Statistical NSE and Stochastic Representation

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Outline:

1. Inviscid limit: time matters
2. Limit for steady solutions
3. Limit for Stationary Statistical Solutions
4. Stochastic Representation and Applications

Inviscid Limit: Time Matters

- $S^E(t)u_0$, solution of incompressible Euler eqns.
- $S_\nu^{NS}(t)u_0$, solution of incompressible NSE

Finite Time Zero Viscosity Limit:

$$\lim_{\nu \to 0} S_\nu^{NS}(t)u_0 = S^E(t)u_0, \text{ for } t \le T.$$

Infinite Time Zero Viscosity Limit:

$$\lim_{\nu \to 0} \lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi(S_\nu^{NS}(t)u_0)dt = \int \Phi(u) d\mu^E(u),$$

Time and zero viscosity limits do not commute.

Finite Time Zero Viscosity Limit

•Smooth regime: Swann (1971) and Kato (1972) : short time. Constantin (1986): As long as solution exists, convergence in $H^{s'}$, with $s' < s$, for $s > d/2 + 1$. Optimal rate.

•Smooth regime: Convergence in H^s: Kato (1975) short time, Masmoudi (2006), as long as solution remains in H^s. Optimal rate.

•Nonsmooth regime = Vortex patches: Constantin and Wu (1996), Abidi and Danchin (2004), Masmoudi (2006). Rates of convergence exist, but deteriorate with loss of smoothness.

Damped Driven NSE

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u + \gamma u + \nabla p = f, \\ \nabla \cdot u = 0 \end{cases}$$

with $\gamma > 0$ a fixed damping coefficient, $\nu > 0$, f time independent with zero mean and $f \in (W^{1,\infty} \cap H^1)(\mathbb{R}^2)$.

Theorem 1 *Let u_0 be smooth, divergence-free, $u_0 \in W^{1,p}(\mathbb{R}^2)^2$, $p \geq 2$. Then the solution with initial datum u_0 exists for all time, is unique, smooth, and obeys the energy equality*

$$\frac{d}{2dt} \int_{\mathbb{R}^2} |u|^2 dx + \gamma \int_{\mathbb{R}^2} |u|^2 dx + \nu \int_{\mathbb{R}^2} |\nabla u|^2 dx = \int_{\mathbb{R}^2} f \cdot u dx.$$

The kinetic energy is bounded uniformly in time, with bounds independent of viscosity:

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le e^{-\gamma t}\left\{\|u(\cdot,0)\|_{L^2(\mathbb{R}^2)} - \frac{1}{\gamma}\|f\|_{L^2(\mathbb{R}^2)}\right\} + \frac{1}{\gamma}\|f\|_{L^2(\mathbb{R}^2)}$$

The vorticity ω *(the curl of the incompressible two dimensional velocity)*

$$\omega = \partial_1 u_2 - \partial_2 u_1 = \nabla^{\perp}\cdot u$$

obeys

$$\partial_t\omega + u\cdot\nabla\omega - \nu\Delta\omega + \gamma\omega = g,$$

with $g \in (L^2 \cap L^{\infty})(\mathbb{R}^2)$ *the vorticity source,* $g = \nabla^{\perp}\cdot f$*. The p-enstrophy is bounded uniformly in time, with bounds independent of viscosity*

$$\|\omega(\cdot,t)\|_{L^p(\mathbb{R}^2)} \le e^{-\gamma t}\left\{\|\omega(\cdot,0)\|_{L^p(\mathbb{R}^2)} - \frac{1}{\gamma}\|g\|_{L^p(\mathbb{R}^2)}\right\} + \frac{1}{\gamma}\|g\|_{L^p(\mathbb{R}^2)}$$

Stationary Solutions

Let $u^{(\nu)}$ be a sequence of solutions of

$$\left\{ \begin{array}{c} -\nu\Delta u + \gamma u + \nabla p + u\cdot\nabla u = f, \\ \nabla\cdot u = 0 \end{array} \right.$$

with vorticities $\omega^{(\nu)}$ obeying

$$\left\{ \begin{array}{c} \gamma\omega + u\cdot\nabla\omega - \nu\Delta\omega = g, \\ \omega = \nabla^{\perp}\cdot u. \end{array} \right.$$

We let $\nu \to 0$ but keep f, g, γ fixed.

