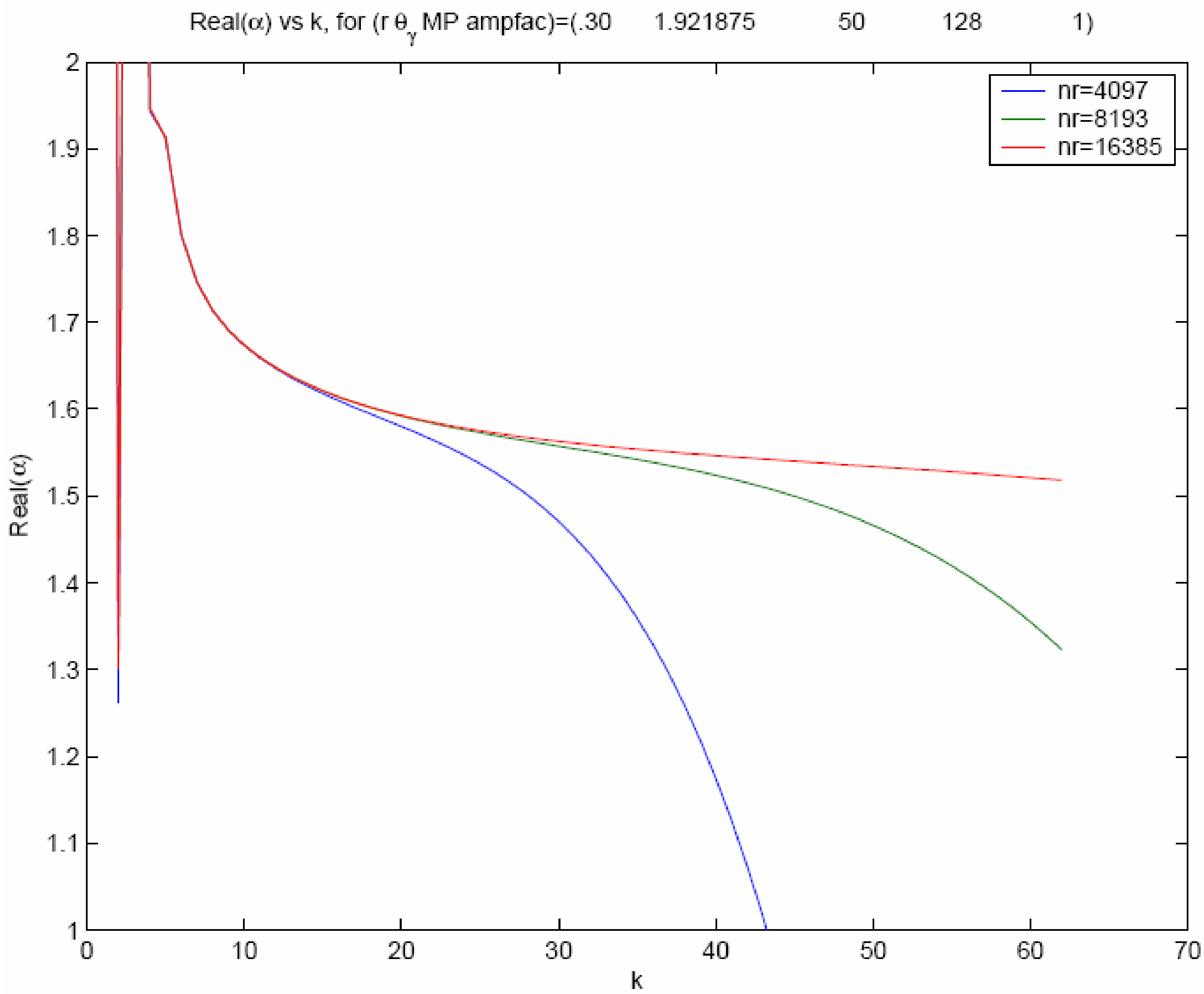
  -Instability controlled using high-precision arithemetic (10^{-100})

