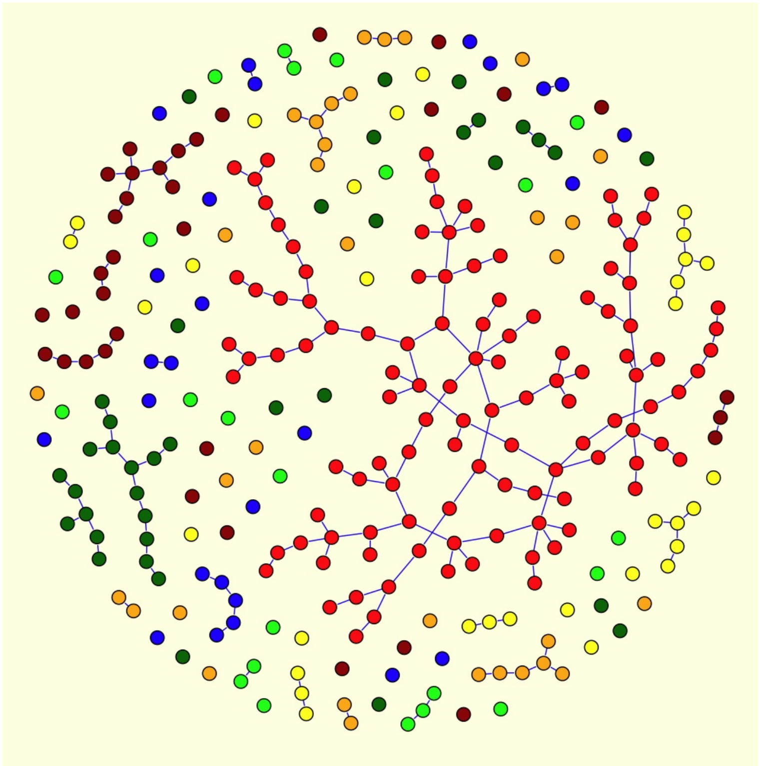
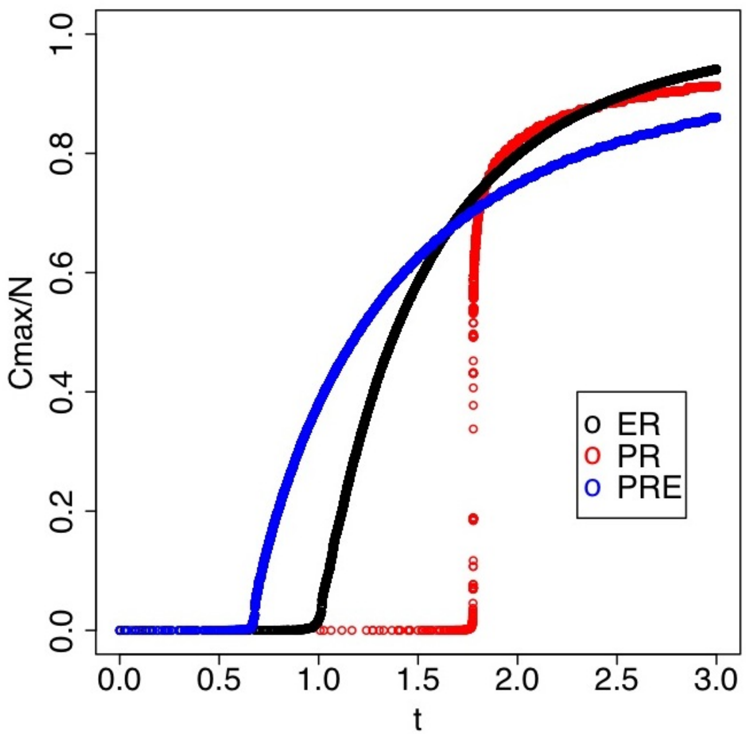
Explosive percolation in random graphs





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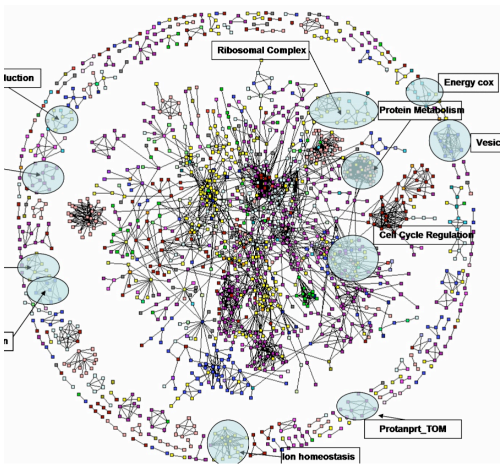
Networks are ubiquitous:

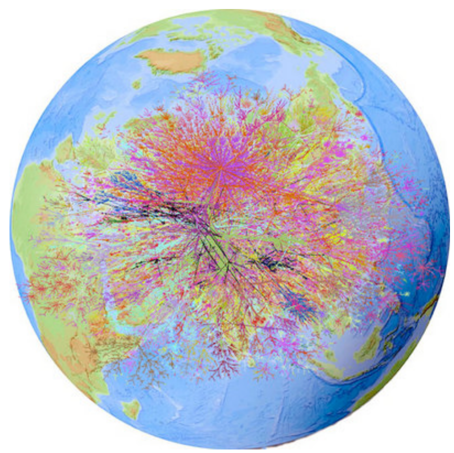
Networks:

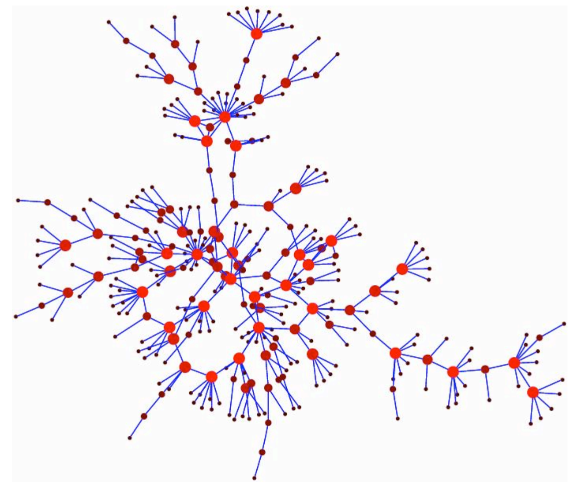
Transportation Networks/ Power grid
(distribution/ collection networks)



Biological networks
- protein interaction
- genetic regulation
- drug design

Computer networks



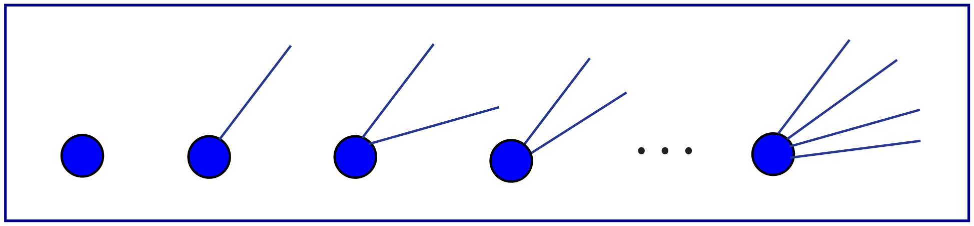


Social networks
- Immunology
- Information
- Commerce

A collection of <u>interacting</u> networks

Modeling networks as random graphs

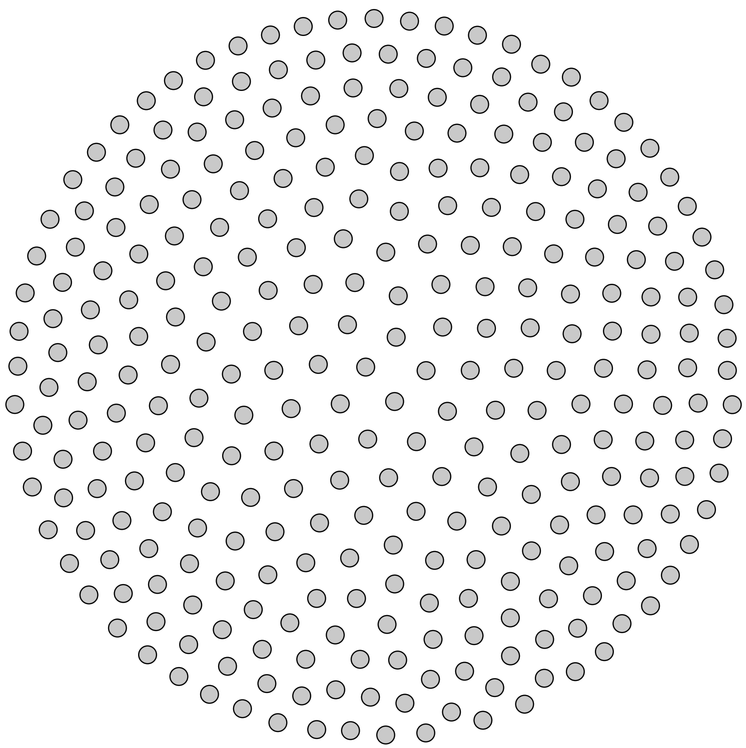
- Erdős and Rényi random graphs (1959, 1960). Phase transition.
- Configuration models (Bollobas 1980, Molloy and Reed *RSA* 1995). Enumerating over all networks with specified $\{p_i\}$.



- Preferential attachment (de Sola Price 1976, Barbási-Albert *Science* 1999, etc). Equilibrium model.
- Growth by copying (Kumar, Raghavan, Rajagopalan, Sivakumar, Tomkins, Upfal *FOCS* 2000), including duplication/mutation (Vazquez, Flammini, Maritan, Vespignani, *ComPlexUs* 2003)
- Many more . . .

Building a random instance of a network

- P. Erdős and A. Rényi, "On random graphs", *Publ. Math. Debrecen.* 1959.
- P. Erdős and A. Rényi, "On the evolution of random graphs", *Publ. Math. Inst. Hungar. Acad. Sci.* 1960.
- E. N. Gilbert, "Random graphs", *Annals of Mathematical Statistics*, 1959.



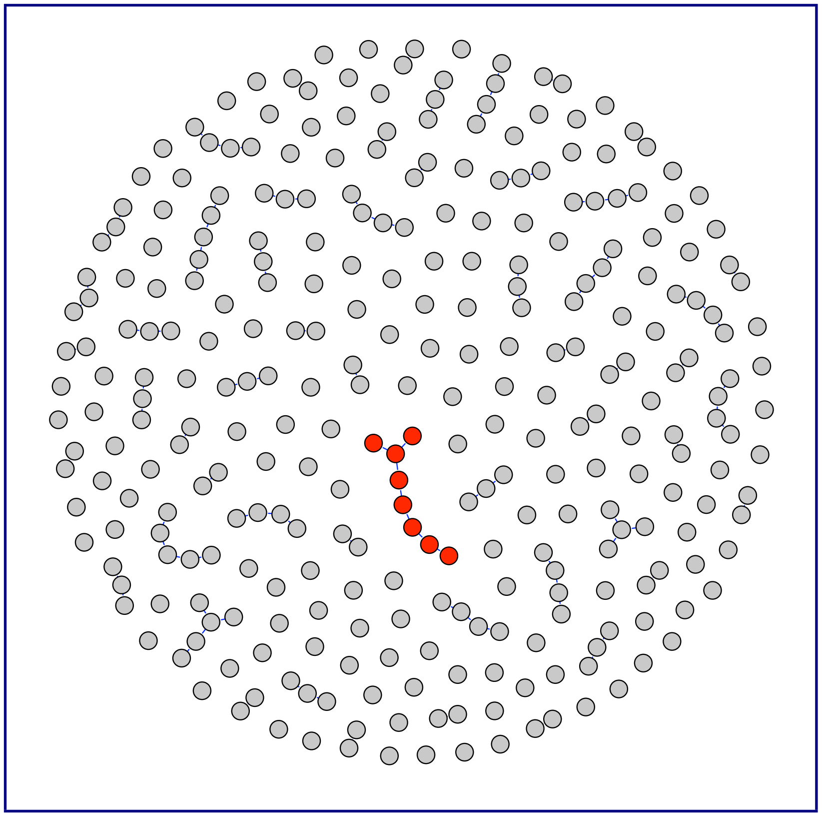
- Start with N isolated vertices.
- Add random edges one-at-a-time.

  $E = N(N-1)/2$ total edges possible.
- After $\mathcal{E}$ edges, probability p of any edge is $p = \mathcal{E}/E = 2\mathcal{E}/N(N-1)$

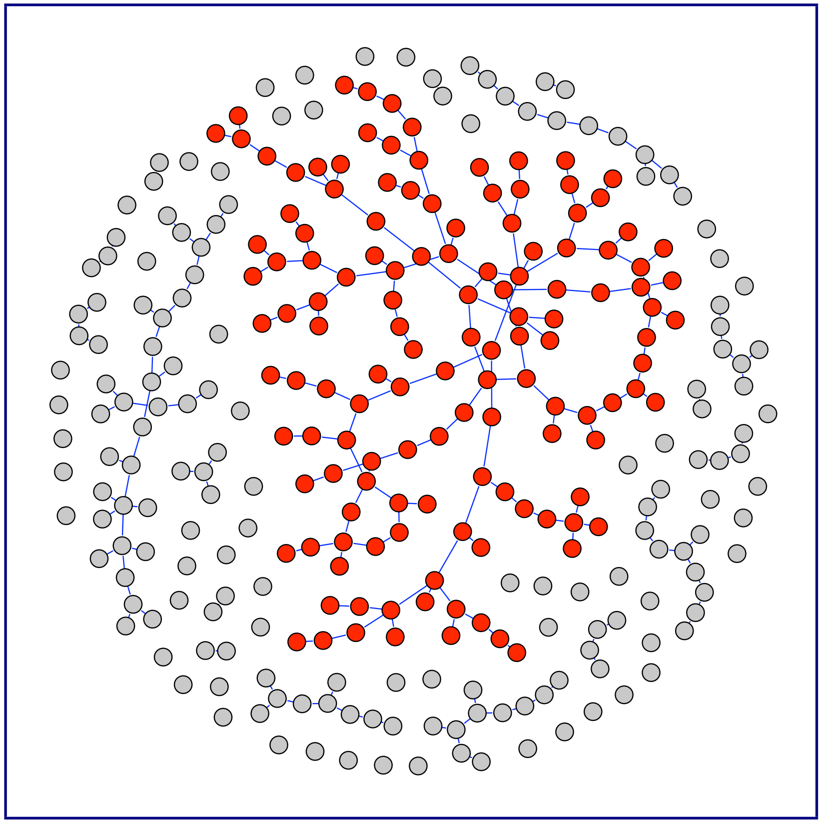
What does the resulting graph look like?

