# Pursuit & Collective Behavior

P. S. Krishnaprasad
University of Maryland

Joint with Kevin Galloway, Eric Justh
and Matteo Mischiati

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This is primarily joint work with my long-term collaborator Eric Justh and my Ph.D. students Kevin Galloway, and Matteo Mischiati.

Geometric methods in control theory have had a useful role in the investigation of problems of collective behavior. In this talk, we discuss recent progress in understanding nonlinear phenomena in small networks governed by feedback control laws that fall in the category of pursuit laws. We also examine applications of these ideas to biological networks.

## Bracken emergence

www.batcon.org

The question of what mechanisms underlie the ability of large columns of bats to emerge and hunt insect prey in a coherent and seemingly organized manner remains an open and interesting one. There are many superlatives associated with this – very large numbers, long duration of flight (dusk-to-dawn), altitude, quantity of insects consumed, jamming avoidance, etc.

# Theme

- In the animal kingdom, **Pursuit** is associated with play (competitive), predation (aggressive) and mating (non-cooperative); it can also be a **building block** of collaborative/social behavior
- This talk is about pursuit (in which labeled individuals, aware of labels, move according to a **cyclic directed graph** of interaction) and the resulting spatio-temporal patterns

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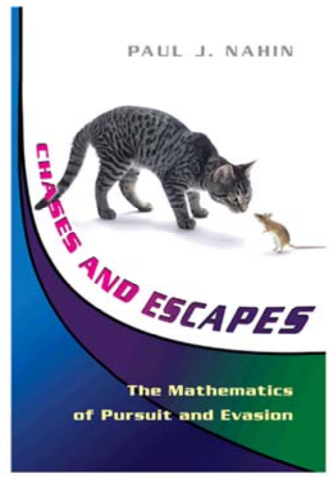
This talk is based on the idea that **pursuit** is a useful and perhaps essential **building block** for achieving **collective spatio-temporal behavior** in the animal kingdom, on many length scales. It is recognizable as a mechanism in a variety of individual behaviors including play, predation and mating. It is mediated by a variety of (active) perceptual modes, including vision, audition, biosonar, somato-sensation, and (collective) sensing of gradients of chemical or nutrient concentrations.

# Pursuit in the 19th century and before

Pursuit has an interesting early history in recreational mathematics.

The problem of a pirate ship chasing a merchant ship.

- Classical pursuit curve
- Cyclic pursuit
  - The problem of 3 bugs (mice, dogs,…)

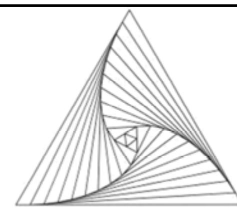



Classical pursuit stimulated a number of papers using calculus-based mathematics, many in the category of recreational mathematics. But it became serious business in WWII.

Paul Nahin’s excellent and entertaining book captures the mathematical spirit underlying classical pursuit, cyclic pursuit and other related modern developments.

The problem of 3 bugs each pursuing the one to its left under strict classical pursuit pre-occupied some writers. For initial positions at the vertices of a triangle (or regular polygon) and all bugs moving at constant speed, there is a common meeting point.

# Cyclic Pursuit



Klamkin, Newman (1971): Three bugs problem

Bruckstein, Cohen, Efrat (1991): Ants, crickets, frogs

Richardson (2001): Non-mutual capture

Lin, Broucke, Francis (2004): Cyclic pursuit to achieve multi-agent formations (linear formulation)

Marshall, Broucke, Francis (2004): Nonlinear dynamics (linear feedback law); proof of relative equilibria; local stability analysis

Smith, Broucke, Francis (2005): Introduces hierarchical scheme

Marshall, Broucke, Francis (2006): Incorporates speed control

Sinha, Ghose (2007): Heterogeneous formations (differing speeds and controller gains)

Smith, Broucke, Francis (2007): Application to curve-shortening and rendezvous problem

Pavone , Frazzoli (2007): Coverage

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Following the initial excitement over pursuit problems in the literature on recreational mathematics, a more modern view based on control-theoretic questions began to emerge. Bugs were replaced by robots.