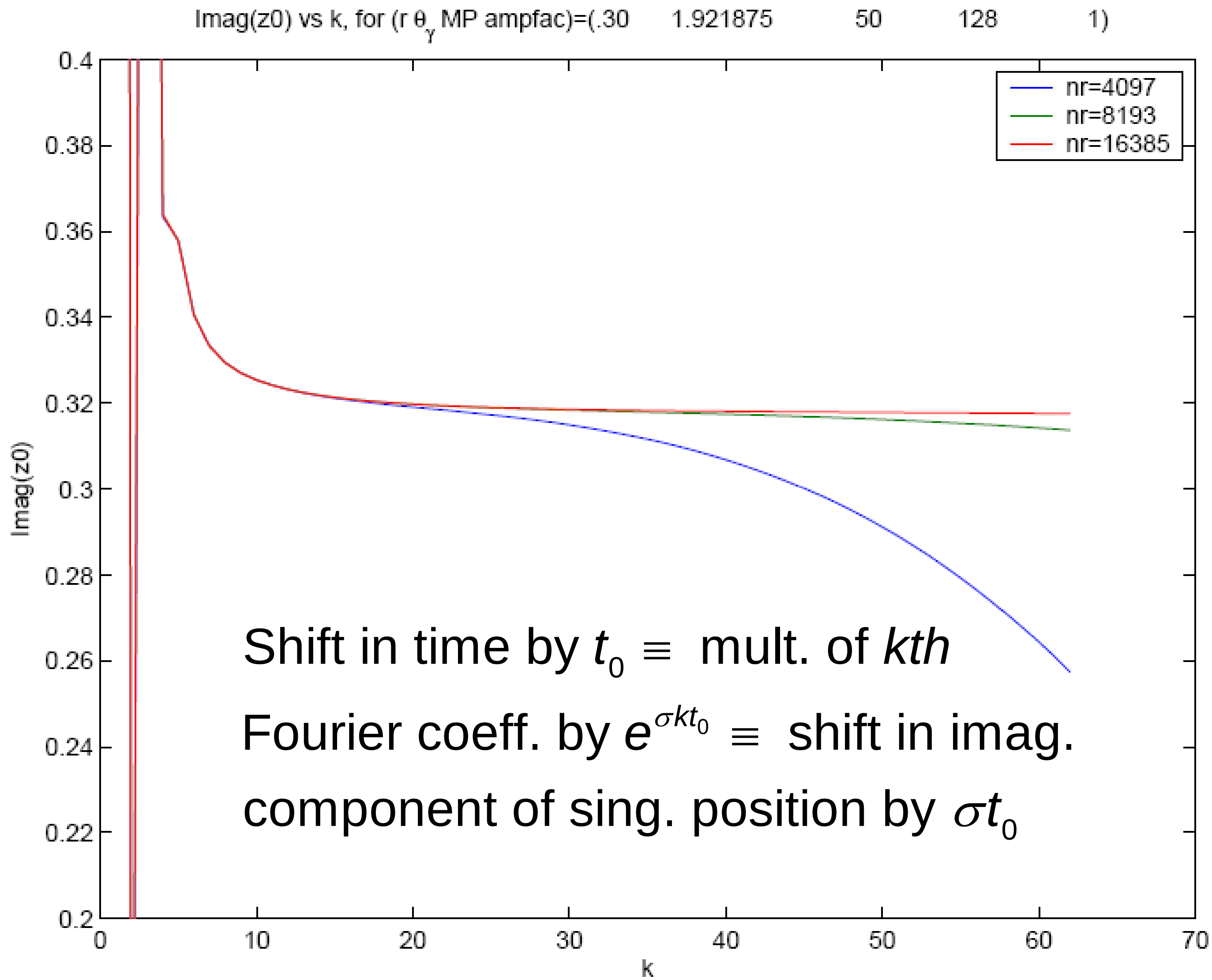
- Singularities detected through asymptotics of Fourier components (Sulem, Sulem, Frisch 1983)

$$u_+ \approx c(z-(\mu - i\ \delta))^{\alpha-1}$$

$$\hat{u}_k \approx c_1 k^{-\alpha} \exp(-k(\delta + i\mu))$$

Caflisch & Siegel (2004)