(Typical member of the ensemble)

N=300

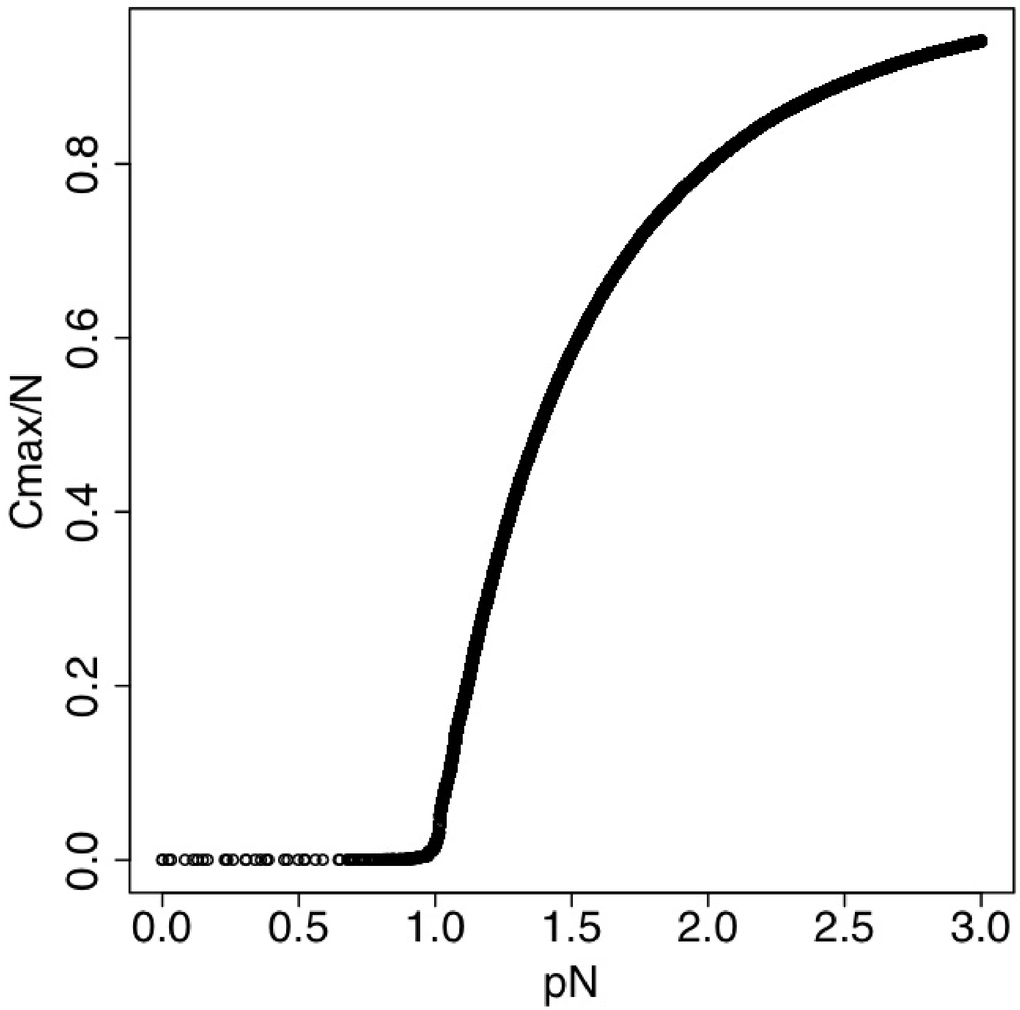


$p = 1/400 = 0.0025$



$p = 1/200 = 0.005$

Emergence of a “giant component”



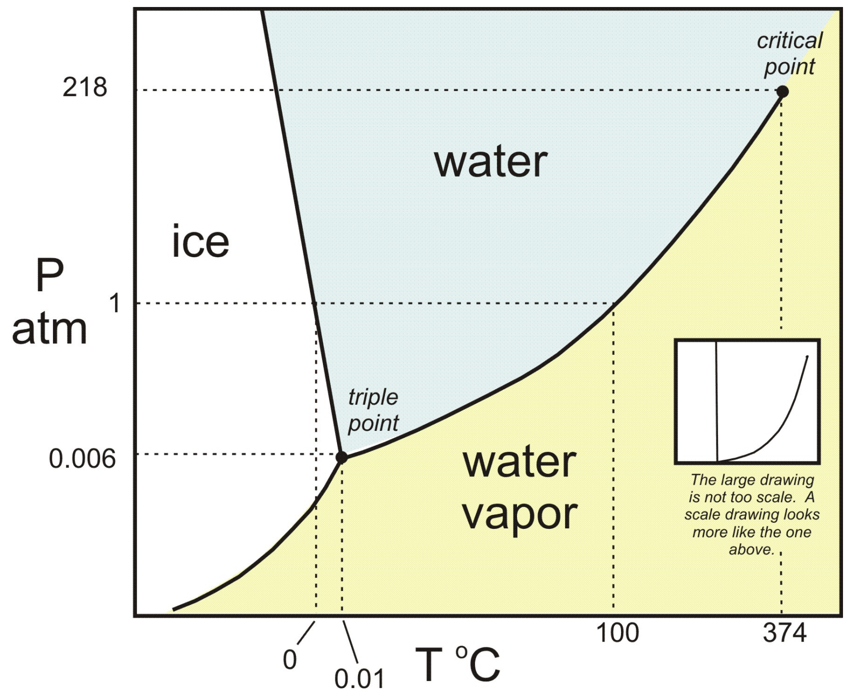
- $p_c = 1/N$.
- $p < p_c$, $C_{\max} \sim \log(N)$
- $p > p_c$, $C_{\max} \sim A \cdot N$

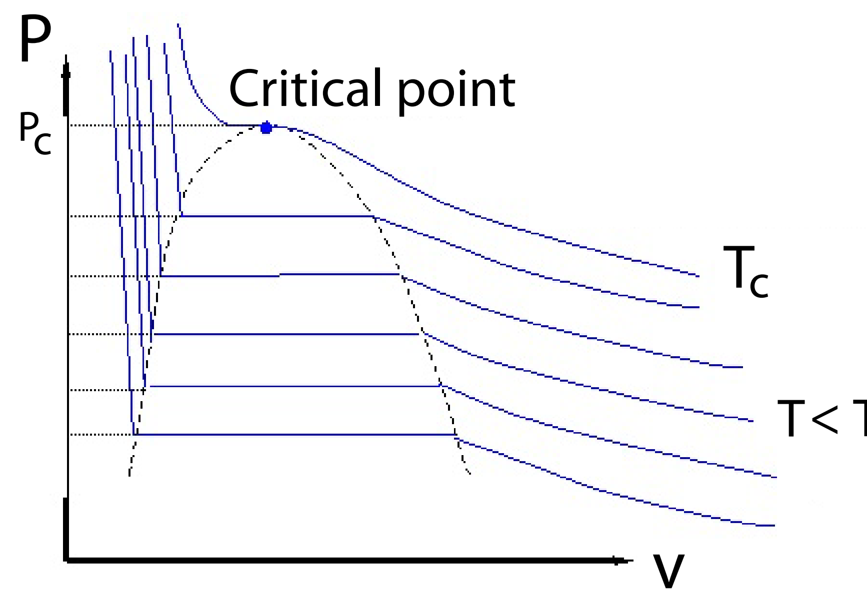
(Ave node degree $t = pN$ so $t_c = 1$.)

Branching process (Galton-Watson); “tree”-like at $t_c = 1$.

Phase transitions

Abrupt change in fundamental property of a system in response to slight change in controlling variable



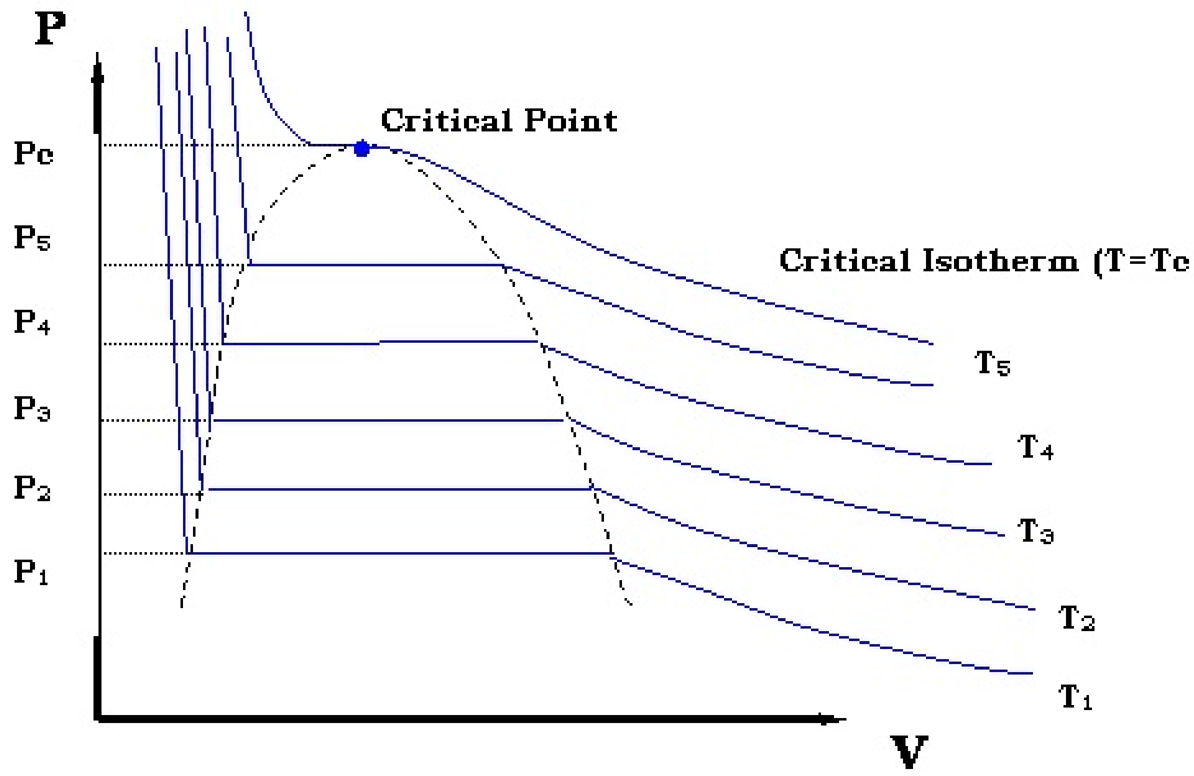


- $T < T_c$, **discontinuous jump** in thermodynamic quantity (*e.g.*, Volume, V)
- $T = T_c$, **continuous change** in V, but derivatives $\left(e.g., \frac{\partial V}{\partial P}\right)$ diverge.
- (For $T > T_c$, "supercritical fluid" state)

Universality classes I

Discontinuous – “first order”

- Phase coexistence / Latent heat
- Finite length scales



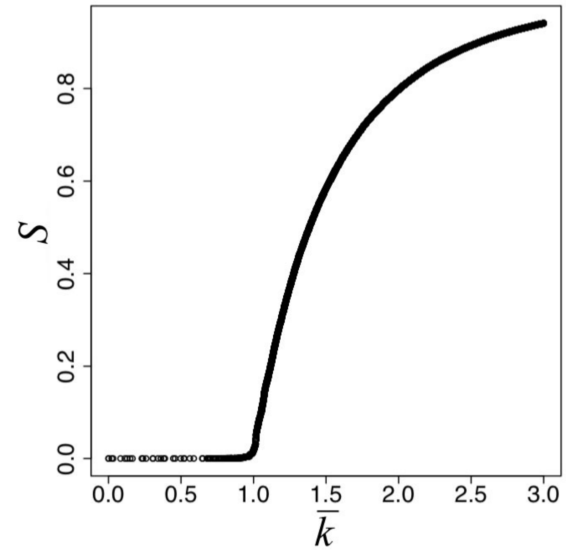
Universality classes II: Continuous – "second order"
Scaling behaviors

- Diverging correlation lengths and response functions.
- Heat capacity: $C_v = \frac{\partial E}{\partial T}\Big|_v \sim |T - T_c|^{-\alpha}$
- Isothermal compressibility: $\kappa_T = -\frac{1}{V}\frac{\partial V}{\partial P}\Big|_T \sim |T - T_c|^{-\gamma}$
- Magnetic susceptibility: $\chi = \frac{\partial m}{\partial h} \sim |T - T_c|^{-\gamma}$

"Mean field" Ising and van der Waals gas $\rightarrow \gamma = 1$.

(Thermodynamic properties depend only on a small number of features – dimensionality, symmetry – insensitive to underlying microscopic properties)

Erdős-Rényi – second order phase transition



- $t < 1, \quad C_{\max} \sim \mathrm{O}(\ln n)$
- $t = 1, \quad C_{\max} = n^{2/3}$
- $t > 1, \quad C_{\max} \sim An,$ with $A > 1$
- The critical window

  Bollobás, *Trans. Amer. Math. Soc.*, **286** (1984).

  Luczak, *Random Structures and Algorithms*, **1** (1990).

$$t = 1 + \lambda n^{-1/3} \qquad \text{(where } t = 2e/n\text{)}$$

- Mean field critical exponents

  e.g., Grimmett, *Percolation*. 2nd Edition. Springer-Verlag. 1999.

$$\chi \sim (t_c - t)^{-\gamma}, \quad \text{with } \gamma = 1.$$

where χ is the expected size of the component to which an arbitrarily chosen vertex belongs.