Theorem 2 *There exists a subsequence $\omega^{(\nu)}$ that converges weakly*

$$\omega^{(0)} = w - \lim_{\nu \to 0} \omega^{(\nu)}$$

in L^2. The function $\omega^{(0)}$ is a renormalized solution of the inviscid equation

$$\begin{cases} \gamma\omega^{(0)} + u^{(0)} \cdot \nabla\omega^{(0)} = g \\ \omega^{(0)} = \nabla^{\perp} \cdot u^{(0)} \end{cases}$$

In addition, $\omega^{(0)} \in L^2(\mathbb{R}^2)$, $u^{(0)} \in H^1(\mathbb{R}^2)$,the equation holds in $W^{-1,q}_{loc}(\mathbb{R}^2)$ for any $1 < q < 2$, and the limit balance

$$\gamma\|\omega^{(0)}\|^2_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} g\omega^{(0)} dx$$

holds.

Absence of anomalous dissipation

Theorem 3 *Let* $\omega^{(\nu)}$ *be a sequence of solutions. Then the enstrophy dissipation vanishes in the limit* $\nu \longrightarrow 0$*:*

$$\lim_{\nu\longrightarrow 0} \nu \int_{\mathbb{R}^2} |\nabla\omega^{(\nu)}|^2 dx = 0$$

Proof Fatou + limit balance:

$$\begin{aligned}\limsup_{\nu\longrightarrow 0} \nu\|\nabla\omega^{(\nu)}\|^2_{L^2(\mathbb{R}^2)} &\le \limsup_{\nu\longrightarrow 0} \int_{\mathbb{R}^2} g\omega^{(\nu)} dx - \liminf_{\nu\longrightarrow 0} \gamma\|\omega^{(\nu)}\|^2_{L^2(\mathbb{R}^2)} \\ &\le \int_{\mathbb{R}^2} g\omega^{(0)} dx - \gamma\|\omega^{(0)}\|^2_{L^2(\mathbb{R}^2)} = 0.\end{aligned}$$

Statistical Stationary Solutions

The Bogoliubov-Krylov method

$$LIM_{T\to\infty}\frac{1}{T}\int_0^T \Phi(S^{NS,\gamma}(t))dt$$

and statistical stationary solutions in the sense of Foias:

Definition A stationary statistical solution (SSS) of the damped, driven Navier-Stokes equation on the phase space of vorticity is a probability measure μ^ν on $L^2(\mathbb{R}^2)$ such that

1. $\int_{L^2(\mathbb{R}^2)} \|\omega\|^2_{H^1(\mathbb{R}^2)}\, d\mu^\nu(\omega) < \infty$;

2. $\int_{L^2(\mathbb{R}^2)} \langle u \cdot \nabla\omega + \gamma\omega - g, \Psi'(\omega)\rangle + \nu\langle \nabla_x \omega, \nabla_x \Psi'(\omega)\rangle d\mu^\nu(\omega) = 0$
for any test functional $\Psi \in \mathcal{T},$ and

3. $\int_{E_1 \le \|\omega\|_{L^2} \le E_2} \left\{\gamma\,|\omega|^2_{L^2} + \nu\,\|\omega\|^2 - \langle \mathbf{g}, \omega\rangle\right\} d\mu^\nu(\omega) = 0.$

where the class of cylindrical test functions $\mathcal{T}$ is the set of functions $\Psi : L^2(\mathbb{R}^2) \to \mathbb{R}$ of the form

$$\Psi(\omega) = \psi\left(\langle\alpha(\omega), \mathbf{w}_1\rangle, \ldots, \langle\alpha(\omega), \mathbf{w}_m\rangle\right), \tag{1}$$

where ψ is a C^1 scalar valued function defined on $\mathbb{R}^m$, $m \in \mathbb{N}$; $\mathbf{w}_1, \ldots, \mathbf{w}_m$ belong to $H^1_0(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is a bounded domain, and

$$\alpha(\omega) = J_\epsilon \beta(J_\epsilon \omega)$$

where $\beta \in C^2$ is a compactly supported function of one real variable and J_ϵ is the convolution operator

$$J_\epsilon(\omega) = j_\epsilon \star \omega$$

with a standard mollifier.

Theorem 4 *For* $u_0 \in (W^\infty \cap H^1)(\mathbb{R}^2)$*, the Banach limit*

$$LIM_{T\to\infty}\frac{1}{T}\int_0^T \Phi(S^{NS,\gamma}(s)u_0)ds = \int_{L^2}\Phi(\omega)d\mu(\omega)$$

is a SSS of the damped, driven NSE. Such limits are supported in

$$\left\{\omega : \left| \; \|\omega\|_{L^p} \le \frac{\|g\|_{L^p}}{\gamma} \right.\right\}$$

for $2 \le p \le \infty$.