# Outline

- Modeling interactions
- Strategies and steering laws
- Symmetry, shape, and reduction
- Special solutions
- Mutual CB pursuit
- Mutual MC Pursuit

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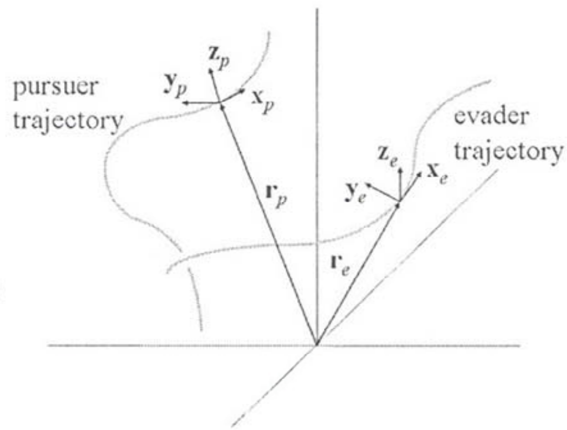
Graph-theoretic techniques are useful, but we focus in this talk on some nonlinear dynamics.

## Modeling Interactions in 3D

The **natural curvatures** are controls. The speeds are time functions dictated by propulsive/lift mechanisms.

$$\dot{\mathbf{r}}_e = \nu_e \mathbf{x}_e$$
$$\dot{\mathbf{x}}_e = \nu_e(\mathbf{y}_e u_e + \mathbf{z}_e \upsilon_e)$$
$$\dot{\mathbf{y}}_e = -\nu_e \mathbf{x}_e u_e$$
$$\dot{\mathbf{z}}_e = -\nu_e \mathbf{x}_e \upsilon_e$$

$$\dot{\mathbf{r}}_p = \nu_p \mathbf{x}_p$$
$$\dot{\mathbf{x}}_p = \nu_p(\mathbf{y}_p u_p + \mathbf{z}_p \upsilon_p)$$
$$\dot{\mathbf{y}}_p = -\nu_p \mathbf{x}_p u_p$$
$$\dot{\mathbf{z}}_p = -\nu_p \mathbf{x}_p \upsilon_p$$




$$r = r_p - r_e$$ baseline vector

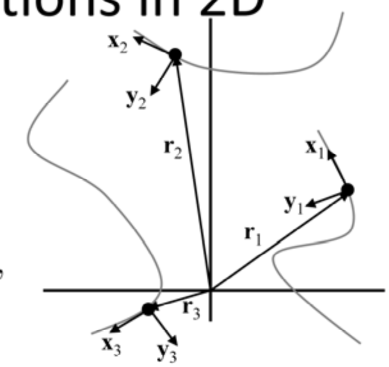
The flight behavior of a bat or a bird, or for that matter an insect is the end result of interaction between (visual, auditory, olfactory, somatosensory, and inertial) sensing, and actuation of a complex network of muscles, mediated by the rapid and learned responses of the neural control substrate. The overwhelming richness of detail present in this feedback loop and in the physics of a multiple-degrees-of-freedom animal needs to be **abstracted** to the right level in seeking answers to questions such as, - What individual behaviors govern collective cohesion? What is the structure of interaction between individuals within a collective? What organizations within a flock enable effective transmission of information across a flock? Based on the results of our prior work on 3D trajectory modeling and analysis of motion camouflage and echolocating bats (Justh and Krishnaprasad 2005; Reddy et. al. 2006, 2007; Reddy 2007; Wei et. al. 2009), we argue that a description with the right level of complexity for modeling an individual in a flock or a swarm is the Newtonian particle model.

The figure presents two particle trajectories as curves with frames, one for the evader/target (denoted as e) and one for the pursuer (denoted as p). The curvatures u and v are controls. The speeds denoted by Greek letter nu  are decided by propulsive/lift considerations.