Erdős Rényi random graph: A continuous phase transition

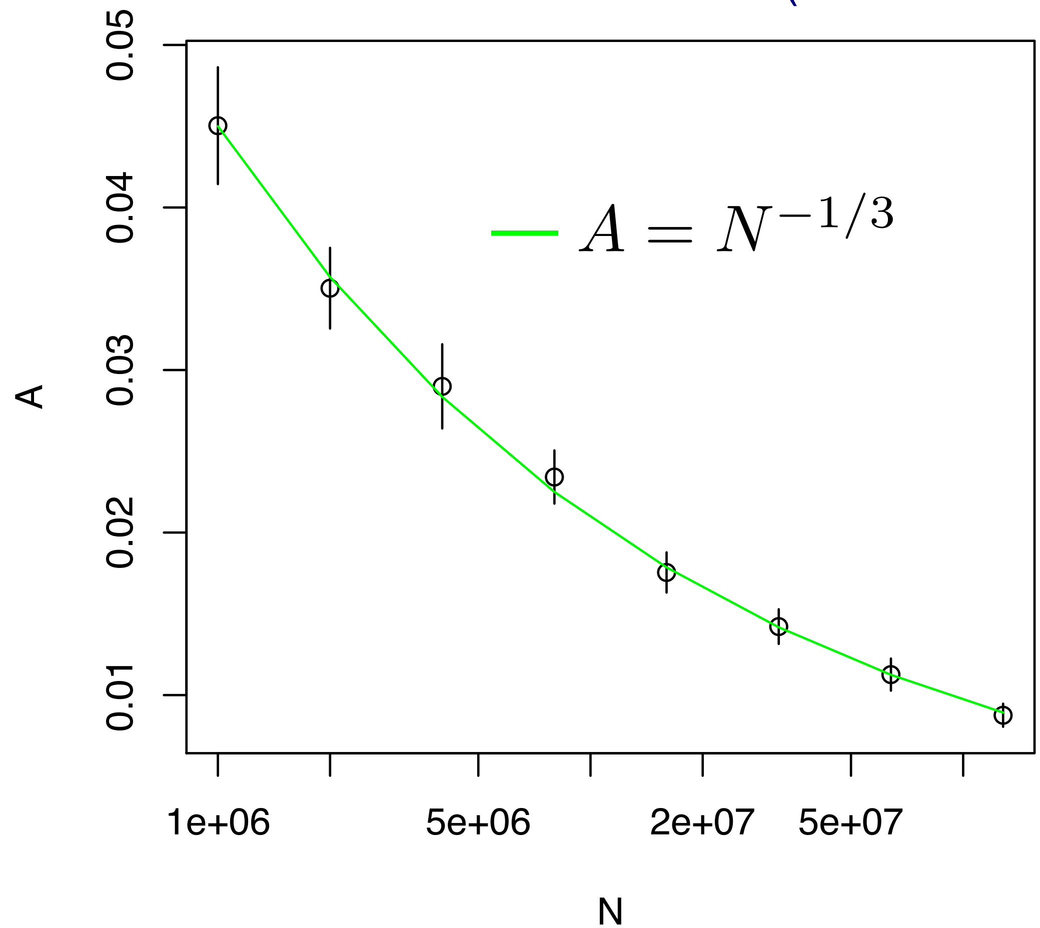
$$C_{\max}(t_c - \epsilon) = N^{2/3}$$

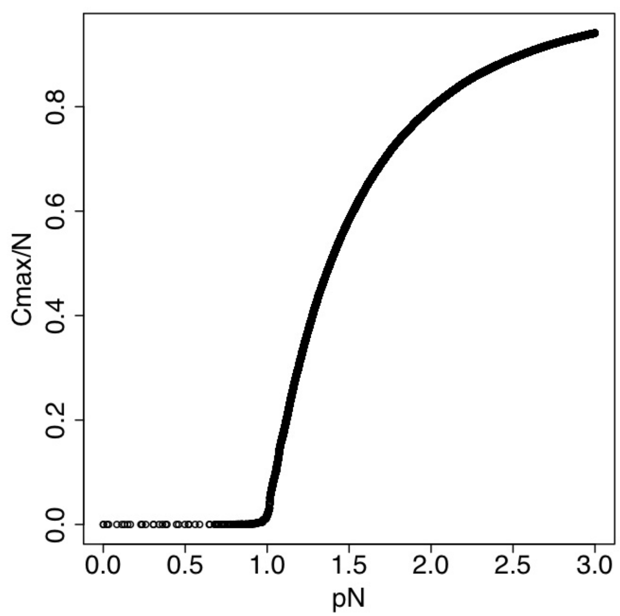
$$C_{\max}(t_c + \epsilon) = N^{2/3} + A$$

$$A = N^{-1/3}$$

$$\lim N \to \infty, A \to 0$$

$(\epsilon = 0.0005)$





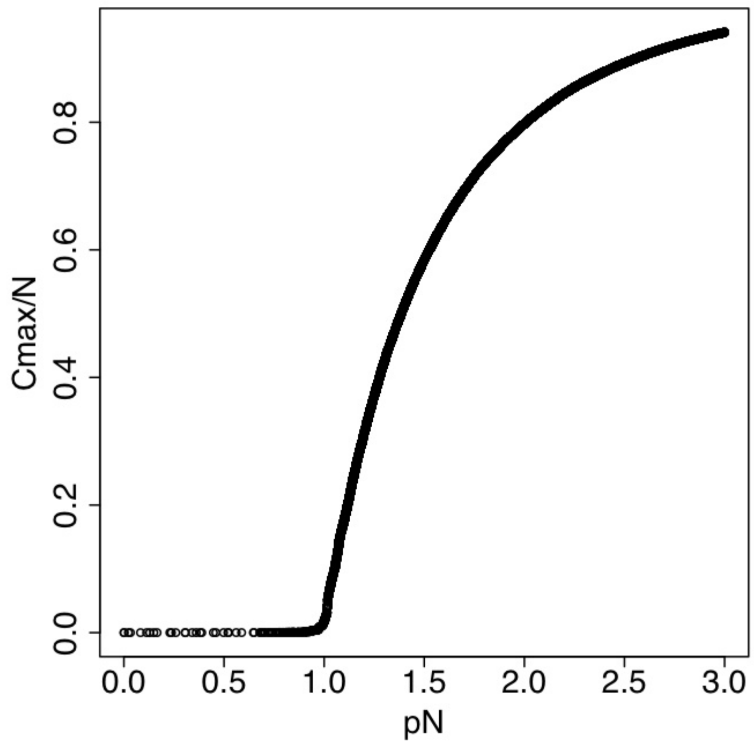
$$\boxed{C_{\max}(t_c + \epsilon) - C_{\max}(t_c) \to 0}$$

Connectivity – good or bad?

- Communications, Transportation, Synchronization, ...

  versus

- Spread of human or computer viruses

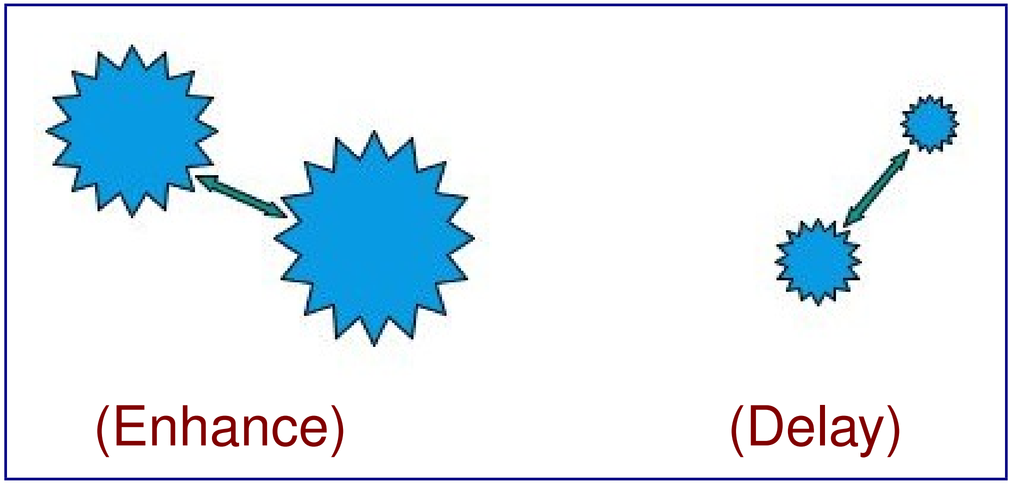


Can any limited perturbation change the phase transition?

[Bohman, Frieze, *RSA* **19**, 2001]

[Achlioptas, D'Souza, Spencer, 2009]

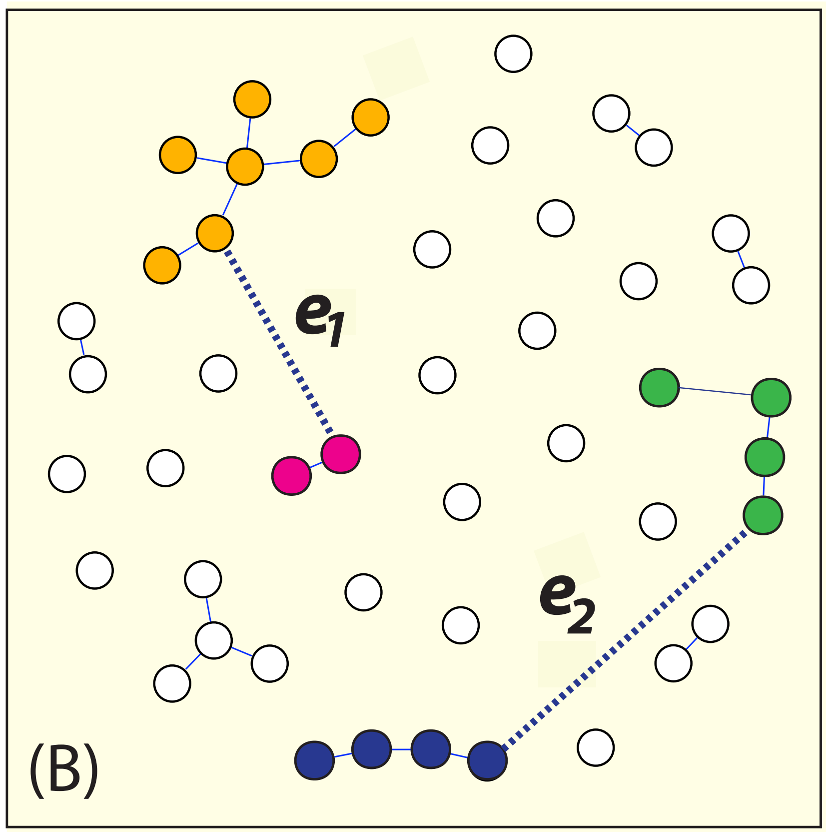
- Possible to **Enhance** or **Delay** the onset?
- The "**Product Rule**"
  - Choose ***two*** edges at random each step.
  - Add only the desirable edge and discard the other.



- The Power of Two Choices

  Azar; Broder; Mitzenmacher; Upfal; Karlin;

ProdRule: Explicit example

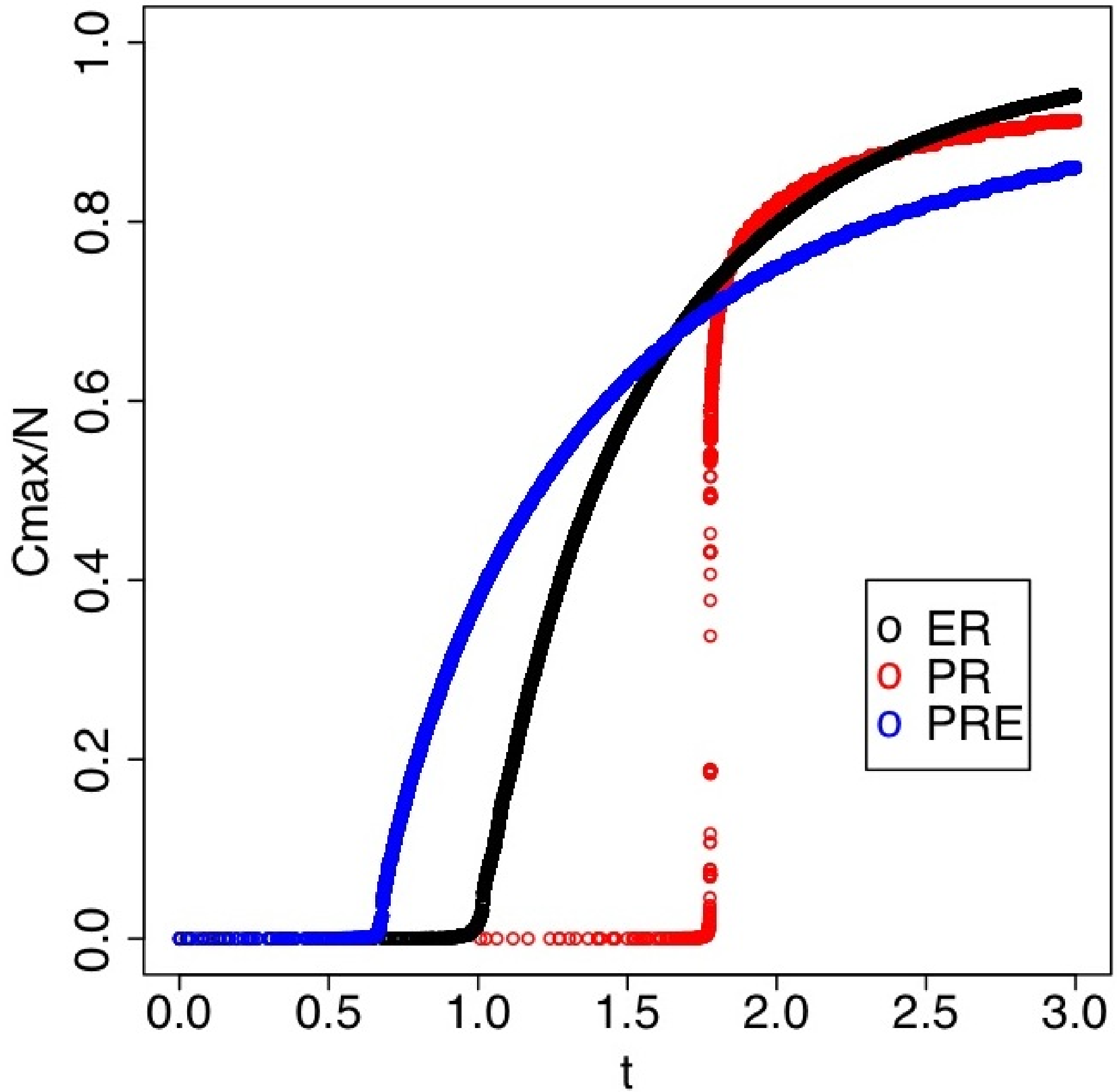


- Prod $e_1 = (7) \times (2) = 14$
- Prod $e_2 = (4) \times (4) = 16$
- To *enhance* choose e_2. To *delay* choose e_1.

Product Rule

- **Enhance** – similar to **ER** but with earlier onset.