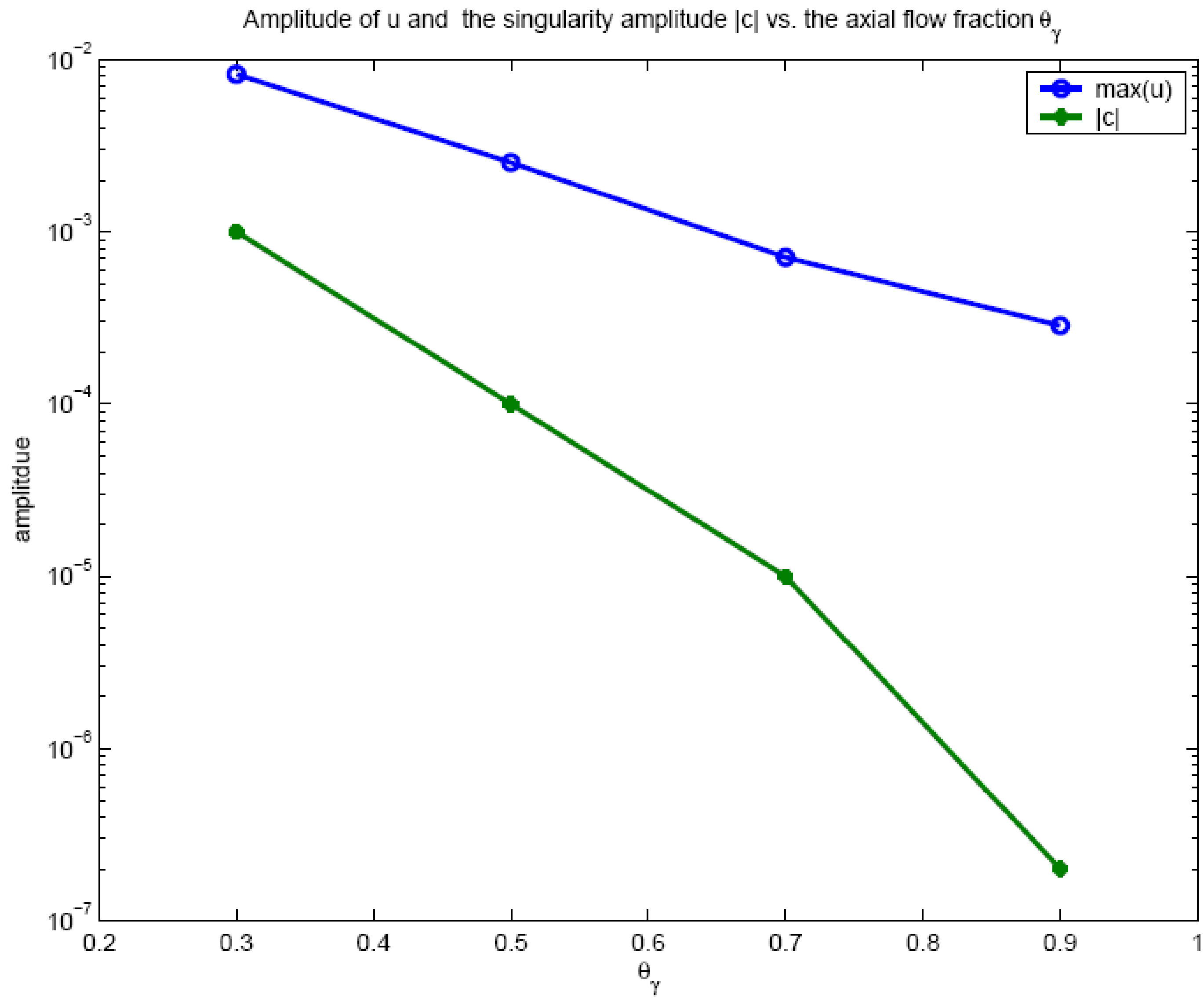




Shift in time by $t_0 \equiv$ mult. of *kth*

Fourier coeff. by $e^{\sigma k t_0} \equiv$ shift in imag. component of sing. position by σt_0

Adjustable parameters: $|\hat{u}_1|, \theta_\gamma$



Amplitude of u and the singularity amplitude |c| vs. the axial flow fraction θγ
max(u)
|c|
amplitdue
θγ
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

- Square root singularity does not satisfy Beale, Kato, Majda theorem

Singularity formation at time T $\Leftrightarrow$

$$\int^{T} \sup_{\mathbf{x}} \left| \omega(\mathbf{x}, t) \right| \, dt = \infty$$

3D traveling wave

- Control numerical instability
- Look for traveling wave solution, periodic in (x, y, z)

$$\mathbf{u} = \sum_{\mathbf{k}>\mathbf{0}} \hat{\mathbf{u}}_k \exp i\mathbf{k}\cdot(\mathbf{x} - i\ \boldsymbol{\sigma}\ t)$$

$$\mathbf{k} = (k, l, m),\ \boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

- Simplify construction

-Base flow $\bar{\mathbf{u}} = \mathbf{0}$

-Instability driven by forcing term

$$\mathbf{F}(\mathbf{x}) = \sum_{\mathbf{k}<\mathbf{N}} \hat{\mathbf{F}}_{\mathbf{k}} \exp i\mathbf{k}\cdot(\mathbf{x} - i\ \boldsymbol{\sigma}\ t)$$

- Euler equations

$$L_k \hat{\mathbf{u}}_{\mathbf{k}} = \mathbf{G}_{\mathbf{k}}(\hat{\mathbf{u}}_{\mathbf{k}_1}, \hat{\mathbf{u}}_{\mathbf{k}_2}, \ldots, \hat{\mathbf{u}}_{\mathbf{k}_n})$$

$$\mathbf{k}_j < \mathbf{k},\ j = 1, \ldots, n$$

$$\begin{pmatrix} \hat{u}_{\mathbf{k}} \\ \hat{v}_{\mathbf{k}} \end{pmatrix} = \{(\boldsymbol{\sigma}\cdot\mathbf{k})(\mathbf{k}\cdot\mathbf{k})\}^{-1} \begin{pmatrix} (l^2+m^2)\hat{M}_{\mathbf{k}}^{(x)} - lk\hat{M}_{\mathbf{k}}^{(y)} - km\hat{M}_{\mathbf{k}}^{(z)} \\ -lk\hat{M}_{\mathbf{k}}^{(x)} + (m^2+k^2)\hat{M}_{\mathbf{k}}^{(y)} - lm\hat{M}_{\mathbf{k}}^{(z)} \end{pmatrix}$$

$$\hat{w}_{\mathbf{k}} = \{(\boldsymbol{\sigma}\cdot\mathbf{k})k\}^{-1}\left(k\hat{M}_{\mathbf{k}}^{(z)} - m\hat{M}_{\mathbf{k}}^{(x)}\right) + mk^{-1}\hat{u}_{\mathbf{k}}$$