This representation of individual trajectory dynamics is known as the natural frame representation, made better known through a well-known paper of R. L. Bishop (1975). Instead of writing Newton's equations as "ma = f" , we are making explicit the role of curvature/steering controls as inputs.

# Modeling interactions in 2D

$$\begin{aligned}\dot{\mathbf{r}}_i &= \nu_i \mathbf{x}_i,\\ \dot{\mathbf{x}}_i &= \nu_i \mathbf{y}_i u_i,\\ \dot{\mathbf{y}}_i &= -\nu_i \mathbf{x}_i u_i,\ i = 1, 2, \ldots, n,\end{aligned}$$




$$M_{state} = \Big\{ (\mathbf{r}_1, \mathbf{x}_1, \mathbf{y}_1, \ldots, \mathbf{r}_n, \mathbf{x}_n, \mathbf{y}_n) \in \mathbb{R}^{6n} \;\Big|\; \mathbf{r}_i \neq \mathbf{r}_{i+1},\ |\mathbf{x}_i| = 1,\ \mathbf{y}_i = \mathbf{x}_i^{\perp},\ i = 1, 2, \ldots, n \Big\}$$



The speed nu_i of each particle may be taken as a time function. Here it is treated as a constant = 1. The state space M_state is the product of n copies of the group SE(2) of rigid motions in the plane **minus** a set of collision states. Only collisions between particles with **consecutive labels** are excluded.

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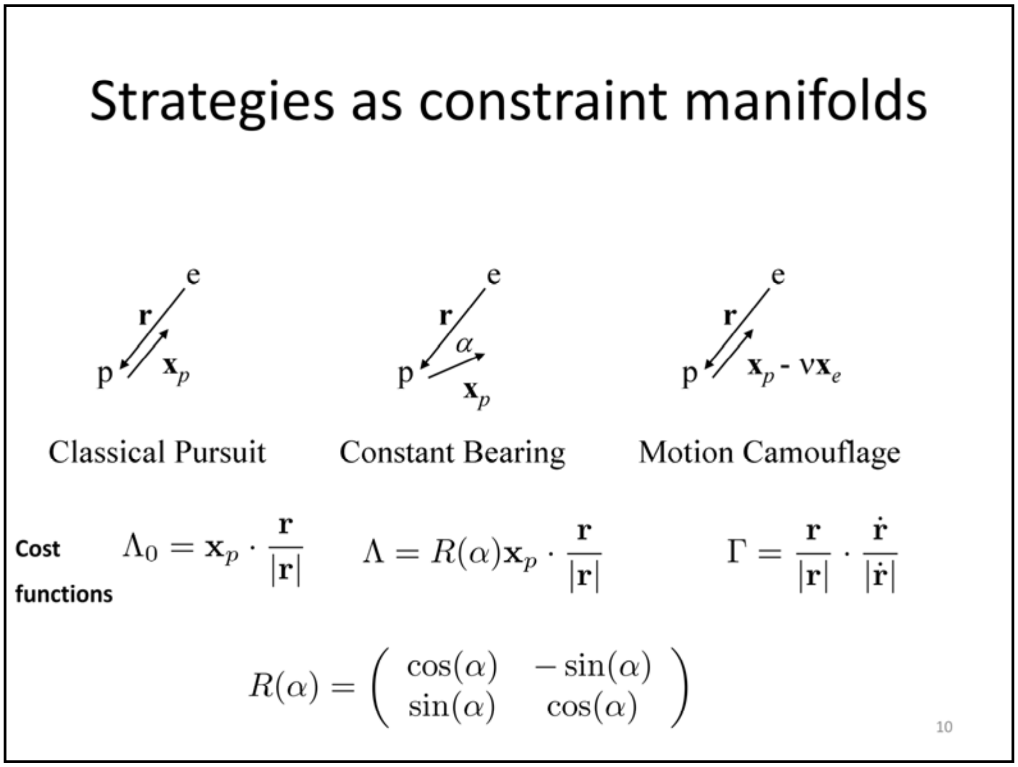



We define a control strategy as the **specification** of a constraint manifold in the joint state space of the pursuer (p) and the target (e). We suggest some typical pursuit strategies.