- **Delay** – changes from ***continuous*** to ***discontinuous*** transition!

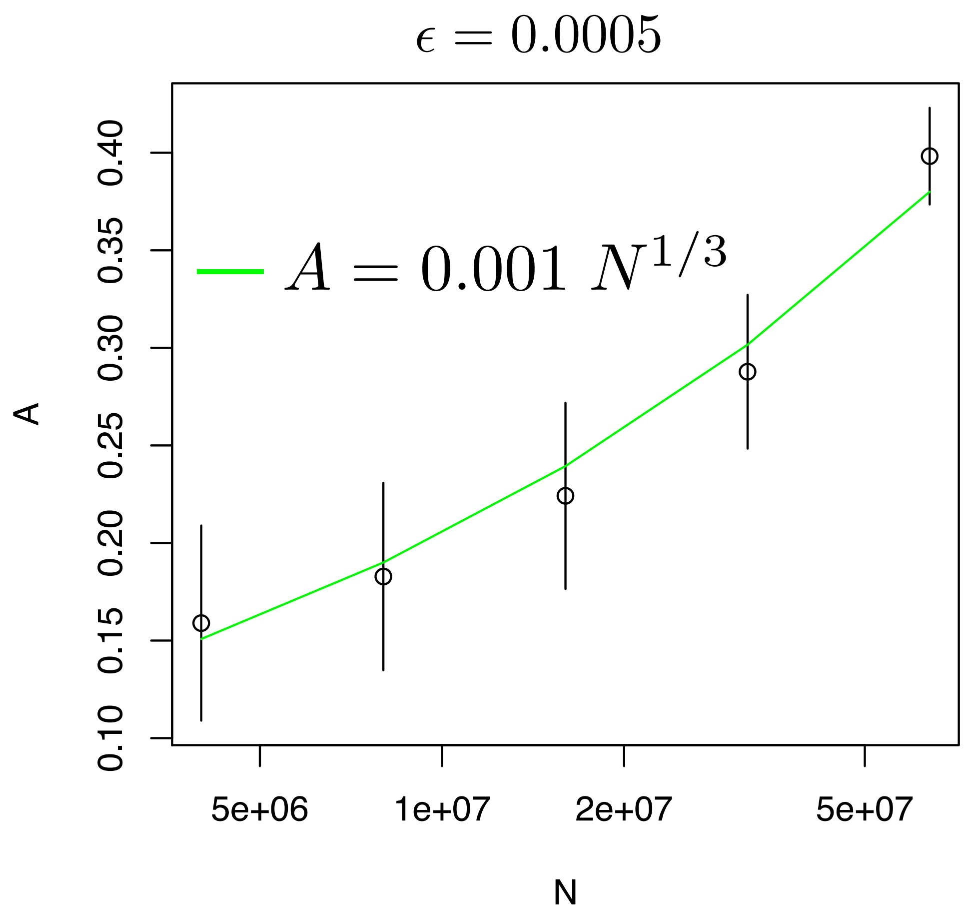


Delayed Product Rule: Discontinuous change

$C_{\max}(t_c - \epsilon) = N^{\gamma}$, with $\gamma < 1$

$C_{\max}(t_c + \epsilon) = A \cdot N$

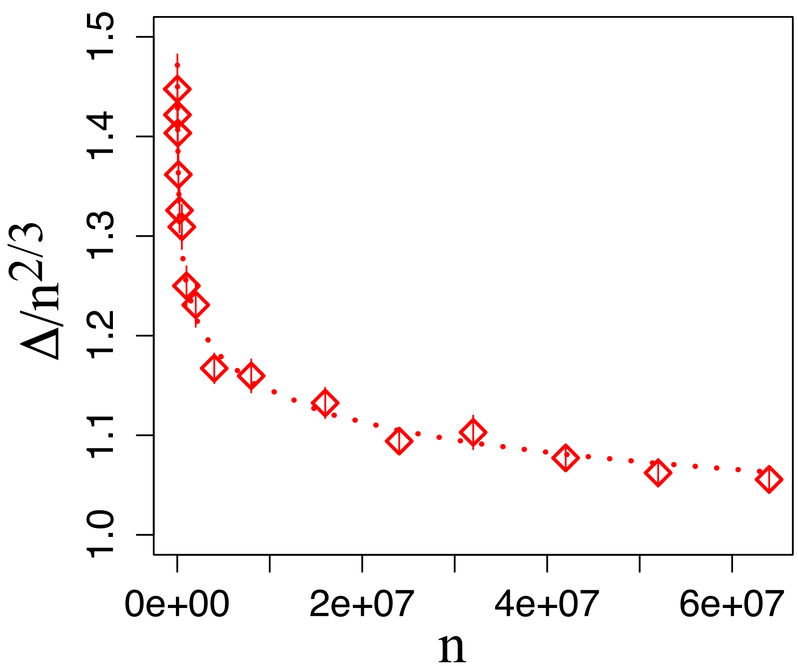
A increases with N!



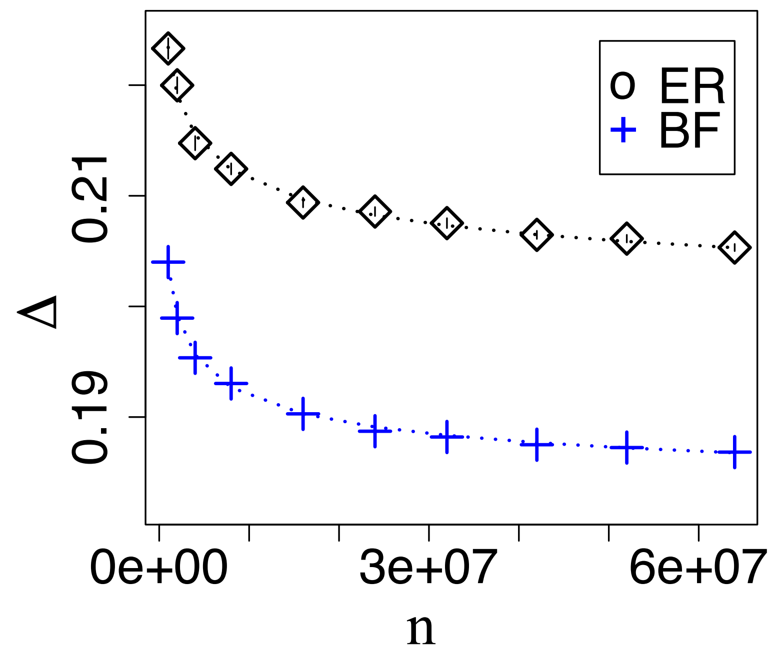
$$\boxed{C_{\max}(t_c + \epsilon) - C_{\max}(t_c) \to A}$$

The window Δ from $n^{1/2}$ to $0.5n$

- Let e_0 denote the last edge added for which $C_{max} < n^{1/2}$. (Recall ER has $n^{2/3}$ at p_c.)
- Let e_1 denote the first edge added for which $C_{max} > 0.5n$.
- Let $\Delta = e_1 - e_0$.



PR $\Delta \sim n^{2/3}$



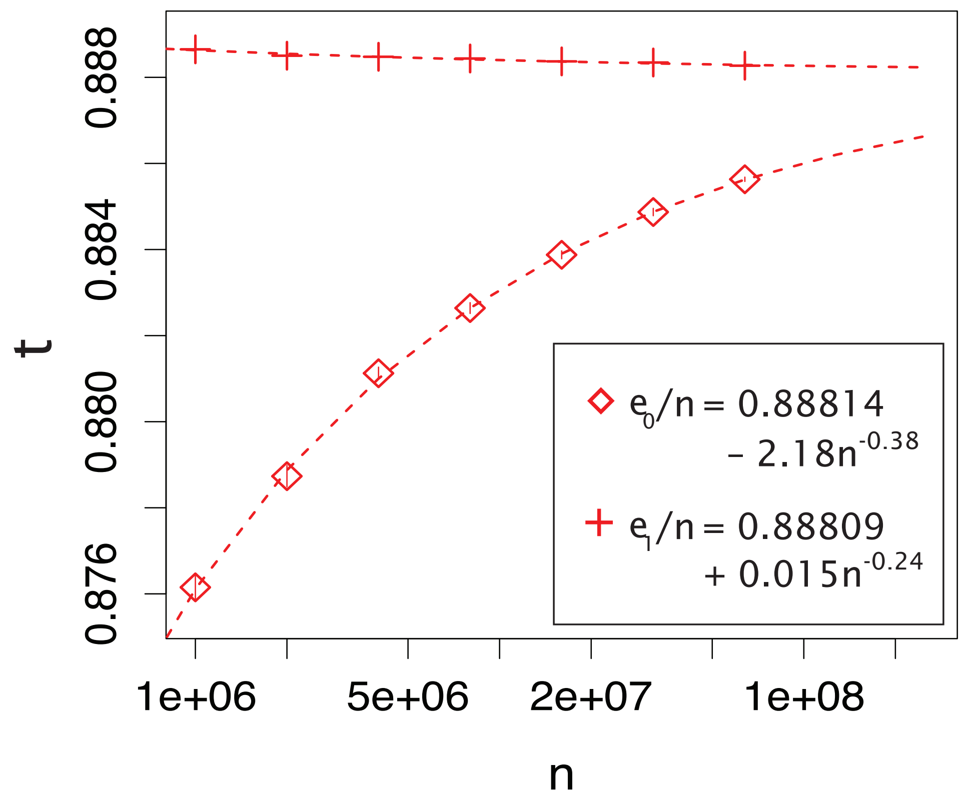
ER (and BF) $\Delta \sim n$.

PR From $n^{1/2}$ to $0.5n$ in number of edges that is sublinear in n.

Bounding t_c, where $t = e/n$

(Note, for ER, $t_c = 1/2$)

- For $t < t_c$, $C_{\max} < n^{1/2}$.
- For $t > t_c, C_{\max} > 0.5n$.

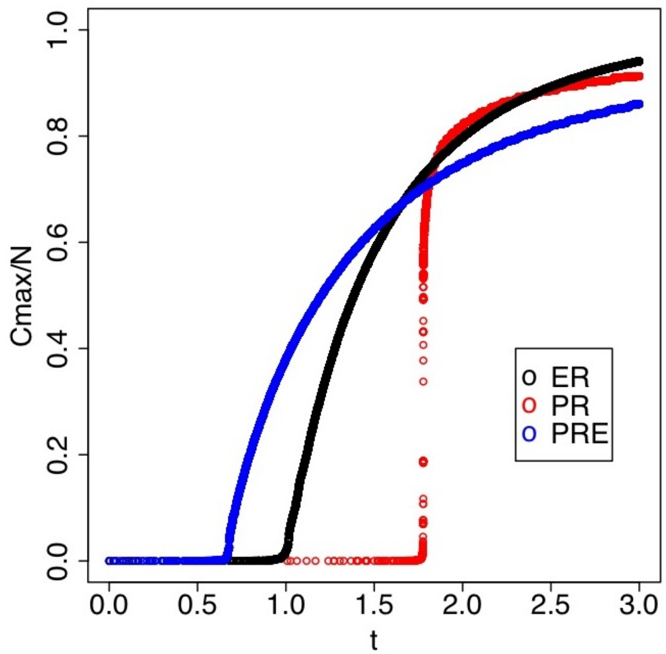


Jumps "instantaneously" from $C_{\max}$ = $n^{1/2}$ to $0.5n$.

“Explosive Percolation in Random Networks”

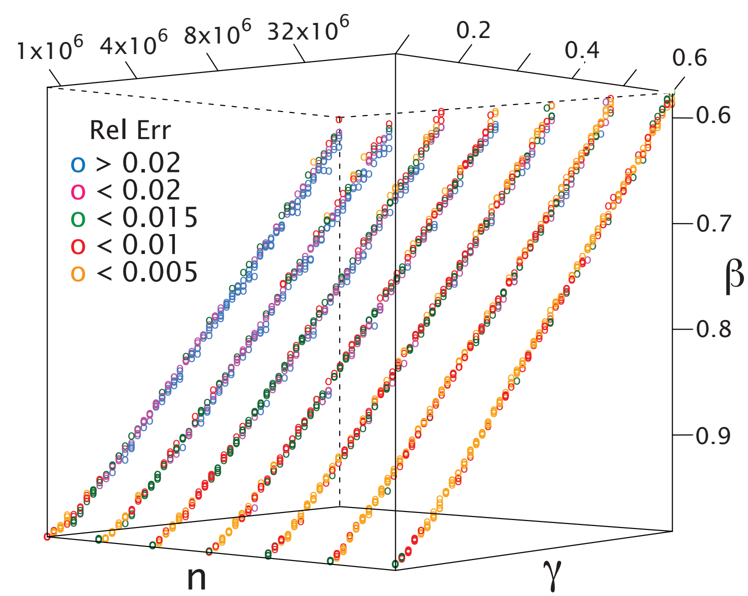
From n^γ to greater than $0.6n$ “instantaneously”

$C_{\max}$ jumps from sublinear n^γ to $\geq 0.5n$ in n^β edges, with $\beta, \gamma < 1$.