Definition A probability measure μ^E on $L^2(\mathbb{R}^2)$ is a renormalized stationary statistical solution of the damped, driven Euler equation if it satisfies

$$\int_{L^2(\mathbb{R}^2)} \langle \gamma\omega - g, \Psi'(\omega)\rangle - \langle u\omega, \nabla_x \Psi'(\omega)\rangle d\mu^E(\omega) = 0;$$

for any test fuctional $\Psi \in \mathcal{T}_0$, where $\Psi \in \mathcal{T}_0$ is a subclass of $\Psi \in \mathcal{T}$, where the functions w_j satisfy $w_j \in C_0^1(\Omega)$, where Ω is bounded in $\mathbb{R}^2$. Furthermore, we say that a renormalized stationary statistical solution of the Euler Equation μ^E satisfies the enstrophy balance if

$$\int_{L^2(\mathbb{R}^2)} \left\{\gamma \left|\omega\right|_{L^2}^2 - \langle \mathbf{g}, \omega\rangle\right\} d\mu^E(\omega) = 0.$$

Theorem 5 *Any sequence of SSS of the damped, driven NSE equation has a weakly convergent subsequence. The limit μ^E is a RSSS of the damped, driven Euler equation. If the supports of the SSS are uniformly bounded in L^∞ then μ^E satisfies the enstrophy balance.*

Idea for the proof of enstrophy balance

From cylindrical test functions one reaches

$$\int_{L^2(\mathbb{R}^2)} \langle \beta(\omega_\epsilon))_\epsilon, (\beta'(\omega_\epsilon)(\gamma\omega - g)_\epsilon)_\epsilon \rangle d\mu^E(\omega)$$
$$+ \int_{L^2(\mathbb{R}^2)} \langle \beta(\omega_\epsilon))_\epsilon, (\beta'(\omega_\epsilon)\partial_k(u_k\omega)_\epsilon)_\epsilon \rangle d\mu^E(\omega) = 0$$

where $h_\epsilon = J_\epsilon h$. The first integral converges to the required enstrophy balance. The second integral converges to zero. In order to see that, we write

$$I_{\beta,\epsilon} = \int_{\mathbb{R}^2} (\beta(\omega_\epsilon))_\epsilon \left[\beta'(\omega_\epsilon)\partial_k(u_k\omega)_\epsilon\right]_\epsilon dx$$

Integrating by parts we write

$$I_{\beta,\epsilon} = J_{\beta,\epsilon} + K_{\beta,\epsilon}$$

with

$$J_{\beta,\epsilon} = -\int \partial_k(\beta(\omega_\epsilon))_\epsilon \left[\beta'(\omega_\epsilon)(u_k\omega)_\epsilon\right]_\epsilon dx$$

and

$$K_{\beta,\epsilon} = -\int (\beta(\omega_\epsilon)_\epsilon) \left[\beta''(\omega_\epsilon)(\partial_k\omega_\epsilon)(u_k\omega)_\epsilon\right]_\epsilon dx$$

We split $J_{\beta,\epsilon}$ further

$$J_{\beta,\epsilon} = L_{\beta,\epsilon} + M_{\beta,\epsilon}$$

with

$$L_{\beta,\epsilon} = -\int \partial_k(\beta(\omega_\epsilon))_\epsilon \left[\beta'(\omega_\epsilon)(u_k)_\epsilon(\omega)_\epsilon\right]_\epsilon dx$$

and

$$M_{\beta,\epsilon} = -\int \partial_k(\beta(\omega_\epsilon))_\epsilon \left[\beta'(\omega_\epsilon)\rho_\epsilon(u_k,\omega)\right]_\epsilon dx$$

We estimate

$$|M_{\beta,\epsilon}| \le C \sup|\beta| \sup|\beta'| \frac{1}{\epsilon} \|\rho_\epsilon(u,\omega)\|_{L^1(\mathbb{R}^2)}$$

We used the fact that

$$\|\partial_k(\beta)_\epsilon\|_{L^\infty} \le C\frac{1}{\epsilon}\|\beta\|_{L^\infty}$$