**Classical pursuit** is the constraint of heading straight for the target.

**Constant bearing pursuit** is heading for the target with a fixed lead or lag (angle alpha).

In **3D** we need a **cone condition**.

**Motion camouflage (with respect to infinity)** is a stealthy pursuit, nulling motion parallax, suggested by the trajectories of dragonflies.

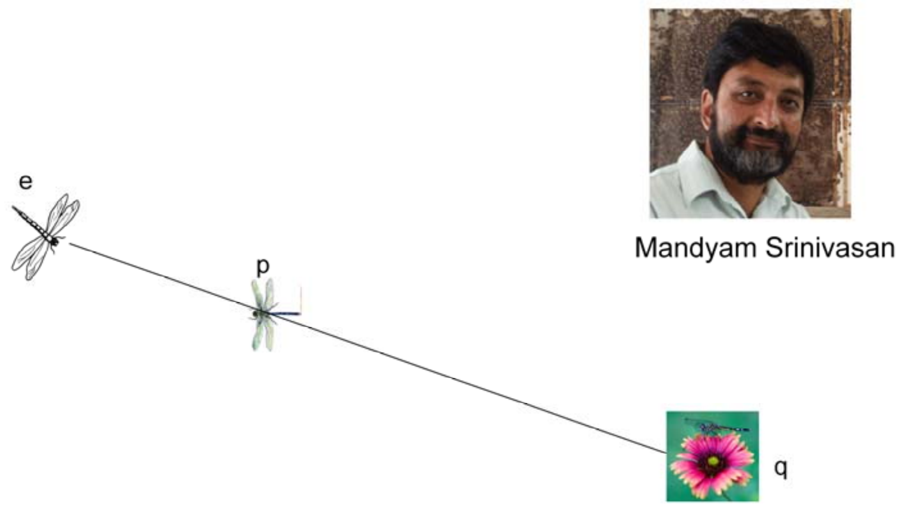
Motion camouflage with respect to infinity is the same as a strategy adopted by bats in pursuit of insects. In that context, we refer to it as the constant absolute target direction strategy (**CATD**).

A pursuer executes a feedback law that (approximately) fulfills the specification -

Pursuer reaches an **epsilon neighborhood** of a constraint manifold in finite time;

Pursuer converges to constraint manifold asymptotically.

# Motion Camouflage (MC)




Srinivasan and Davey (1995), Proc. Roy. Soc. Lond. B 259(1354):19-25
Mizutani, Chahl and Srinivasan (2003), Nature, 423:604

The pursuer (p) moves in such a way that the moving pursuee/target (e) thinks (p) is co-located with a familiar, stationary object (q). Motion camouflage at infinity refers to the case where the object (q) is at infinity. We idealize, p, e and q as points. The idea that insects execute such movement strategies was put forward by Srinivasan and Davey (hoverflies), Mizutani, Chahl and Srinivasan (dragonflies).

## Examples in Nature

- Dragonflies engaged in territorial battles (MC)
- Single echolocating bat hunting prey (MC)
  Ghose et. al. (2006), PLoS Biol., 4(5):865-873
- Two bats engaged in competition for prey (CP)
  Chiu (2008), Ph.D. thesis (advisor: C. Moss)
- Peregrine falcon stoop (CB)
  Tucker et. al. (2000), JEB, 203:3755–3763

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Recall from notes in previous two slides.

The story about single echolocating bat is based on laboratory studies (Cynthia Moss, Kaushik Ghose, Timothy Horiuchi) and analysis (Viswanadha Reddy, Eric Justh, and PSK) – in PLoS Biology (2006) and Reddy's M.S. thesis (2007).