Nontrivial Scaling behaviors

$\gamma + 1.2\beta = 1.3$ for $A \in [0.1, 0.6]$



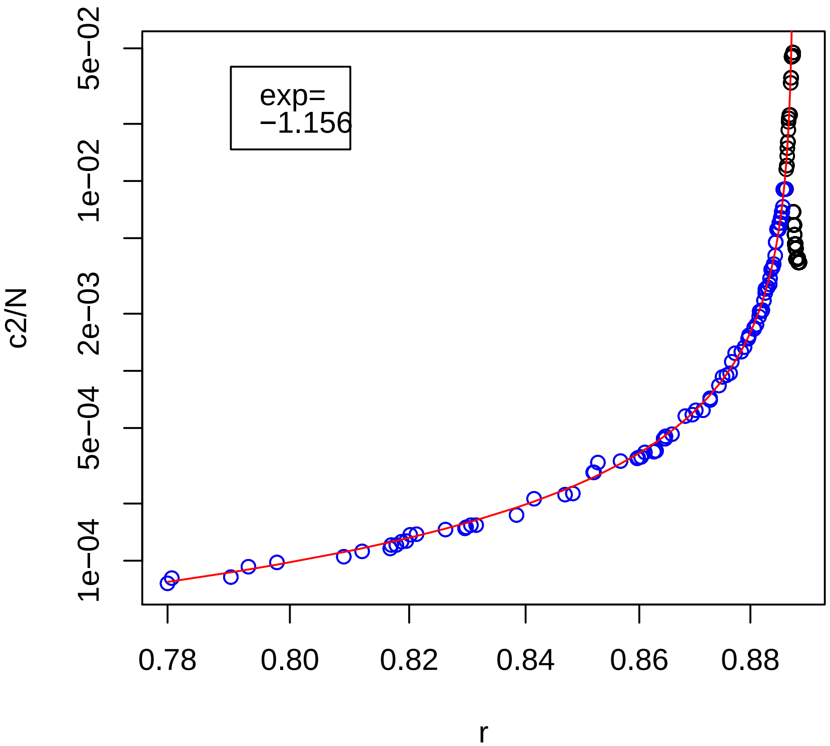
Achlioptas, D’Souza, Spencer, *Science*, **323** (5920), 2009

A Hybrid Transition! Diverging correlation length, I

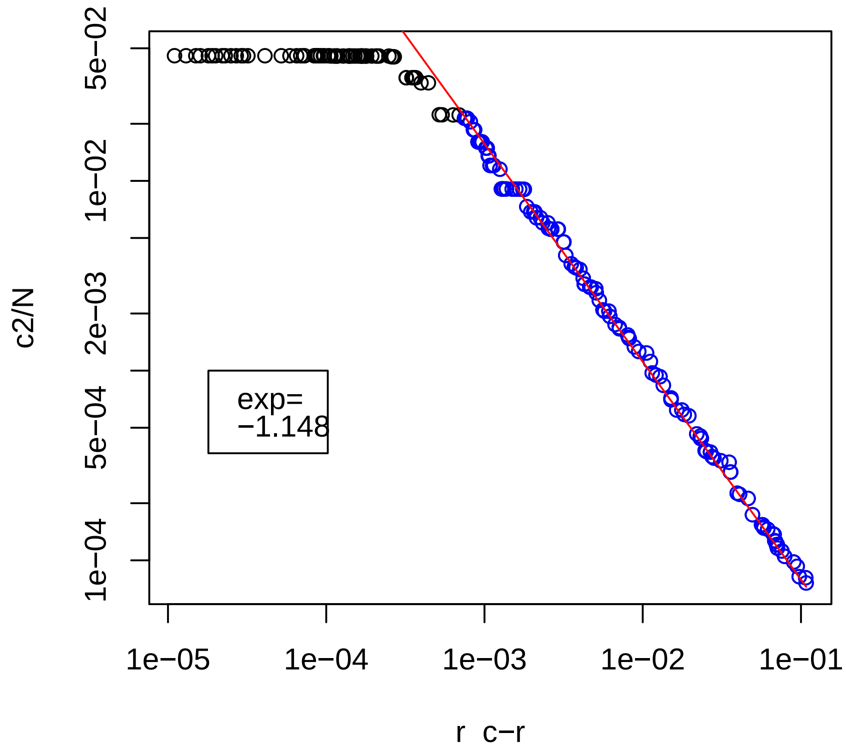
The second largest component, C_2

$$\boxed{C_2 \sim (t_c - t)^{-\gamma}, \text{ with } \gamma \approx 1.15}$$

(No simple corrections to scaling yield mean field $\gamma = 1$)



(t)



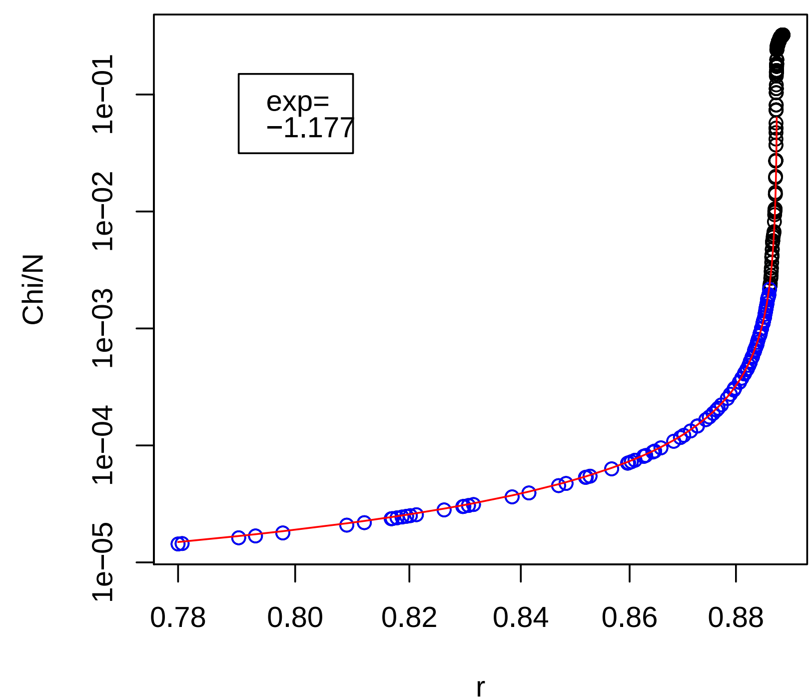
$(t_c - t)$

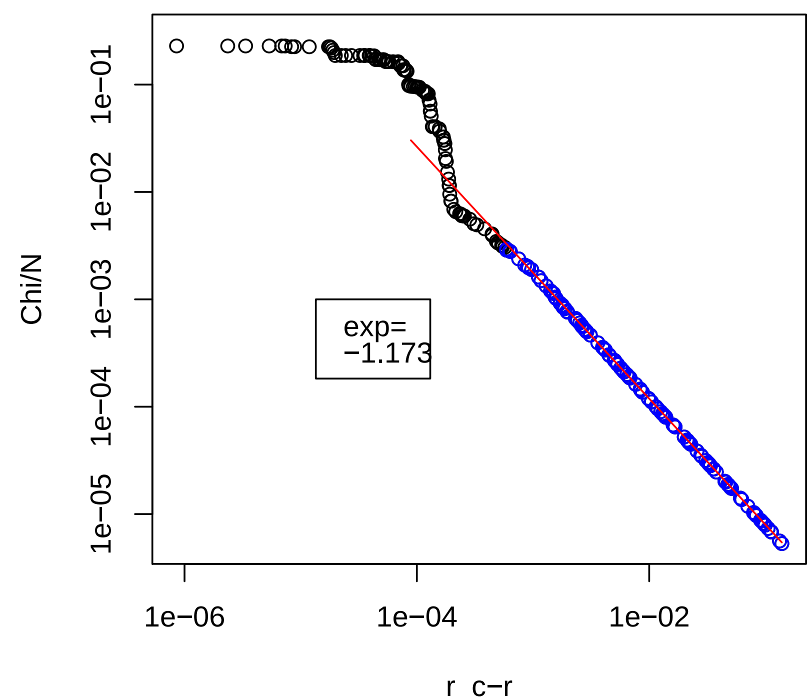
Product Rule: Diverging correlation length, II

- "Susceptibility", the expected size of the component to which an arbitrarily chosen vertex belongs,

$$\chi = \frac{1}{n}\sum_{v \in V(G)} |C(v)| = \frac{1}{n}\sum_{components} |C_i|^2.$$

$$\boxed{\chi \sim (t_c - t)^{-\gamma}, \text{ with } \gamma \approx 1.17}$$



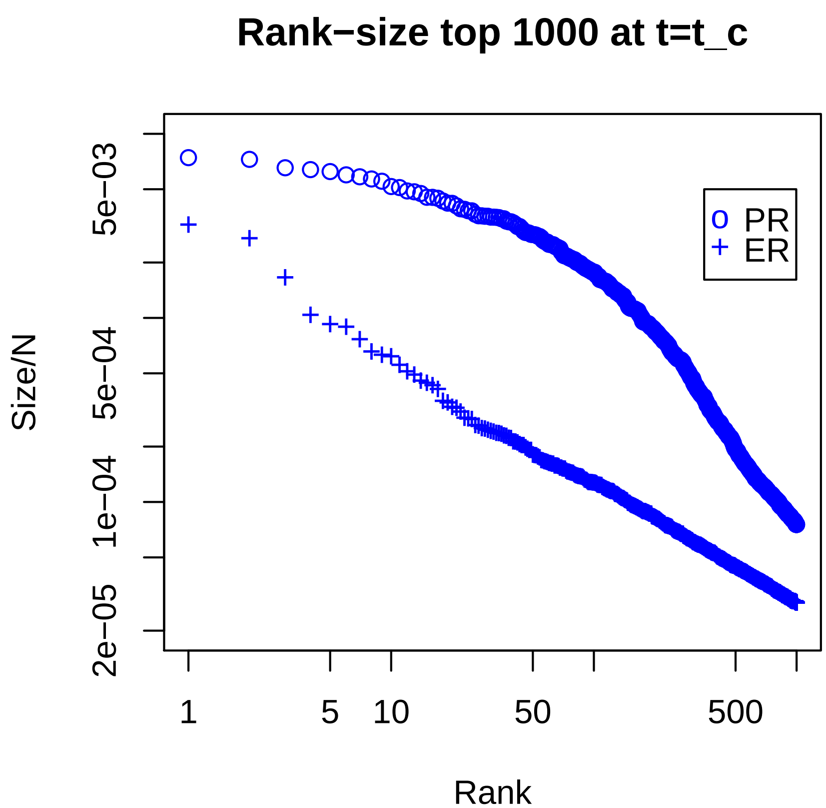


Explosive percolation now observed in ...

- R. Ziff, *Phys. Rev. Lett.* 103, 045701 (2009).
  "Explosive Growth in Biased Dynamic Percolation on Two-Dimensional Regular Lattice Networks"
- Y. S. Cho, J. S. Kim, J. Park, B. Kahng, D. Kim, *Phys. Rev. Lett.* 103, 135702 (2009).
  "Percolation Transitions in Scale-Free Networks under the Achlioptas Process"
  (Chung-Lu weighted node power law growth model)
  - $p_c > 0$ for $\gamma > 2.3$ or 2.4 and discontinuous.
- F. Radicchi, S. Fortunato *Phys. Rev. Lett.* 103, 168701 (2009).
  "Explosive percolation in scale-free networks"
  (Configuration model power law)
  - $p_c > 0$ for $\gamma > 2.2$, discontinuous for $\gamma > 3$.
- E. J. Friedman, A. S. Landsberg *Phys. Rev. Let.* 103, 255701 (2009).
  "Construction and Analysis of Random Networks with Explosive Percolation"
- Y.S. Cho, B. Kahng, D. Kim; *Phys. Rev. E* (R), 2010.
  "Cluster aggregation model for discontinuous percolation transition"
- Rozenfeld, Gallos, Makse; arxiv:0911.4082
  "Explosive Percolation in the Human Protein Homology Network"

Beyond "Product Rule"

- "Sum rule" also works, but delay is smaller.
- In general any rule that keeps components similar in size in subcritical regime should be explosive:
  - "Powder Keg" of Friedman and Landsberg *PRL* (2009).
  - Starting ER from proper initial state; Cho, Khang, Kim *PRE* (2010).



Rigorous techniques only for bounded size (thus far)

- Bounded size rules
  (treat all components of size $\geq K$ as the same)

- Assume cluster aggregation models (two distinct clusters merged with each edge addition).

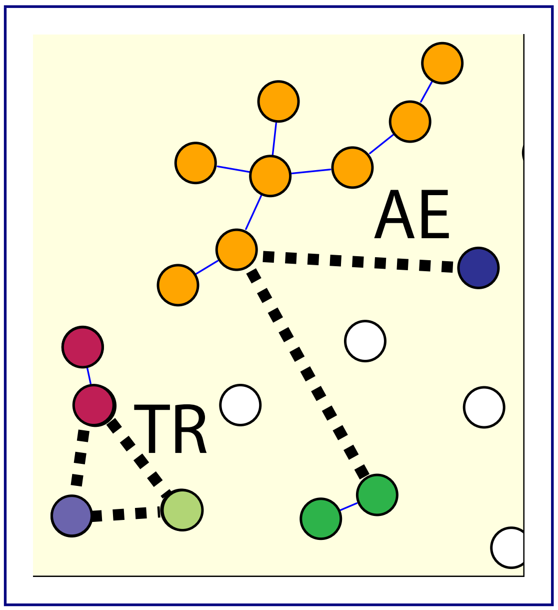
- "Birth Control for Giants",
  J. Spencer, N. Wormald, *Combinatorica* **27**(5), 2007 :

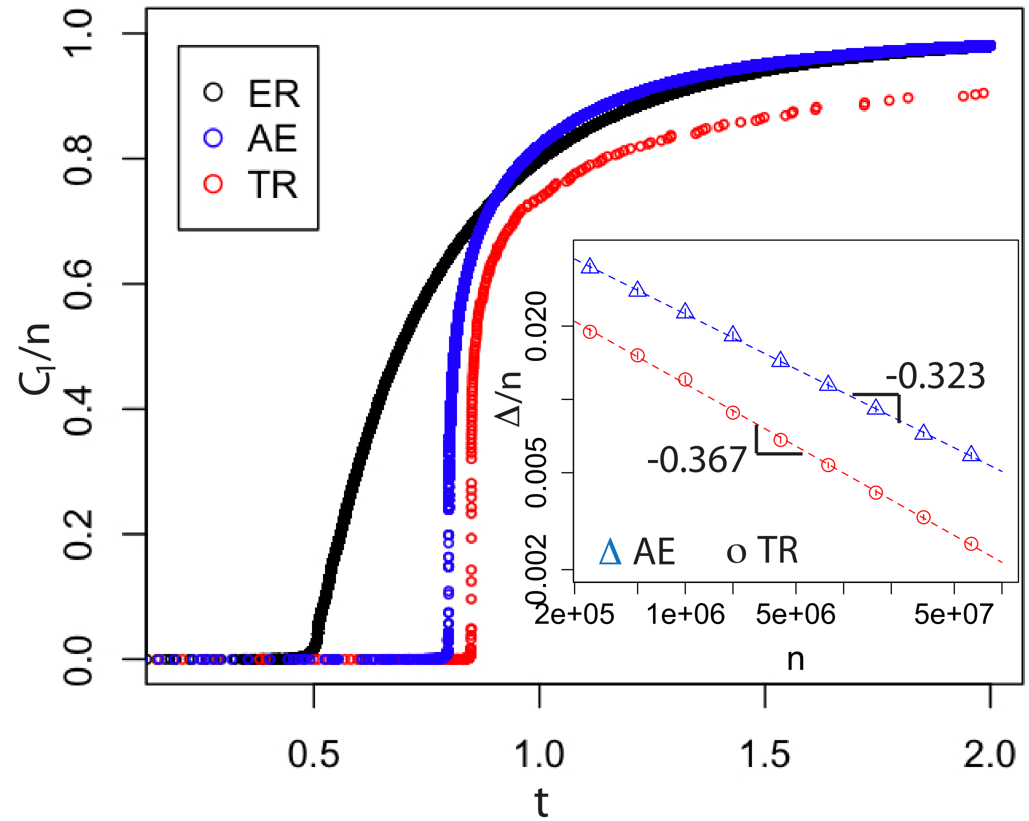
  – Differential equation for evolution
  (rigorous proof that error term is small in subcritical regime)

  – Conjecture that all bounded size rules have continuous PT's

<u>Local</u> Cluster Aggregation Models with Explosive Percolation

(R.D. and M. Mitzenmacher, *Phys. Rev. Lett.*, in press.)