From the properties of ρ it follows that

$$|M_{\beta,\epsilon}| \le C \sup|\beta| \sup|\beta'| \|\omega\|_{L^2} \int j(z)|z| \|\delta_{\epsilon z}\omega\|_{L^2} dz$$

where $(\delta_h\omega)(x) = \omega(x-h) - \omega(x)$. We fix $\epsilon > 0$ and we consider a sequence of compactly supported functions $\beta(y)$ that converge uniformly on the compact $R_\infty = [-2\frac{\|g\|_{L^\infty}}{\gamma}, 2\frac{\|g\|_{L^\infty}}{\gamma}]$ together with two derivatives to the function y , (i.e $\beta \to y$, $\beta' \to 1$, $\beta'' \to 0$) and such that

$$|\beta(y)| + |\beta'(y)| + |\beta''(y)|) \le C.$$

It is easy to see that for fixed $\epsilon > 0$

$$\lim_{\beta \to y} \int (L_{\beta,\epsilon} + K_{\beta,\epsilon}) d\mu^E = 0$$

On the other hand, from

$$\int |K_{\beta,\epsilon}| d\mu^E \leq C \int \int j(z)|z| \|\delta_{\epsilon z}\omega\|_{L^2} dz d\mu^E$$

it follows that

$$\lim_{\epsilon \to 0} \limsup_{\beta \to y} \int |K_{\beta,\epsilon}| d\mu^E = 0$$

We used that the support of μ^E is bounded in L^∞.

Absence of anomalous dissipation

Theorem 6 *Let* $f \in (H^1 \cap W^{1,\infty})(\mathbb{R}^2)$*. Let* $u_0 \in H^1 \cap W^{1,\infty}$ *be divergence free. Let* ω^ν *be the curl of the solution of the damped, driven NSE. Then*

$$\lim_{\nu\to 0} \lim\sup_{T\to\infty} \frac{\nu}{T}\int_0^T \int \left|\nabla \omega^{(\nu)}(x,t)\right|^2 dxdt = 0$$

holds.

Idea of proof

We argue by contradiction. We find $\delta > 0$, a sequence of viscosities, $\nu_j \to 0$ and for each, a sequence of times $T_k^{(j)} \to \infty$ so that at each fixed ν_j, the time averges of the dissipation integrals are bounded below by $\delta > 0$. We use the NSE equation and the balance

$$\frac{\nu_j}{T_k^{(j)}}\int_0^{T_k^{(j)}} \|\nabla\omega^{\nu_j}\|_{L^2}^2 dt + \frac{1}{T_k^{(j)}}\int_0^{T_k^{(j)}} \left\{\gamma\|\omega^{\nu_j}\|_{L^2}^2 - \langle g, \omega^{\nu_j}\rangle\right\} dt = O(\frac{1}{T_k^{(j)}})$$

Passing to a subsequence of times, still at fixed ν_j we obtain a SSS of NSE, μ^{ν_j}, with

$$\int \left\{\gamma|\omega|_{L^2}^2 - \langle g, \omega\rangle\right\} d\mu^{\nu_j} \leq -\delta$$

The support of the sequence is bounded apriori in $L^\infty \cap L^2$, uniformly in j. Passing to a subsequence we have a RSSS of the damped, driven Euler equations μ^E that satisfies

$$\int \left\{ \gamma |\omega|^2_{L^2} - \langle g, \omega \rangle \right\} d\mu^E(\omega) \leq -\delta$$

and that is absurd in view of the enstrophy balance of the limit.

Remarks

- result conjectured by D. Bernard in 2000.

- Finite time zero viscosity limit without damping: Eyink, and Lopes-Filho, Mazzucato, Nussenzveig Lopes.

- No vanishing rates known.

- No infinite time result w/o damping known.

Stochastic Lagrangian Representation: Navier-Stokes

Theorem 7 *Let W be an n-dimensional Wiener process. Let $k \geq 1$ and assume $u_0 \in C^{k+1,\alpha}$ is a deterministic divergence-free vector field. Let (u, X) solve the stochastic system*

$$\begin{cases} dX = u dt + \sqrt{2\nu} dW, \\ A = X^{-1}, \\ u = \mathbb{E}\mathbb{P}\left\{\left(\nabla^T A\right)(u_0 \circ A)\right\} \end{cases}$$

Then u solves the deterministic incompressible NSE:

$$\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = 0,$$

$$\nabla \cdot u = 0$$

•When $\nu = 0$, all is deterministic, and we recover the Eulerian-Lagrangian deterministic representation based on the Weber formula.

Remarks

•$A = X^{-1}$ is the spatial inverse ("back-to-labels"). It exists, and it is as smooth as X. Both are stochastic.

•**Forced NSE**

$$\begin{cases} dX = udt + \sqrt{2\nu}dW, \\ A = X^{-1} \\ u = \mathbb{EP}\left\{(\nabla^T A)\left[u_0 + \int_0^t (\nabla^t X) f(X_s, s) ds\right] \circ A(t)\right\} \end{cases}$$

represents

$$\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = f, \quad \nabla \cdot u = 0.$$

•Representations for Lans-alpha, Burgers. No direct representation for Leray regularization.