The story about two bats engaged in competition for prey was first investigated in the Ph.D. thesis of Chen Chiu (2008). Recent analysis, submitted to the Journal of Experimental Biology, reveals that a CP **constraint** appears to hold during the periods when a bat **follows** another bat instead of heading towards a mealworm.

The peregrine falcon has two foveae (deep and shallow). The deep fovea has higher acuity while the shallow fovea is equipped for stereopsis. In the stoop maneuver, initially the deep fovea is used, later trading acuity for target localization. The spiral stoop is consistent with **CB pursuit** of a fixed target. Aerodynamic drag considerations also are involved.

## Steering laws for pursuit

**Classical Pursuit:**

$$u_{CP} = -\mu \left( \mathbf{y}_p \cdot \frac{\mathbf{r}}{|\mathbf{r}|} \right) - \frac{1}{|\mathbf{r}|} \left( \frac{\mathbf{r}}{|\mathbf{r}|} \cdot \dot{\mathbf{r}}^{\perp} \right)$$

$$\text{Drives } \Lambda_0 \longrightarrow -1$$

**Constant Bearing:**

$$u_{CB(\alpha)} = -\mu \left( R(\alpha)\mathbf{y}_p \cdot \frac{\mathbf{r}}{|\mathbf{r}|} \right) - \frac{1}{|\mathbf{r}|} \left( \frac{\mathbf{r}}{|\mathbf{r}|} \cdot \dot{\mathbf{r}}^{\perp} \right)$$

$$\text{Drives } \Lambda \longrightarrow -1$$

**Motion Camouflage:**

$$u_{MC} = -\mu \left( \frac{\mathbf{r}}{|\mathbf{r}|} \cdot \dot{\mathbf{r}}^{\perp} \right)$$

$$\text{Drives } \Gamma \longrightarrow -1 + \varepsilon$$

E. Wei, E.W. Justh, and P.S. Krishnaprasad, "Pursuit and an evolutionary game", *Proc. R. Soc. A (London)*, Vol. 465, pp. 1539-1559, May 2009.
P.V. Reddy, E.W. Justh, and P.S. Krishnaprasad, "Motion camouflage in three dimensions", *Proc. 45th IEEE Conf. Decision and Control,* pp. 3327-3332, 2006.

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The control laws for classical pursuit and constant bearing pursuit include in them a (second) term that corresponds to motion camouflage. For a **faster** pursuer, these two laws drive the state asymptotically to the specified constraint manifold.; the control law for motion camouflage drives the state to an epsilon neighborhood of the states satisfying the motion camouflage (with respect to infinity) constraint.

Feedback laws that incorporate sensorimotor delays are also known and have been analyzed
(Reddy, Justh and Krishnaprasad (2007), Proc. 46th IEEE Conference on Decision and Control).

Extensions to three dimensions are known. (Galloway, Justh and Krishnaprasad, 2010, submitted).