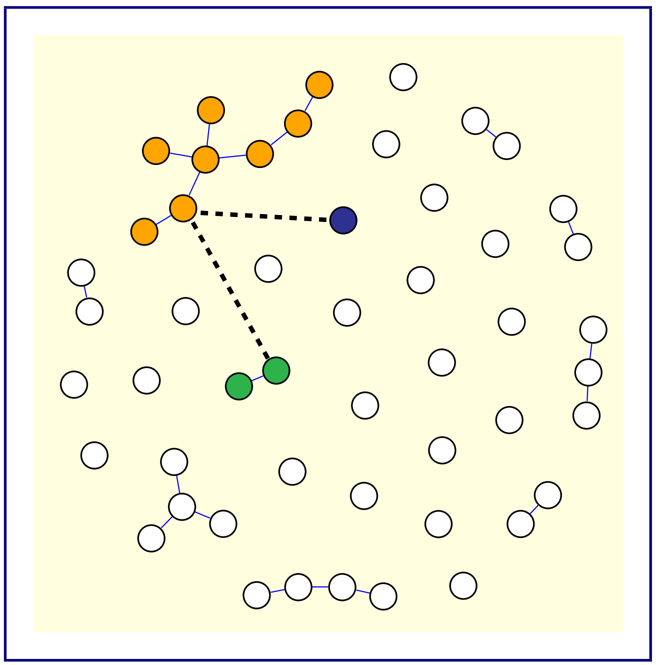


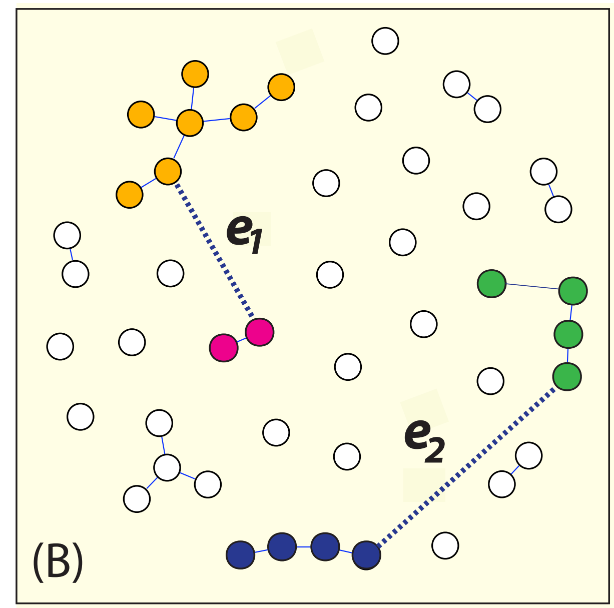


- "Adjacent Edge (AE)" – 2 candidate edges share a common vertex.
- "Triangle Rule" (TR) – choose 3 vertices at random, 3 candidate edges.

Locality: more physical and simplifies analytic treatment

Adjacent Edge Rule (3 components) **vs** **Product Rule (4 components)**

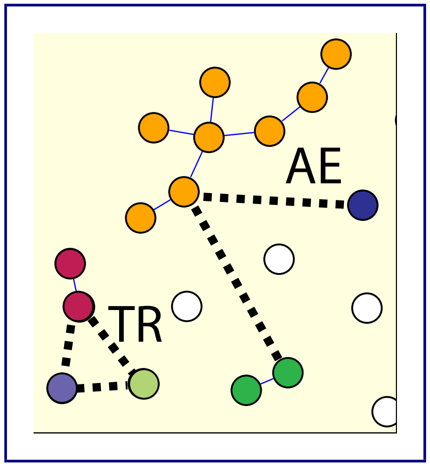




(TR also depends only on 3 components)

Evolution equations for the bounded size AE rule:

- Let x_i denote fraction of nodes in components of size i, for $1 \leq i \leq K$.



- Let $S_i = \sum_{j=i}^{\infty} x_j$
  (the weight in the tail starting at size i)
  ($S_{K+1} = 1 - \sum_{j=1}^{K} x_j$ is useful)
- Probability first node is in component of size i is x_i.
- Probability the smaller of the two additional components has size j is $S_j^2 - S_{j+1}^2$.
- Expected evolution of x_i's:

$$\boxed{\frac{dx_i}{dt} = -ix_i - i(S_i^2 - S_{i+1}^2) + i\sum_{j+k=i} x_j(S_k^2 - S_{k+1}^2)}$$

“Susceptibility”: $W = \sum_{i=1}^{\infty} ix_i = \sum_{i=1}^{\infty} i^2 n_i$

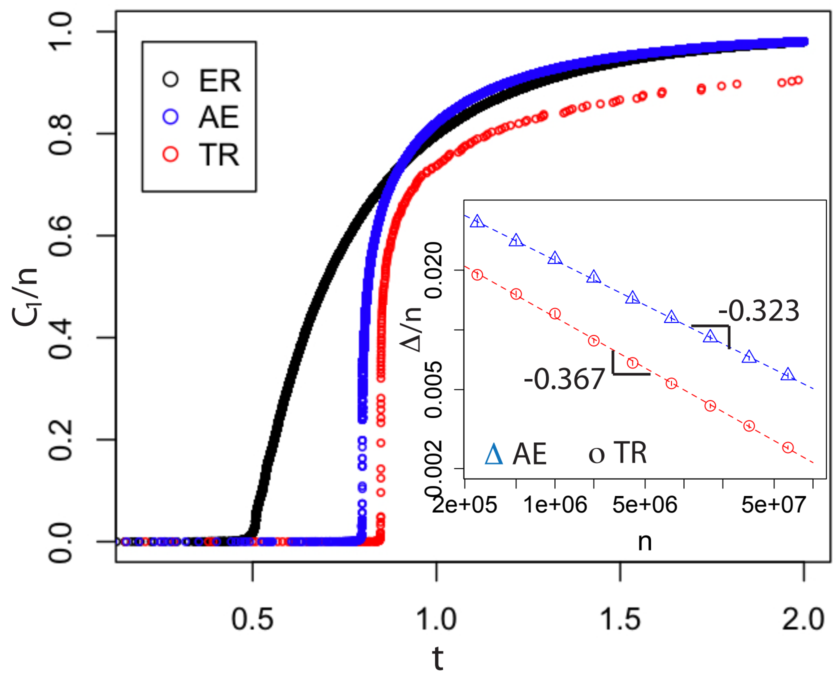
(Expected size of component to which arbitrary vertex belongs)

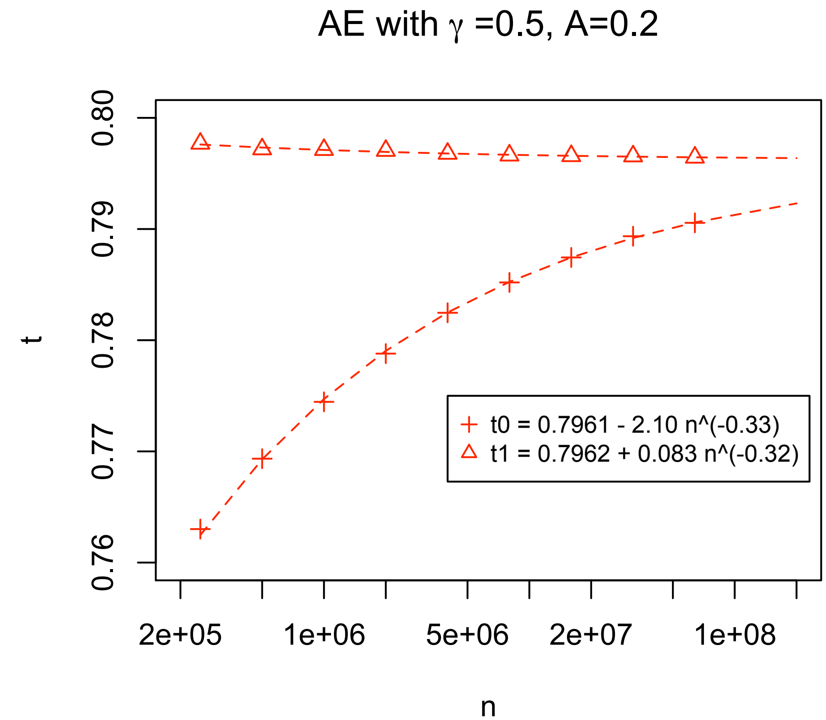
<u>The evolution of W:</u>

$$\begin{aligned}\frac{dW}{dt} \;=\; & \sum_{j=1}^{K}\sum_{k=1}^{K} 2jkx_j(s_k^2 - s_{k+1}^2) + \sum_{j=1}^{K} 2jW^* x_j s_{K+1} \\ & + \sum_{k=1}^{K} 2kW^*(s_k^2 - s_{k+1}^2) + 2(W^*)^2 s_{K+1}.\end{aligned}$$

- Where $W^* = W - \sum_{i=1}^{K} ix_i$ (contributions from comps of size greater than K).
- $W \to \infty$ at the phase transition. Simple Euler's method numerics yields $\boxed{t_c = 0.796}$ (with $K = 600$).

Direct simulation of the AE graph evolution process





- Find agreement to three digits, $\boxed{t_c = 0.796}$.

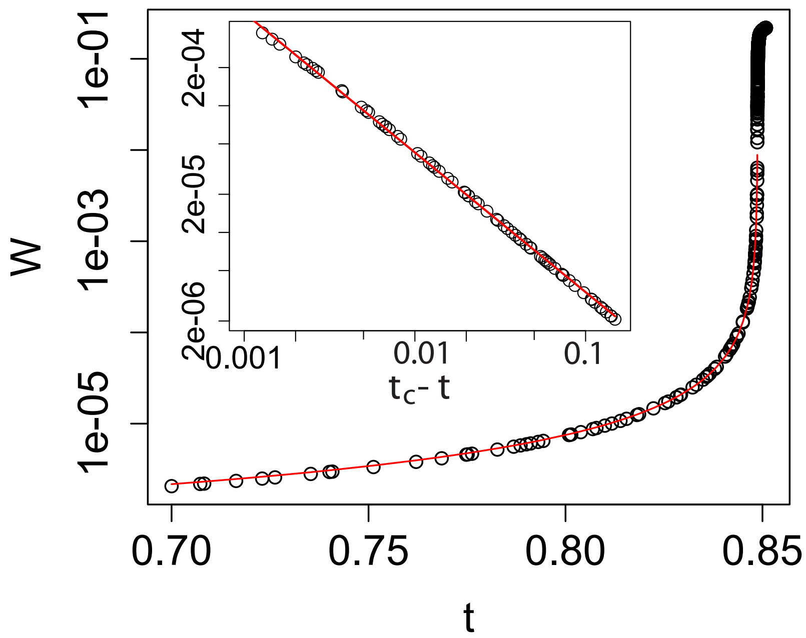
- Δ sublinear in n, where t_0 is last time $C_1 \leq n^\gamma$, and t_1 first time $C_1 \geq An$. (Denote this by $\Delta(\gamma, A)$)

  (For AE $\Delta(0.5, 0.2)$ sublinear. For PR $\Delta(0.5, 0.6)$ sublinear.)

AE, PR, TR are Hybrid Transitions!

(Discontinuous change, but scaling behavior)

- Component density $n_i \sim i^{-\tau}$
- $W \sim |t - t_c|^{-\alpha}$
- $C_2 \sim |t - t_c|^{-\mu}$



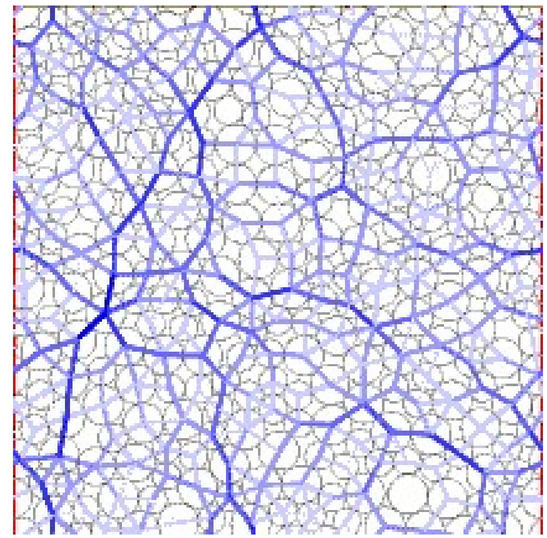
	PR	AE	TR
t_c	0.888	0.796	0.848
τ	2.1	2.1	2.1
α	1.17	1.13	1.13
μ	1.17	1.13	1.13

Other Hybrid Transitions

- **k-sat** (constraint satisfaction) for $k \geq 3$, Infinite dimensional
  (Monasson, Zecchina, Kirkpatrick, Selman, Troyansky, *Nature*, 1999)

- **Jamming in models of granular materials**
  Finite dimensional / spatial constraints

  (OHern, Langer, Liu, Nagel, *Phys. Rev. Lett.* 2002)
  (Henkes, Chakraborty, *Phys. Rev. Lett.* 2005)
  (Toninelli, Biroli, Fisher, *Phys. Rev. Lett.* 2006)
  (Schwarz, Liu, Chayes, *Europhys. Lett.* 2006)

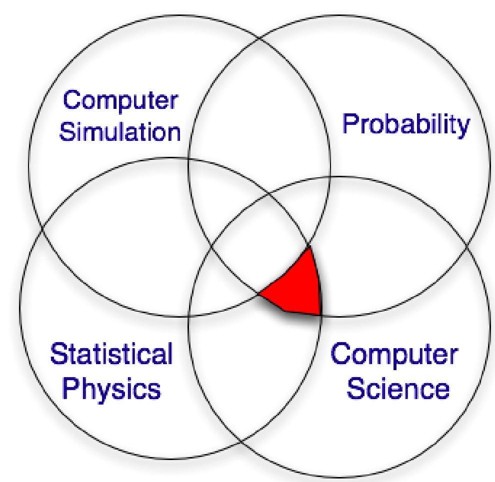


- **Spin glasses** glassy systems, slow relaxation time
  (D. Gross and M. Mezard. *Nucl. Phys. B*, 1984)
  (Kirkpatrick and Thirumalai, *Phys. Rev. Lett.* 1987)

Mixed transitions: geometry, disorder, computation

- Critical slowing down and computational complexity?
- *Nature* 1999: 2-sat <u>continuous</u> $\in$ P, 3-sat <u>discontinuous</u> $\in$ NP
- Hard instances, applications to Cryptography?