where $\boldsymbol{\sigma}\cdot\mathbf{k} \neq 0,\ k \neq 0$

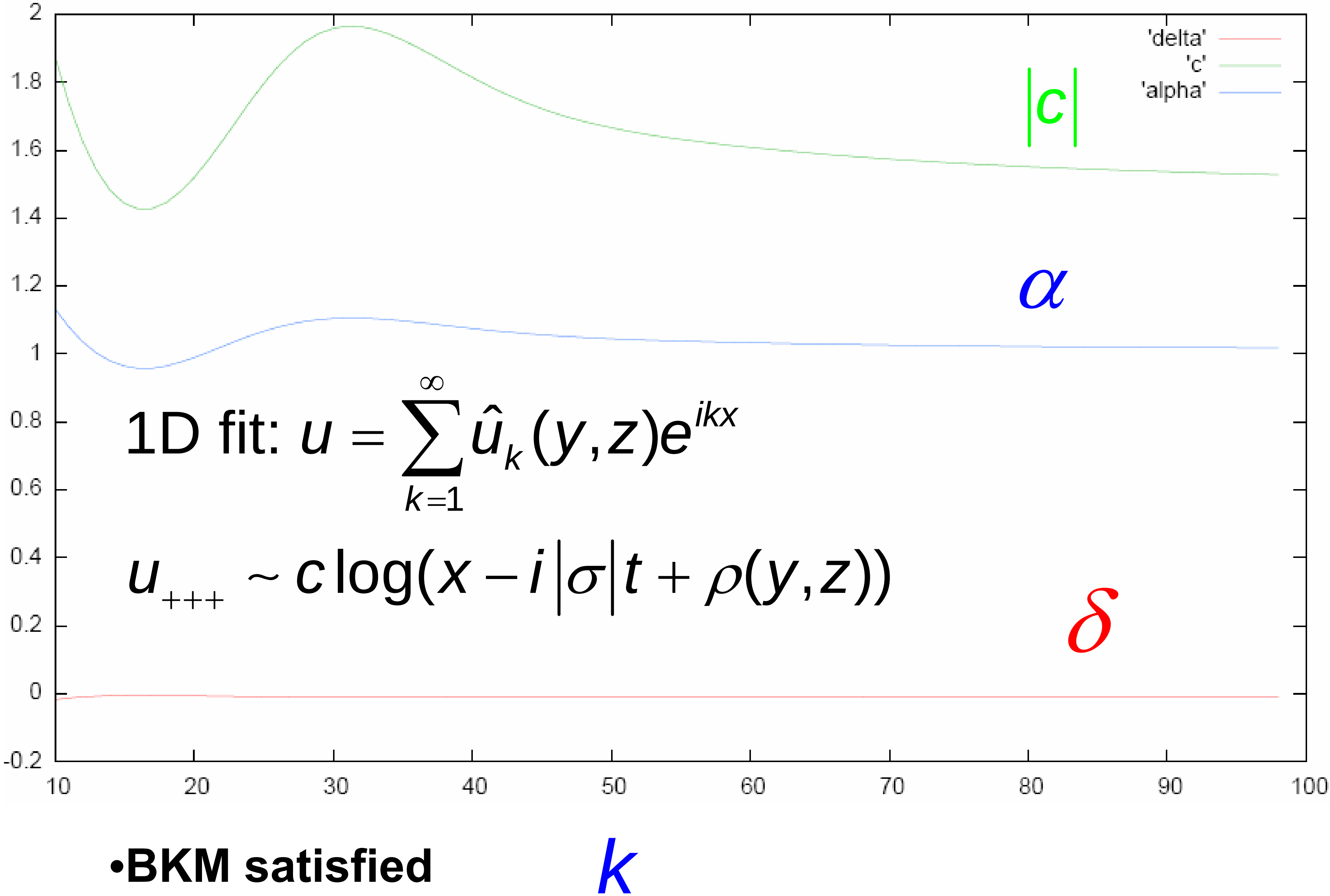
$$\hat{\mathbf{M}}_{\mathbf{k}} = \hat{\mathbf{F}}_{\mathbf{k}} + \hat{\mathbf{N}}_{\mathbf{k}} \longleftarrow \wp_{\mathbf{k}}(-\mathbf{u}\cdot\nabla\mathbf{u})$$

- Small amplitude singularity by choice of forcing
- Introduce ε into forcing; when ε=0, solution **u** is entire.
- For small ε, singularity amplitude is O(ε)

Numerical method

- Nonlinear terms $\hat{N}_k$ evaluated by pseudospectral method
- No truncation error in restriction to finite **k**
- Since N is quadratic, padding with zeroes eliminates aliasing error from pseudospectral part of calculation
- Extreme numerical instability eliminated; very mild instability controlled by spectral filtering
- We compute traveling wave $\mathbf{u}_{+++}$, $\mathbf{u}_{+++} + \mathbf{u}_{---}$ is real