Local Existence for the Stochastic System, Remarkable Formulae

Theorem 8 *Let $u_0 \in C^{k+1,\alpha}$ be divergence-free. There exists a $T > 0$ depending on the norm of u_0, but independent of viscosity, so that a solution (u, X) of the stochastic system exists on $[0, T]$. Moreover, $\|u\|_{C^{k+1,\alpha}} \leq U$ for $t \in [0, T]$ with U dependent on the norm of the initial data and T.*

Theorem 9 *Let $\omega = \nabla \times u$, $\omega_0 = \nabla \times u_0$. Then*

$$\omega = \mathbb{E}\left\{((\nabla X)\omega_0) \circ A\right\}.$$

In two dimensions,

$$\omega = \mathbb{E}\left[\omega_0 \circ A\right].$$

For forced systems in $n = 2, 3$, replace in the formulae above ω_0 by

$$\xi_t = \omega_0 + \int_0^t (\nabla X_s)^{-1} g(X_s, s) ds$$

with $g = \nabla \times f$.

•Circulation is conserved.

Let

$$\widetilde{u} = \mathbb{P}\left\{(\nabla^t A)(u_0 \circ A)\right\}$$

This is a stochastic incompressible velocity, with initial data u_0 and

$$u = \mathbb{E}\widetilde{u}$$

$$\oint_{X(\gamma)} \widetilde{u} \cdot dr = \oint_{\gamma} u_0 \cdot dr.$$

Stochastic Lagrangian Transport

- The "back-to-labels" process obeys

$$dA_t + [u \cdot \nabla A - \nu \Delta A]\, dt + \sqrt{2\nu} \nabla A dW = 0$$

For any smooth function $\phi(a, t)$, $v(x, t) = \phi(A(x, t), t)$ obeys

$$dv_t + [u \cdot \nabla v - \nu \Delta v]\, dt + \sqrt{2\nu} \nabla v dW = \partial_t \phi \circ A$$

- Cancellation, chain rule as if it were a first order PDE, due to the joint quadratic variation.

- Valid if u is smooth, not necessarily divergence-free.

Stochastically Passive Scalars

$$d\theta_t + [u \cdot \nabla\theta - \nu\Delta\theta]\, dt + \sqrt{2\nu}\nabla\theta dW = 0$$

- $\theta_1,\ \theta_2,$ sps $\Rightarrow \theta_1\theta_2$sps

- with viscosity, inviscid invariants become stochastically passive

Stochastic Particles

Let

$$m = M(a, \alpha, t)$$

solve

$$dM = (u(X,t) + G(X, M, t))dt + \sqrt{2\kappa} dW$$

with

$$M(a, \alpha, 0) = \alpha.$$

Let

$$(A(x,t), R(x, m, t)) = (X(a,t), M(a, \alpha, t))^{-1}$$

It exists and a.s. for all t

$$A(X(a,t), t) = a, \quad R(X(a,t), M(a, \alpha, t)) = \alpha.$$

Then

$$f(x,m,t) = f_0(A(x,t), R(x,m,t))\,\mathrm{det}\left(\nabla_m R\right)(x,m,t)$$

solves

$$df + (u\cdot\nabla_x f + \mathrm{div}_g(Gf) - \kappa\Delta_g f - \nu\Delta_x f)dt = \\ -\sqrt{2\kappa}\nabla_g f\cdot dW - \sqrt{2\nu}\nabla_x f\cdot dW = 0$$

and so

$$\overline{f} = \mathbb{E}f$$

solves

$$\partial_t \overline{f} + u \cdot \nabla_x \overline{f} + \mathrm{div}_g (G\overline{f}) = \kappa \Delta_g \overline{f} + \nu \Delta_x \overline{f}.$$

- $\nu \geq 0$, $\kappa \geq 0$.

- Cartesian noise.

Idea of proof

If

$$dX = U(X,t)dt + \sqrt{2\nu} M dW$$

with M a constant matrix and if we set
$A = X^{-1}$ and $P(D) = \mathrm{Tr}(MM^T(\nabla \otimes \nabla))$, then

$$f(x,t) = f_0(A(x,t)) exp\{\int_0^t V(X(a,s),s)ds_{|a=A(x,t)}\}$$

solves

$$df + (u \cdot \nabla_x f - \nu P(D) f - V(x,t) f)dt + \sqrt{2\nu} \nabla_x f M dW = 0$$