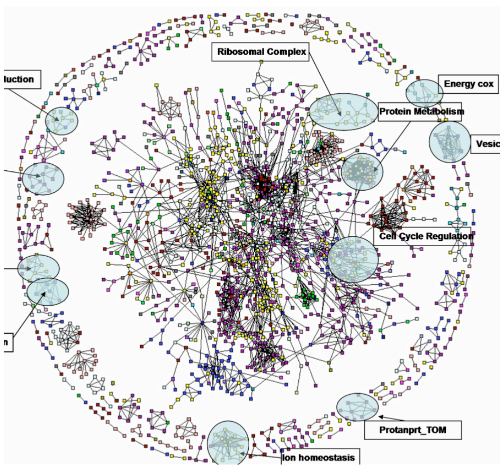
- Aspen Center for Physics: "**Complexity, Disorder, and Algorithms**"
  Organizers: S. Coppersmith, A. Middleton, J. Machta, C. Moore
  May 25 - June 22, 2008.
- American Institute for Mathematics: "**Phase Transitions**" Aug 21-25, 2006
  Organizers: P. Diaconis, D. Fisher, C. Moore, C. Radin
- Mathematical Sciences Research Institute (MSRI)
  "**Probability, Algorithms and Statistical Physics**"
  Organizers: Y. Peres, A. Sinclair, D. Aldous,
  C. Kenyon, H. Kesten, J. Kleinberg, F. Martinelli,
  A. Sokal, P. Winkler, U. Zwick
  Jan 3 - May 15, 2005.



A collection of interacting networks

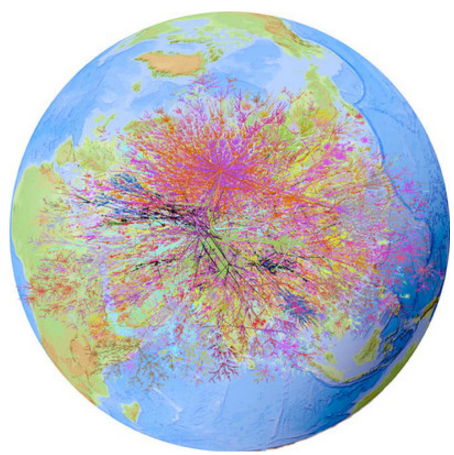
Networks:

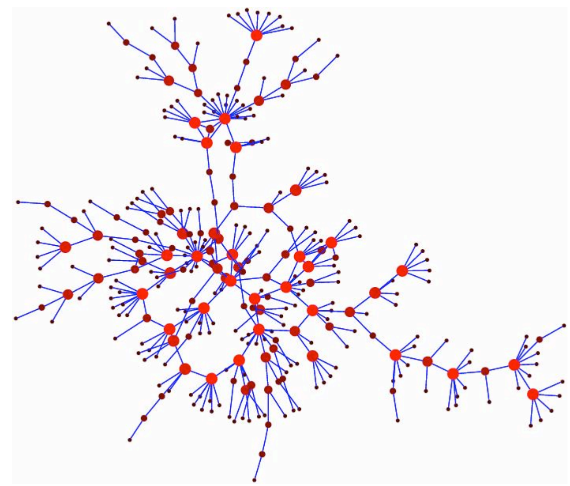
Transportation Networks/ Power grid
(distribution/ collection networks)



Biological networks
- protein interaction
- genetic regulation
- drug design

Computer networks

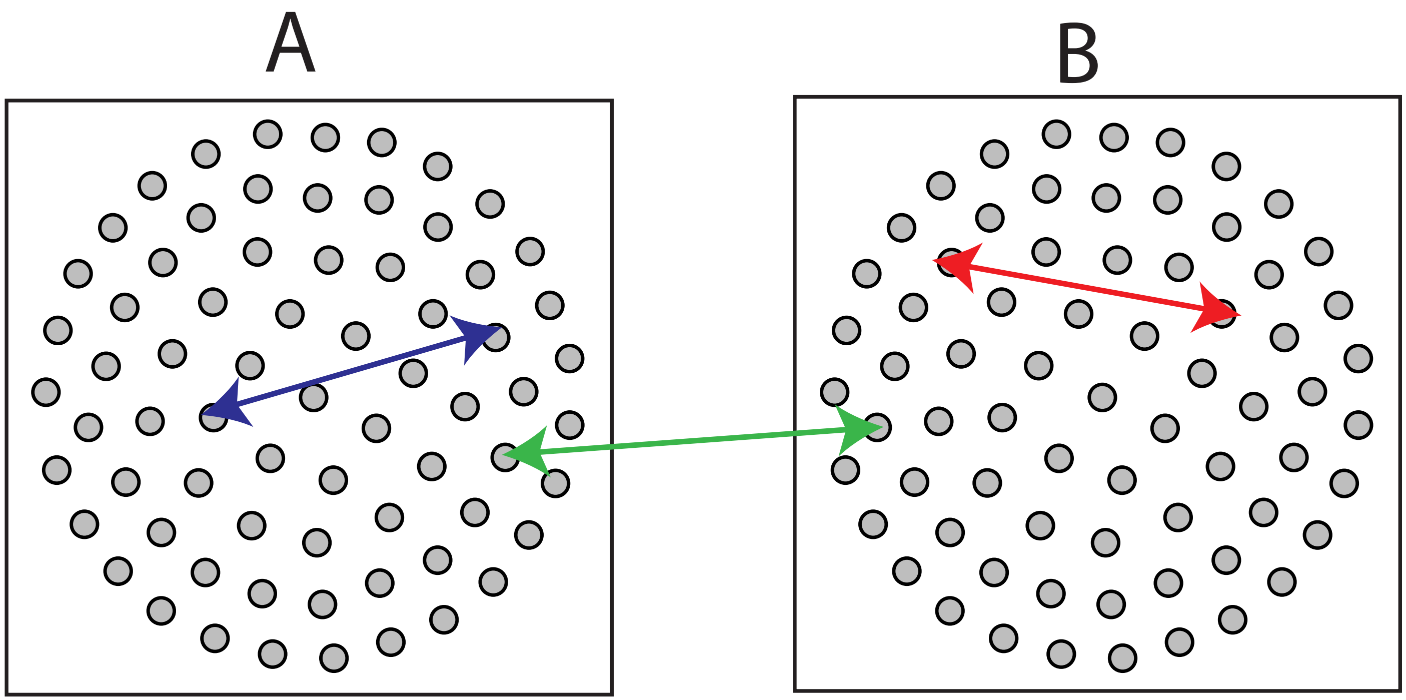




Social networks
- Immunology
- Information
- Commerce

Modular Erdős-Rényi

- Divide nodes initially into two groups (A and B):

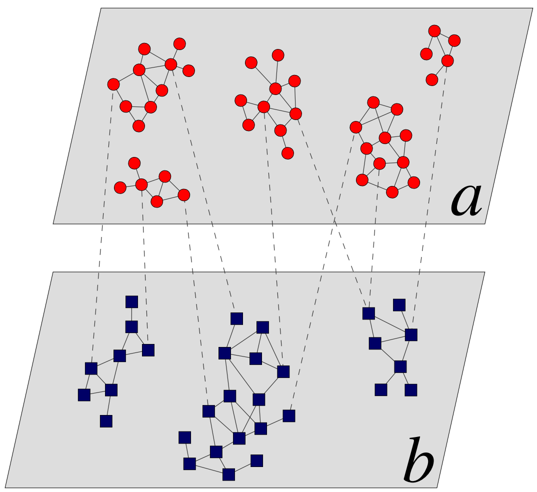


- Add internal a-a edges with rate λ.
- Add internal b-b edges with rate λ/r_1, with $r_1 > 1$.
- Add intra-group a-b edges with rate λ/r_2, with $r_2 > 1$, $r_2 \neq r_1$.

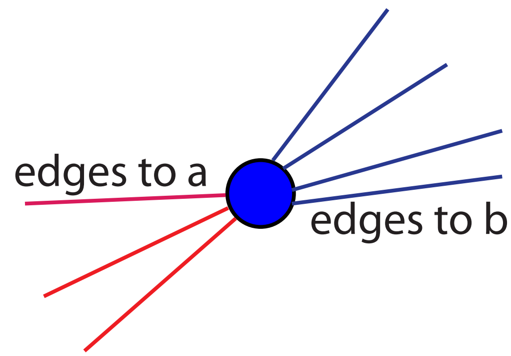
What happens? (Anything different?)

Percolation on interacting networks, using random graph models

(E. Leicht and R. D'Souza, arXiv:0907.0894)



System of two networks



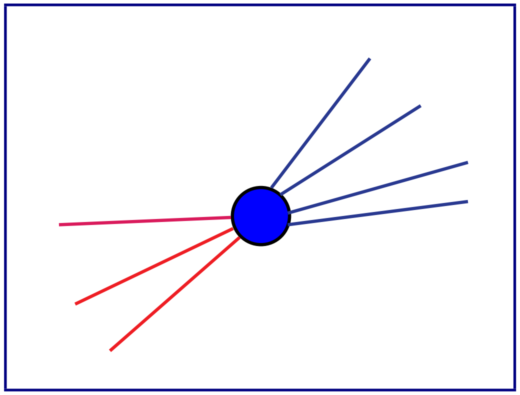
Connectivity for an individual node

- Probability distribution nodes in network a: $p^a_{k_a k_b}$
- For the the system: $\{p^a_{k_a k_b}, p^b_{k_a k_b}\}$
- Build generating function formalism for interacting networks.

Generating Functions – Distribution of component sizes:

(Extending Newman, Strogatz, Watts *PRE* **64**, (2001))

Three step process:



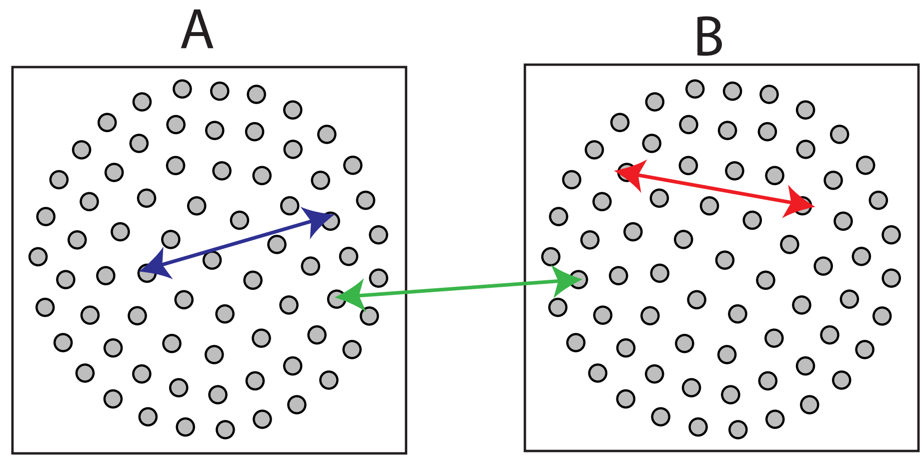
1. G.F. for connectivity of a node connected to a random edge, $G_{ab}(x_a, x_b)$.
2. G.F. for the size of the component to which that node belongs, $H_{ab}(x_a, x_b)$.
3. G.F. for the size of the component to which an arbitrary node belongs, $H_a(x_a, x_b)$.

Moments of GFs provide information, e.g., the expected number of a-nodes in the component of an arbitrary a-node:

$$\langle s_a \rangle_a = \frac{\partial}{\partial x_a} H_a(x_a, x_b) \Big|_{x_a = x_b = 1}$$

Distributions for modular Erdős-Rényi

- $p^a_{k_a k_b} = p^a_{k_a} p^a_{k_b}$ (uncorrelated)
- Independent Poisson distributions with related means:



(aa rate λ, bb rate λ/r_1, ab rate λ/r_2)

$$\overline{k}_{bb} = \frac{1}{r_1}\overline{k}_{aa},$$

$$\overline{k}_{ab} = \overline{k}_{ba} = \frac{1}{2r_2}\overline{k}_{aa}.$$

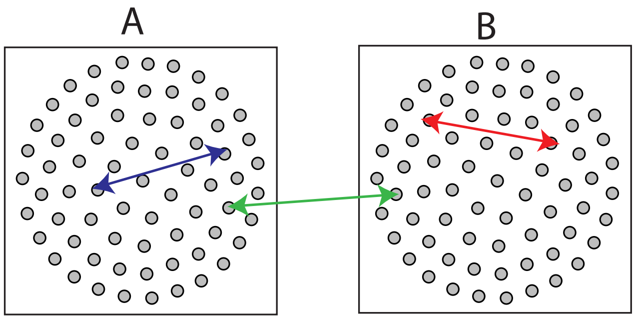
$$p^a_{k_a k_b} = \frac{\overline{k}_{aa}{}^{k_a} e^{-\overline{k}_{aa}}}{k_a!} \cdot \frac{\overline{k}_{ab}{}^{k_b} e^{-\overline{k}_{ab}}}{k_b!}$$