Fit of singularity parameters $\sigma = (1,0,0), \varepsilon = 1$

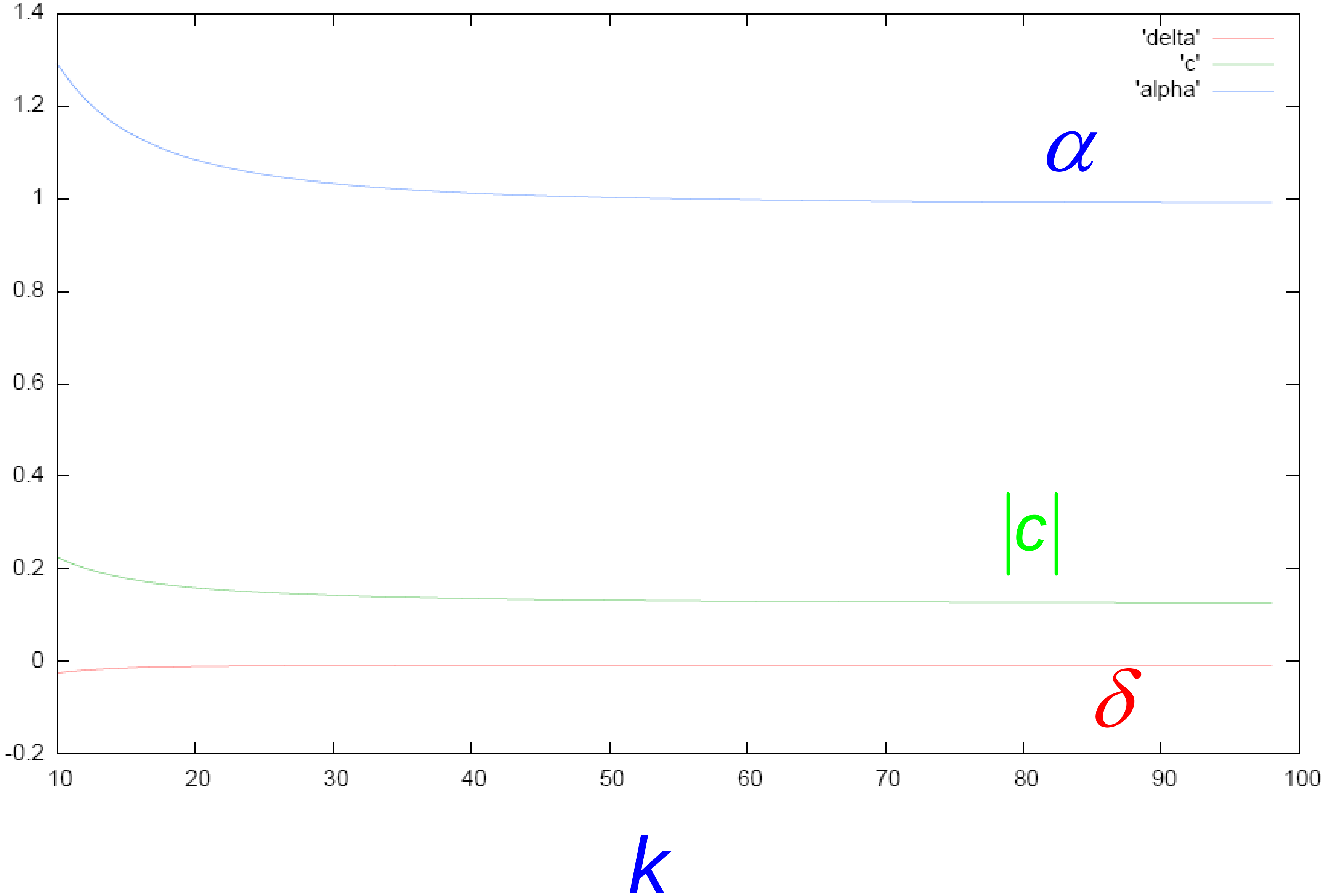


1D fit: $u = \sum_{k=1}^{\infty} \hat{u}_k(y,z)e^{ikx}$

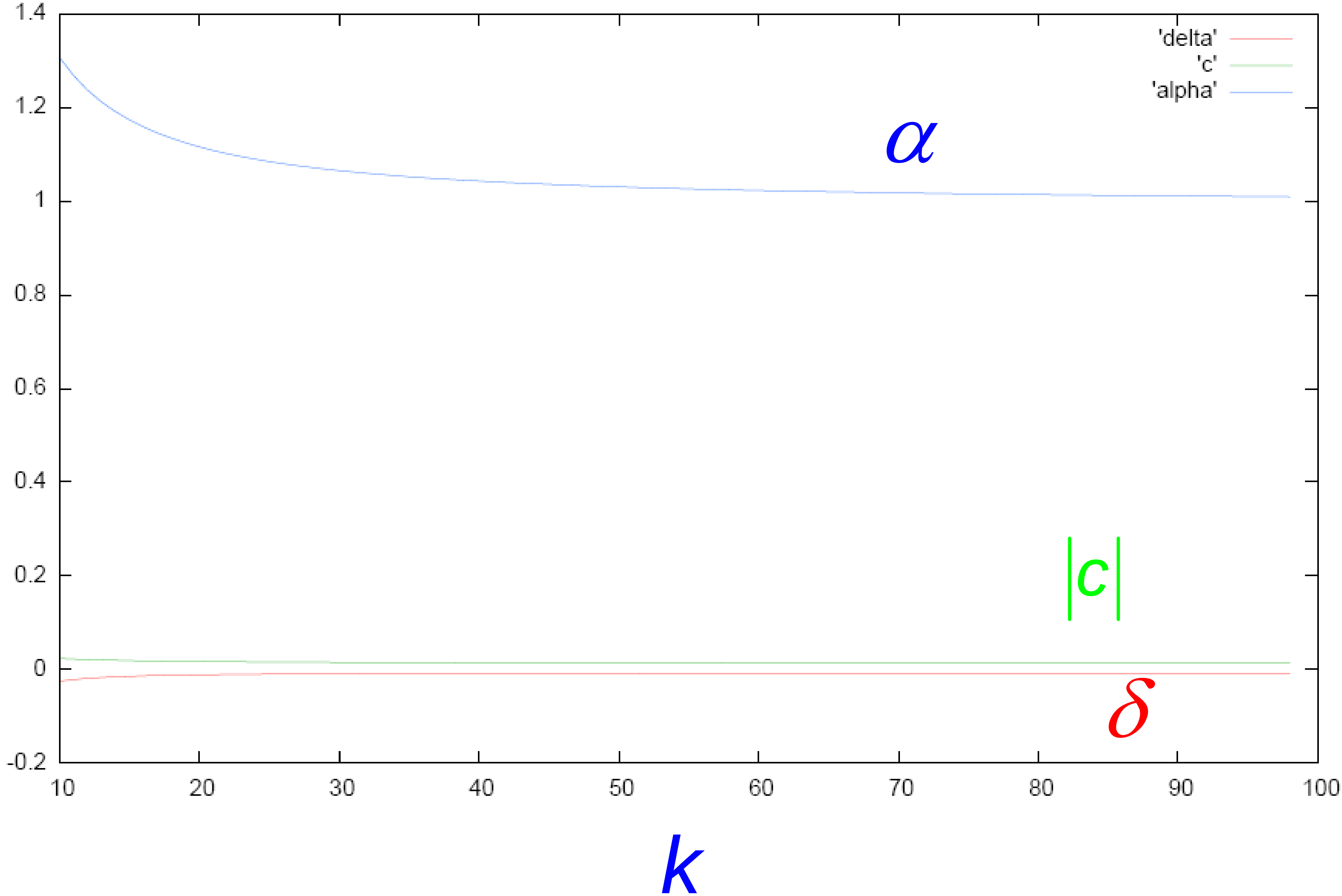
$u_{+++} \sim c\log(x - i|\sigma|t + \rho(y,z))$