Application: generalized relative entropies

Linear Fokker-Planck with potential

$$\mathcal{D}\rho = \Delta_x \rho - \mathrm{div}_x(U\rho) + V\rho$$

$$\partial_t f = \mathcal{D} f, \quad \partial_t \rho = \mathcal{D}\rho, \quad \rho > 0$$

$$\partial_t \phi + \mathcal{D}^* \phi = 0, \quad \phi \geq 0.$$

Michel, Mischer, Perthame: if H is convex, then

$$\frac{d}{dt} \int H\left(\frac{f}{\rho}\right) \phi \rho dx \leq 0.$$

Stochastic understanding and proof

$$dX = U dt + \sqrt{2} dW$$

$$\psi_{f_0}(x,t) = f_0(A(x,t)) \exp\left\{\int_0^t V(X(a,s),s)ds_{|a=A(x,t)}\right\} \det(\nabla_x A)(x,t)$$

Then $\psi = \psi_{f_0}$ solves

$$d\psi + (\nabla_x \cdot (U\psi) - \Delta_x \psi - V\psi)\, dt + \sqrt{2} \nabla_x \psi \cdot dW = 0$$

with initial datum $\psi(x,0) = f_0(x)$. Therefore $\mathbb{E}(\psi_{f_0})$ solves the Fokker-Planck eqn.

Deterministic

$$\partial_t \phi + \mathcal{D}^* \phi = 0, \quad \phi \geq 0 \Rightarrow$$

$$M(a,t) = \phi(X(a,t),t)\,\mathrm{exp}\left\{\int_0^t V(X(a,s),s)ds\right\}$$

is a martingale.

$$\psi_{\rho_0}(x,t)\phi(x,t)H\left(\frac{\psi_{f_0}(x,t)}{\psi_{\rho_0}(x,t)}\right) =$$
$$\rho_0(A(x,t))H\left(\frac{f_0(A(x,t))}{\rho_0(A(x,t))}\right)M(A(x,t),t)\mathrm{det}(\nabla_x A)$$

Consequently we have almost surely

$$\int \psi_{\rho_0}(x,t)H\left(\frac{\psi_{f_0}(x,t)}{\psi_{\rho_0}(x,t)}\right)\phi(x,t)dx = \int \rho_0(a)H\left(\frac{f_0(a)}{\rho_0(a)}\right)M(a,t)da.$$

(stochastically passive scalar)(martingale $\circ A$)(det$(\nabla_x A)$), integrated dx

$$\frac{d}{dt}\mathbb{E}\left\{\int \psi_{\rho_0} H\left(\frac{\psi_{f_0}}{\psi_{\rho_0}}\right)\phi dx\right\} = 0.$$

MMP follows from

$$\mathbb{E}\left(\psi_{\rho_0}\right) H\left(\frac{\mathbb{E}(\psi_{f_0})}{\mathbb{E}(\psi_{\rho_0})}\right) \le \mathbb{E}\left\{\psi_{\rho_0} H\left(\frac{\psi_{f_0}}{\psi_{\rho_0}}\right)\right\}$$

which follows from Jensen for

$$Pf = \mathbb{E}\left(\frac{\psi_{\rho_0}}{\mathbb{E}\psi_{\rho_0}} f\right).$$

Variable diffusivity

$$\mathcal{D}\rho = \nu \partial_i (a_{ij} \partial_j \rho) - \mathrm{div}_x (U\rho) + V\rho$$

$$a_{ij}(x,t) = \sigma_{ip}(x,t)\sigma_{jp}(x,t)$$

$$A(D)\rho = a_{ij}\partial_i \partial_j \rho$$

$$u_j(x,t) = U_j(x,t) - \nu \partial_i (a_{ij}(x,t))$$

$$P = V - \mathrm{div}_x(U)$$

$$\mathcal{D}\rho = \nu A(D)\rho - u \cdot \nabla_x \rho + P\rho.$$

Stochastic Lagrangian Flow

In order to represent solutions of equations with variable diffusivity we need to modify the drift:

$$v_j(x,t) = u_j + 2\nu(\partial_k \sigma_{jp})\sigma_{kp} = U_j - \nu(\partial_k \sigma_{kp})\sigma_{jp} + \nu(\partial_k \sigma_{jp})\sigma_{kp}$$