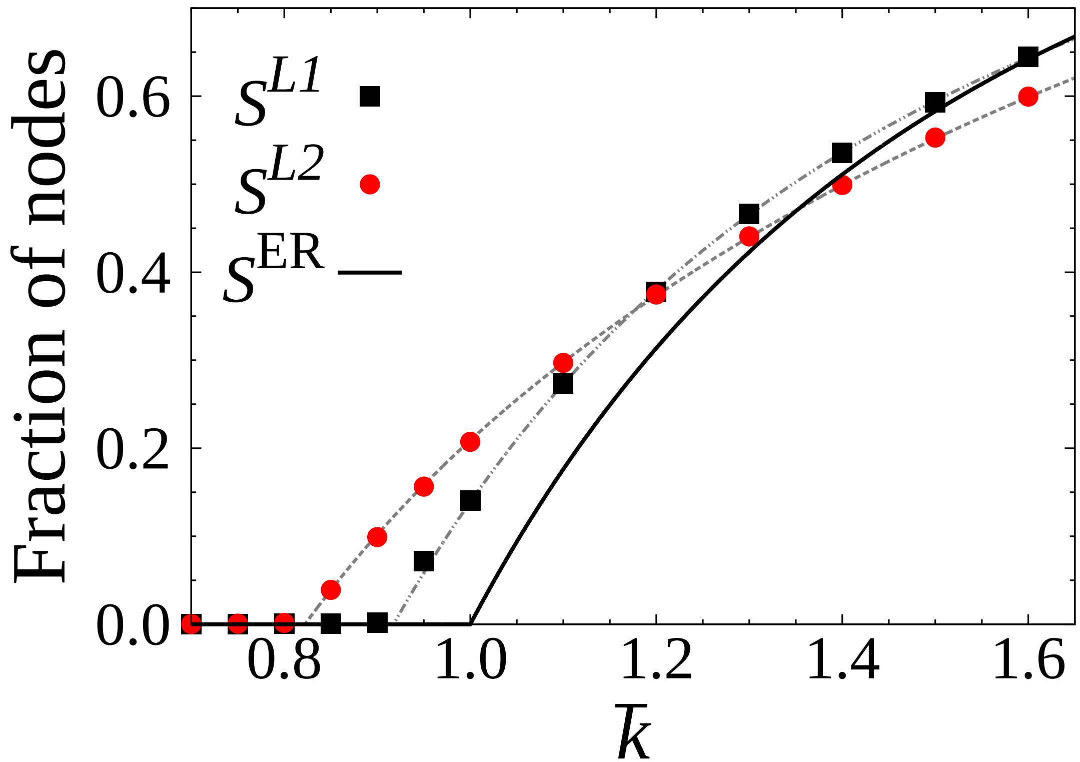
$$p^b_{k_a k_b} = \frac{\overline{k}_{bb}{}^{k_b} e^{-\overline{k}_{bb}}}{k_b!} \cdot \frac{\overline{k}_{ba}{}^{k_a} e^{-\overline{k}_{ba}}}{k_a!}$$

$$\boxed{\langle s_a \rangle_a = 1 + \frac{\overline{k}_{aa} - \overline{k}_{aa}\overline{k}_{bb} + \overline{k}_{ab}\overline{k}_{ba}}{(1-\overline{k}_{aa})(1-\overline{k}_{bb}) - \overline{k}_{ab}\overline{k}_{ba}}}$$

(Bollobas, Janson, Riordan *RSA*, 2007
Ostilli, Mendes *J Stat Mech* 2009)

Wiring which respects group structures percolates earlier!





aa rate λ

bb rate λ/r_1, w/ $r_1 = 2$

ab rate λ/r_2, w/ $r_2 = 6$

"Cooperative enhancement"

Other distributions

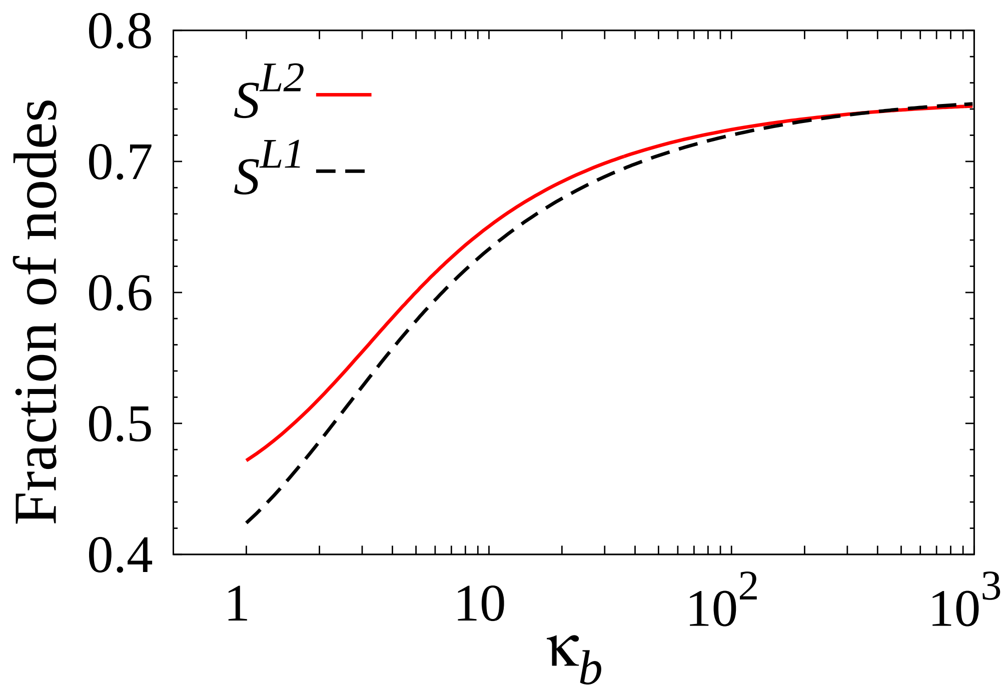
1. **Two interacting trunc PLs**

$p_k \sim k^{-\tau} \exp(-k/\kappa)$

a) $\tau_a = 2.8, \kappa_a = 100$

b) $\tau_b = 2.0, \kappa_b$ varied

ab) $p^a_{k_b} = p^b_{k_a}$ Poisson $\overline{k}_{ab} = 0.4$



- Approximates two loosely coupled human contact networks.
- Giant component is larger for $L2$.

Other distributions, cont.

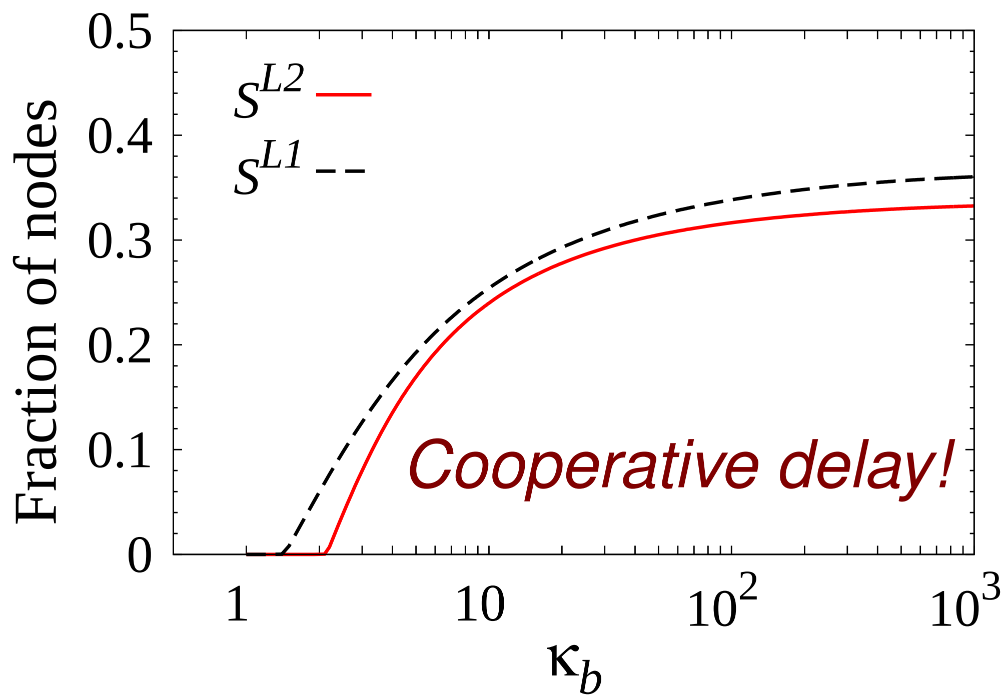
2. **Trunc PL and Poisson**

with $n_a = 4n_b$

a) Poisson, $\overline{k}_{aa} = 0.5$

b) $\tau_b = 2.5, \kappa_b$ varied

ab) $p^a_{k_b}$ Poisson $\overline{k}_{ab} = 0.2$

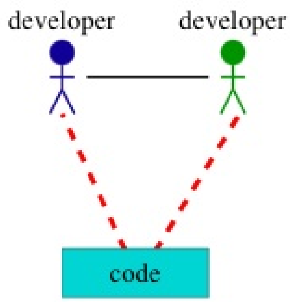


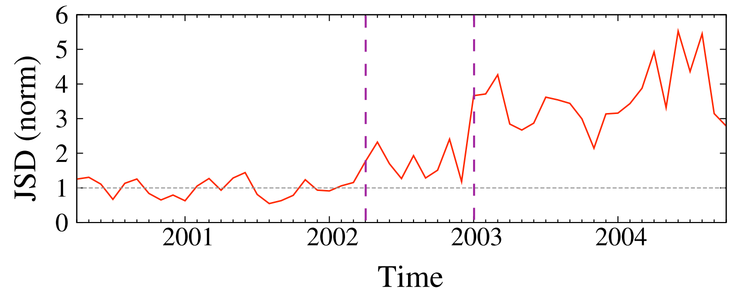
- Approximates critical infrastructure:
  - power grid nodes having narrow degree distribution.
  - Internet routers have broad degree distribution.

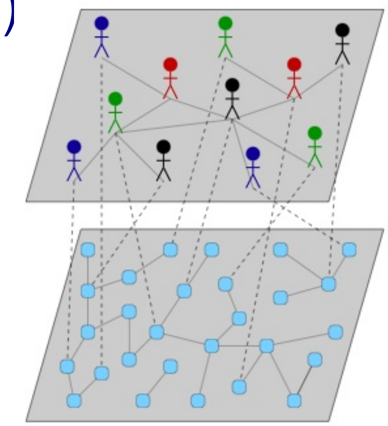
Statistical signatures of interacting networks:

(How and when do they differ from random or each-other?)

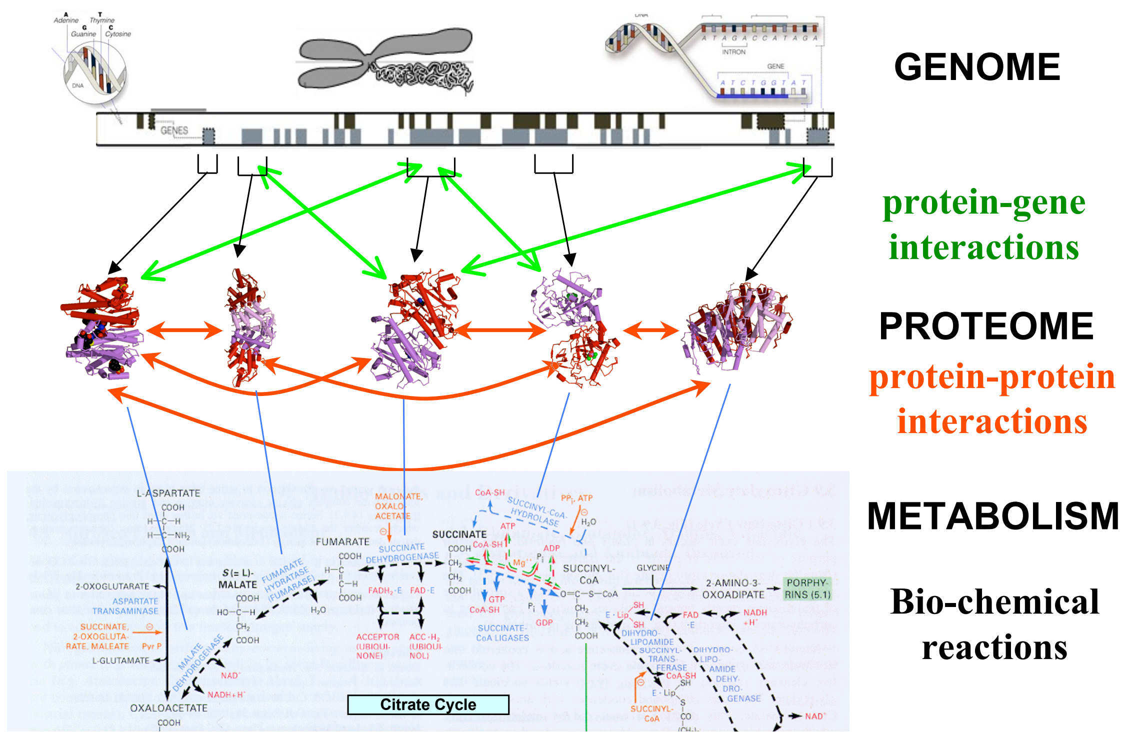
- **Socio-technical congruence** (e.g., Open Source Software)



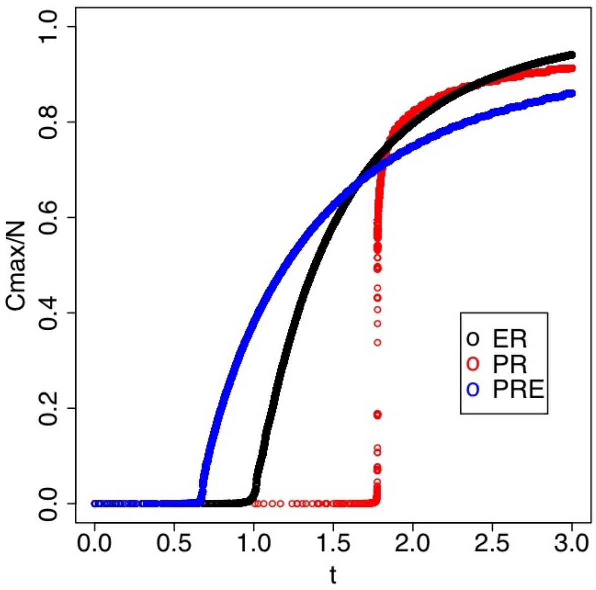




- **Searching for biomarkers of disease:**



Explosive percolation in random graphs – Conclusions



- Controlling phase transitions with choice
  – Delay or enhance
  – Changing speed of onset
  – Changing universality classes
  – Including locality

- Network interactions
  – Can change the onset of phase transitions
  – Modular treatments can percolate sooner
  (or later, depending on specifics of degree distributions involved)

- Random graphs and real-world networks
  – Can the differences serve as diagnostic tools?