- **BKM satisfied**

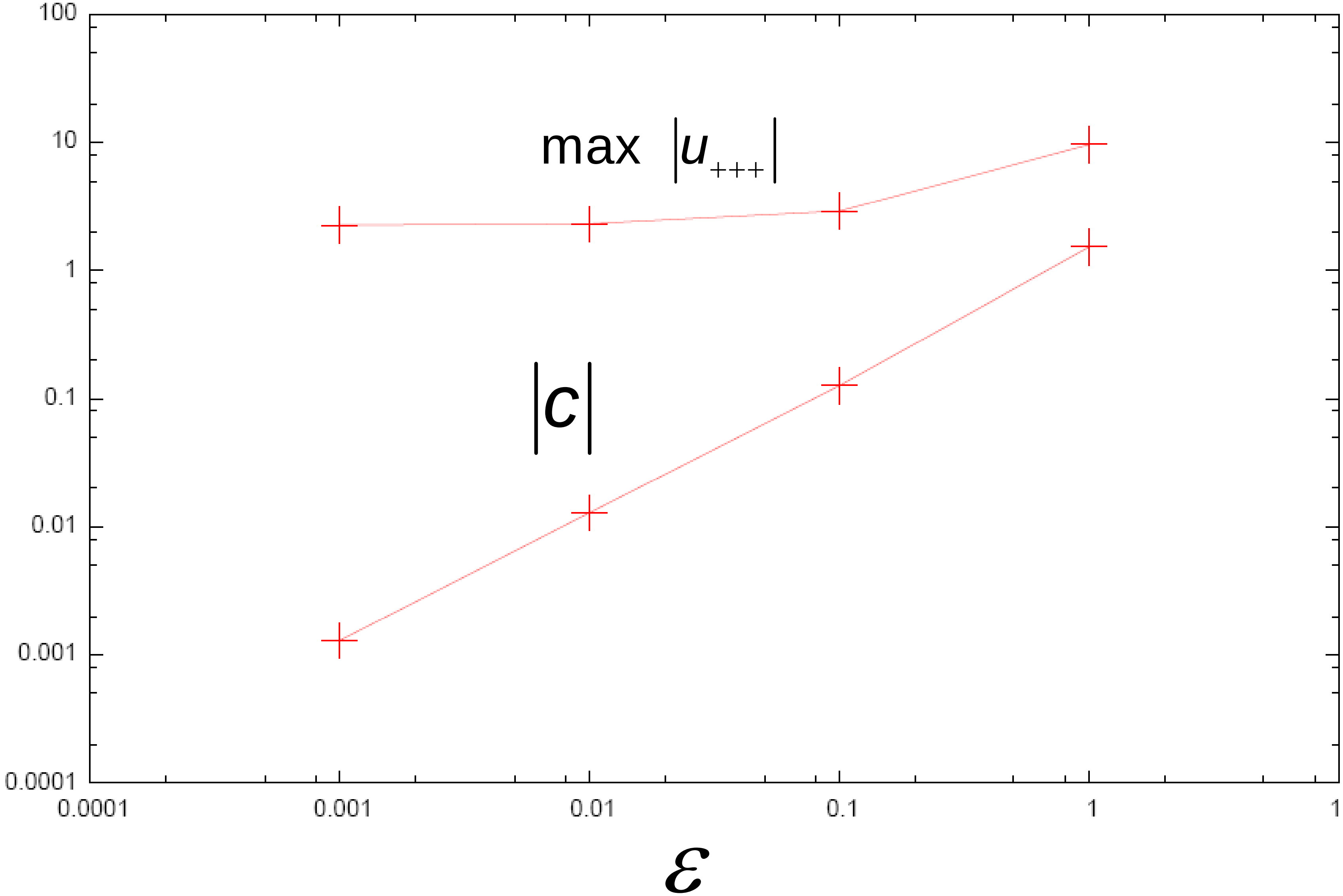
Fit of singularity parameters $\varepsilon = 0.1$