Let $X(a,t)$ be the strong solution of the stochastic differential system

$$dX_j(t) = v_j(X,t)dt + \sqrt{2\nu}\sigma_{jp}(X,t)dW_p$$

with initial data $X(a,0) = a$. The map X is smooth and the determinant

$$D(a,t) = det(\partial_a X(a,t))$$

obeys the SDE

$$d(D(a,t)) = [D(a,t)]\times$$
$$\left\{[(\mathrm{div}_x v)(x,t) + 2\nu E(x,t)]_{|x=X(a,t)}\, dt + \sqrt{2\nu}(\partial_k(\sigma_{kp}))(x,t)_{|X(a,t)} dW_p\right\}$$

with

$$E(x,t) = \sum_{i<j} \sum_p det(\partial_i \sigma_{jp})_{ij}.$$

The map $A(x,t)$ satisfies the stochastic partial differential system

$$dA_j + \left(u \cdot \nabla_x A_j - \nu A(D) A_j\right) dt + \sqrt{2\nu}(\partial_k A_j)\sigma_{kp} dW_p = 0$$

The process $\psi = \psi_{f_0}$ given by

$$\psi(x,t) = f_0(A(x,t)) \exp\left\{ \int_0^t P(X(a,s),s) ds_{|a=A(x,t)} \right\}$$

solves

$$d\psi - (\mathcal{D}\psi)\, dt + \sqrt{2\nu}\nabla_x \psi \sigma dW = 0$$

with initial datum $\psi(x,0) = f_0(x)$.

If ϕ solves $\partial_t \phi + \mathcal{D}^* \phi = 0$ then

$$M(a,t) = \phi(X(a,t),t) \, \mathrm{det} \, (\partial_a X(a,t)) \, \mathrm{exp} \left\{ \int_0^t P(X(a,s),s) ds \right\}$$

is a martingale.

Theorem 10 *Let h_0 and ρ_0 be smooth time independent deterministic func tions. Consider the stochastically passive scalar $\theta_{h_0}(x,t) = h_0(A(x,t))$ and the process ψ_{ρ_0} with initial datum ρ_0. Consider also $\phi(x,t)$, a deterministic solution of $\partial_t \phi + \mathcal{D}^* \phi = 0$. Then the random variable*

$$\mathcal{E}(t) = \int_{\mathbb{R}^n} \phi(x,t)\psi_{\rho_0}(x,t)\theta_{h_0}(x,t)dx$$

is a martingale. Consequently, if H is convex, $\rho = \mathbb{E}\psi_{\rho_0}$, $f = \mathbb{E}\psi_{f_0}$, then

$$\int_{\mathbb{R}^n} \phi\rho H\left(\frac{f}{\rho}\right) dx \le \int_{\mathbb{R}^n} \phi(0)\rho H\left(\frac{f_0}{\rho_0}\right) dx$$

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Stochastic Representation: Language

$$f \in BV[0,T] \Leftrightarrow \sup_{\mathcal{P}} \sum_k |f(t_{k+1} - f(t_k))| < \infty$$

$$f \in BV \Rightarrow df \quad \text{is a Radon measure}$$

$$\int_0^T \phi df \quad \text{usual Stieltjes integral}$$

- Brownian motion is not BV.

$$\sum_k |W(t_{k+1}) - W(t_k)| = \infty \quad \text{a.s.}$$

Stochastic (Itô) Integral

$$\int_0^t \phi dW = \lim_{N\to\infty} \sum_{k=0}^{N-1} \phi(t_k)(W(t_{k+1}) - W(t_k))$$

$$\mathbb{E} \sum_k (W(t_{k+1}) - W(t_k))^2 = T$$

X_t continuous $(\mathcal{F}_t)$ adapted. Martingale:

$$E(X_t|\mathcal{F}_s) = X_s, \;\; a.s \;\; t > s.$$

Example:

$$X_t = x + \int_0^t \phi dW$$

Semimartingale:

$S_t = M_t + B_t$, (Martingale + BV).

$$\int_0^t \phi dS = \lim_{N\to\infty} \sum_{k=0}^{N-1} \phi(t_k)(M(t_{k+1}) - M(t_k)) + \int_0^t \phi dB$$

Example: SDE

$$dX = u(X)dt + \sigma(X)dW$$

is the semimartingale eqn

$$X_t = x_0 + \int_0^t \sigma(X_s)dW + \int_0^t u(X_s)ds$$

Quadratic Variation

$$\langle X \rangle_t = \lim_{\|\Delta\| \to 0} \sum_{k=0}^{N-1} \left(X_{t \wedge t_{k+1}} - X_{t \wedge t_k} \right)^2$$

Examples:

- Continuous BV process: $\langle X \rangle_t = 0$.
- Brownian motion: $\langle W \rangle_t = t$.
- If X_t is a continuous bounded (r. local) martingale then $\langle X \rangle_t$ exists, and

$$X_t^2 - \langle X \rangle_t$$

is a bounded (r. local) martingale.

Joint Quadratic Variation

$$\langle X, Y\rangle_t = \lim_{\|\Delta\|\to 0} \sum_{k=0}^{N-1} (X_{t\wedge t_{k+1}} - X_{t\wedge t_k})(Y_{t\wedge t_{k+1}} - Y_{t\wedge t_k})$$

•If M is a continuous local martingale and $f \in L^2(\langle M\rangle)$ then the Itô integral

$$\int_0^t f dM$$

is a continuous local martingale and

$$\langle \int f dM,\ N\rangle_t = \int_0^t f_s d\langle M, N\rangle_s$$

$\forall\ N$ continuous local martingale.

Itô Formula

•If $F(x_1, \ldots, x_n, t)$ is deterministic and smooth and $X = (X_1, \ldots X_n)$ is a continuous (vector valued) semimartingale then

$$F(X_t, t) - F(X_0, 0) = \int_0^t \partial_s F(X_s, s) ds + \int_0^t \nabla F(X_s, s) \cdot dX_s + \frac{1}{2} \int_0^t \frac{\partial^2 F(X_s, s)}{\partial x_i \partial x_j} d\langle X_i, X_j \rangle_s$$

In differential form:

$$d(F(X,t)) = (\partial_i F) dX_i + \left(\frac{1}{2} \frac{\partial^2 F}{\partial x_i \partial x_j} d\langle X_i, X_j \rangle + \partial_t F dt \right)$$

Generalized Itô -Wentzell- Bismut- Kunita Formula

•If $F(x,t)$ is a continuous C^2 process and a C^1 semimartingale, and if g_t is a continuous vector-valued predictable process, then the composition $F(g_t,t)$ is a continuous predictable process and

$$F(g_t,t) - F(g_0,0) = \int_0^t \partial_i F(g_s,s) dg_s^i + \frac{1}{2}\int_0^t \frac{\partial^2 F(g_s,s)}{\partial x_i \partial x_j} d\langle g_s^i, g_s^j\rangle \\ + \int_0^t F(g_s, ds) + \langle \int_0^t \partial_i F(g_s, ds), g_t^i\rangle$$

In differential form:

$$d(F(g_t,t)) = \partial_i F(g_t,t) dg_t^i + \frac{1}{2}\frac{\partial^2 F(g_t,t)}{\partial x_i \partial x_j} d\langle g_t^i, g_t^j\rangle \\ +(dF)(g_t,t) + d\langle \int_0^t \partial_i F(g_s, ds), g_t^i\rangle.$$

Here

$$\int F(g_s, ds) = \lim_{N\to\infty} \sum_{k=0}^{N-1} \Big(F(g_{t_k}, t_{k+1}) - F(g_{t_k}, t_k) \Big)$$

and

$$\int \partial_i F(g_s, ds) = \lim_{N\to\infty} \sum_{k=0}^{N-1} \Big(\partial_i F(g_{t_k}, t_{k+1}) - \partial_i F(g_{t_k}, t_k) \Big)$$

are usual Itô integrals.

Proof explains it:

$$F(g_t, t) - F(g_0, 0) = \sum_{k=0}^{N-1} \Big\{ F(g_{t_k}, t_{k+1}) - F(g_{t_k}, t_k) \Big\} + \\ + \sum_{k=0}^{N-1} \Big\{ F(g_{t_{k+1}}, t_{k+1}) - F(g_{t_k}, t_{k+1}) \Big\}$$

and

$$\sum_{k=0}^{N-1}\left\{F(g_{t_{k+1}},t_{k+1})-F(g_{t_k},t_{k+1})\right\}=I+J+K$$

with

$$I=\sum_{k=0}^{N-1}\left\{\partial_i F(g_{t_k},t_{k+1})-\partial_i F(g_{t_k},t_k)\right\}(g^i_{t_{k+1}}-g^i_{t_k}),$$

$$J=\sum_{k=0}^{N-1}\partial_i F(g_{t_k},t_k)(g^i_{t_{k+1}}-g^i_{t_k})$$

$$K=\frac{1}{2}\sum_{k=0}^{N-1}\frac{\partial^2 F(\xi_k,t_{k+1})}{\partial x_i\partial x_j}(g^i_{t_{k+1}}-g^i_{t_k})(g^j_{t_{k+1}}-g^j_{t_k})$$

$$\lim_{\|\Delta\|\to 0} I = \langle \int \partial_i F(g, ds), g^i \rangle, \quad \lim_{\|\Delta\|\to 0} J = \int \partial_i F(g, s) dg^i$$

$$\lim_{\|\Delta\|\to 0} K = \frac{1}{2} \int \frac{\partial^2 F(g, s)}{\partial x_i \partial x_j} d\langle g^i, g^j \rangle.$$