Fit of singularity parameters $\varepsilon = 0.01$

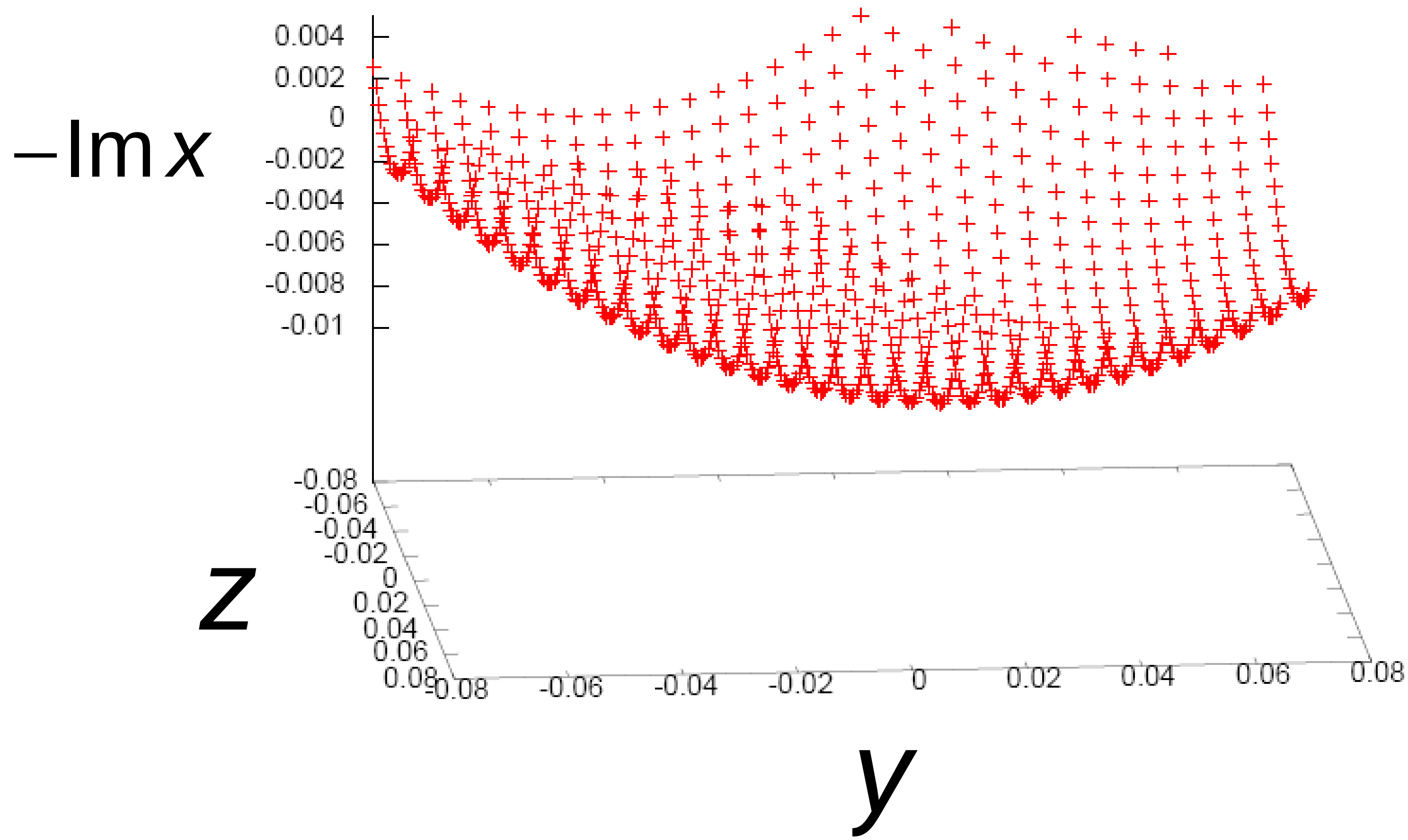


Singularity amplitude

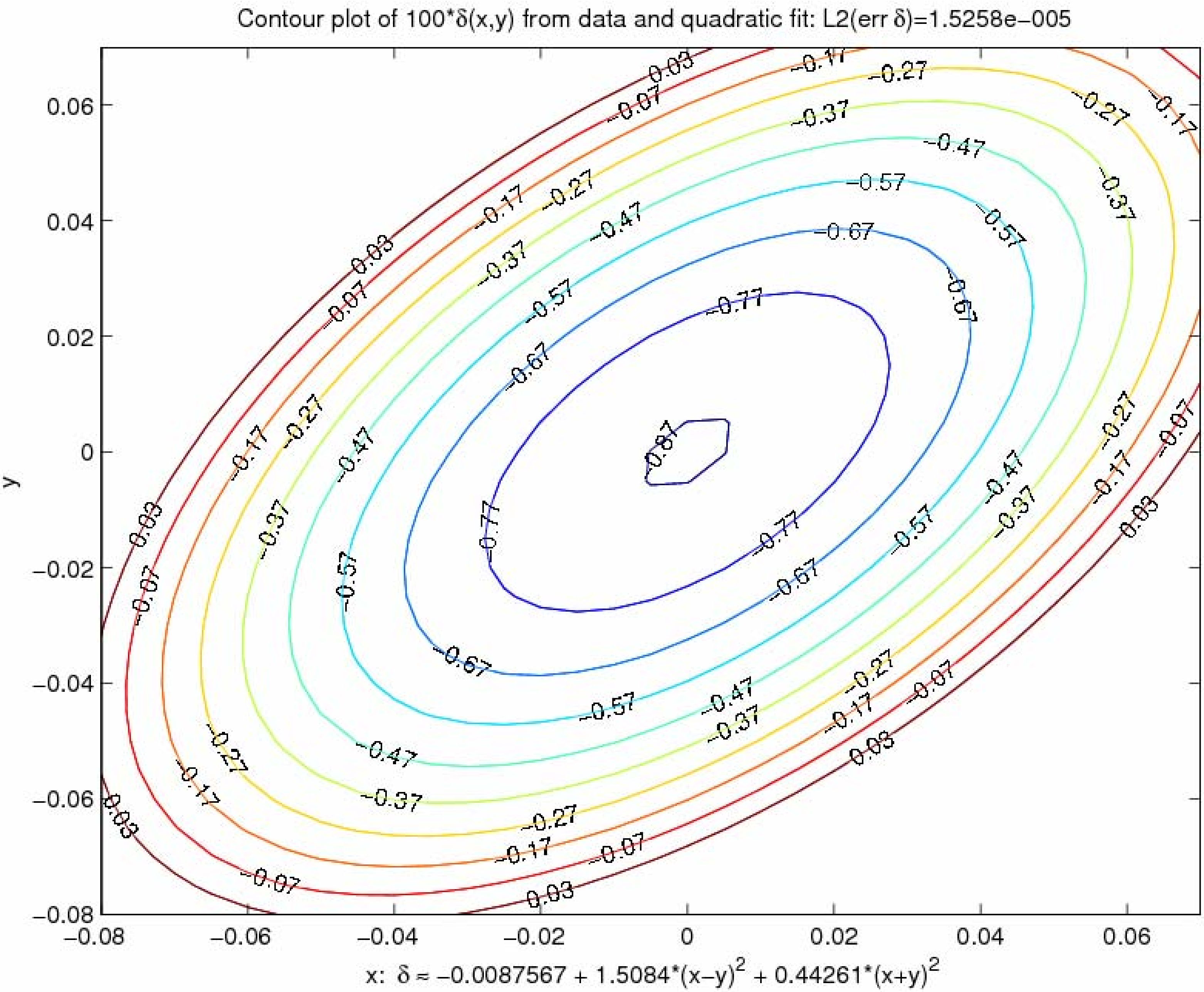


Singular surface

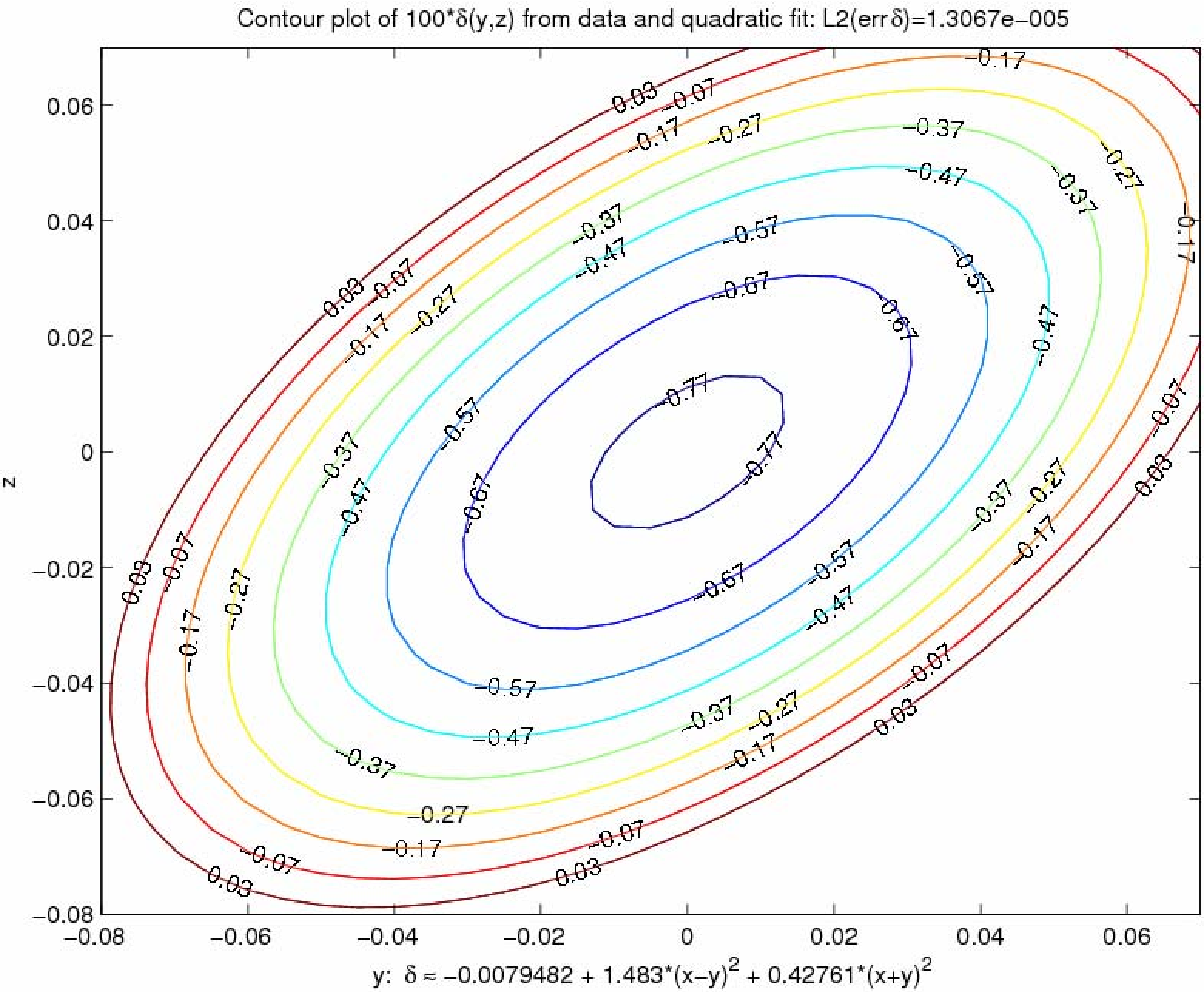
$$-\mathrm{Im}\, x = \rho(y,z) \qquad \boldsymbol{\sigma} = (1,0,0)$$



- Geometry of singular surface is useful for analysis



Contour plot of 100*δ(x,y) from data and quadratic fit: L2(err δ)=1.5258e−005
y
0.06
0.04
0.02
0
−0.02
−0.04
−0.06
−0.08
−0.08
−0.06
−0.04
−0.02
0
0.02
0.04
0.06
x: δ ≈ −0.0087567 + 1.5084*(x−y)^2 + 0.44261*(x+y)^2
−0.03
−0.07
−0.17
−0.27
−0.37
−0.47
−0.57
−0.67
−0.77
−0.87



Contour plot of 100*δ(y,z) from data and quadratic fit: L2(errδ)=1.3067e−005
z
y: δ ≈ −0.0079482 + 1.483*(x−y)^2 + 0.42761*(x+y)^2

Conclusion

- Introduced new method to compute singular solutions to 3D Euler equations with complex velocity

- Eliminated numerical instability observed in earlier calculations; introduced techniques to achieve small amplitude singularity

- Results suggest a traveling wave singularity to 3D complex Euler equations in which the velocity blows up; satisifies Beale, Kato, Majda theorem, smooth singular surface

- Easily generalized to other problems, e.g., 2D and 3D MHD, quasi-geostrophic equation, etc.