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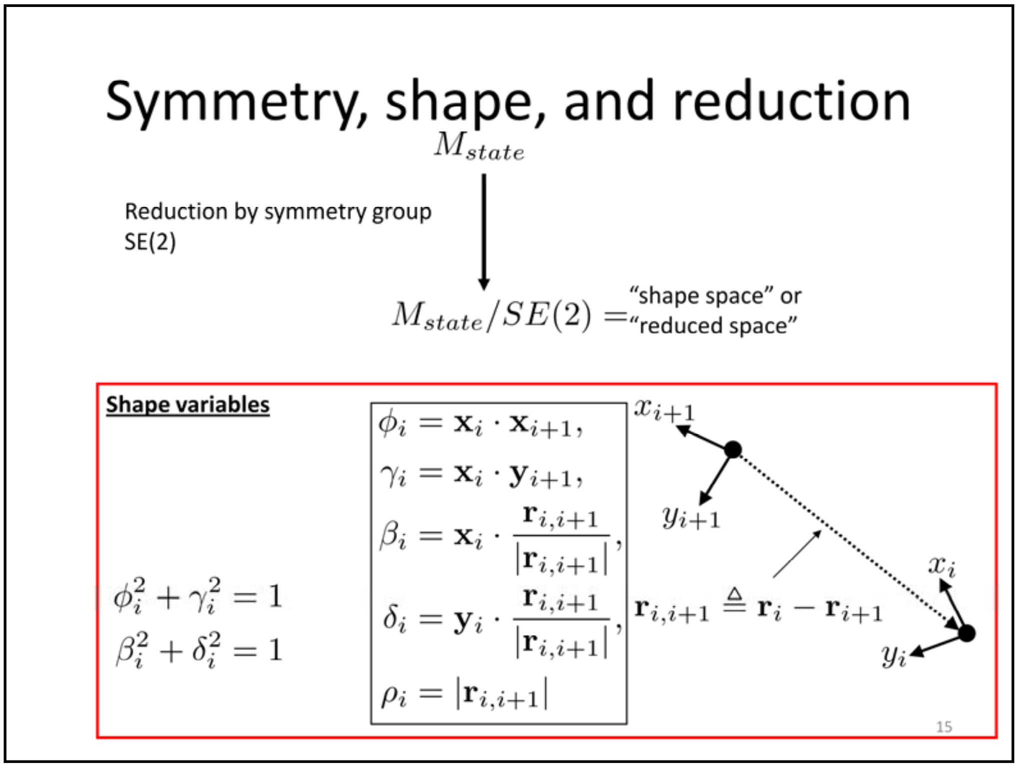


With the pursuit laws for CB and MC, the closed loop dynamics is invariant under the diagonal action of the rigid motion group SE(2); hence one can reduce to the quotient manifold or shape space. The given shape variables give a complete description of the shape space.

# Shape dynamics

For any SE(2)-invariant steering law $u_i$

$$
\begin{aligned}
\dot{\phi}_i &= -\gamma_i(u_i - u_{i+1}), \\
\dot{\gamma}_i &= \phi_i(u_i - u_{i+1}), \\
\dot{\beta}_i &= u_i\delta_i + \frac{1}{\rho_i}\left[\delta_i^2(1-\phi_i) - \beta_i\gamma_i\delta_i)\right], \\
\dot{\delta}_i &= -u_i\beta_i + \frac{1}{\rho_i}\left[\gamma_i\beta_i^2 - \beta_i\delta_i\,(1-\phi_i)\right], \\
\dot{\rho}_i &= \beta_i\,(1-\phi_i) + \gamma_i\delta_i, \quad i = 1, 2, \ldots, n,
\end{aligned}
$$

Closed-loop dynamics under CB cyclic pursuit

$$
\begin{aligned}
\dot{\phi}_i &= -\gamma_i(u_{CB(\alpha_i)} - u_{CB(\alpha_{i+1})}), \\
\dot{\gamma}_i &= \phi_i(u_{CB(\alpha_i)} - u_{CB(\alpha_{i+1})}), \\
\dot{\beta}_i &= \mu_i\left[\sin(\alpha_i)\beta_i\delta_i - \cos(\alpha_i)\delta_i^2\right], \\
\dot{\delta}_i &= \mu_i\left[\cos(\alpha_i)\beta_i\delta_i - \sin(\alpha_i)\beta_i^2\right], \\
\dot{\rho}_i &= \beta_i\,(1-\phi_i) + \gamma_i\delta_i, \quad i = 1, 2, \ldots, n,
\end{aligned}
$$

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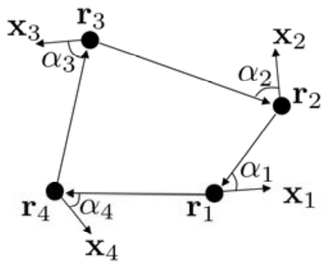
Here we use the expressions for CB cost functions and steering laws obtained from expressing various dot products in terms of shape variables.

# An invariant submanifold

$$M_{Joint\ CB^-(\alpha)} = \Big\{(\mathbf{r}_1, \mathbf{x}_1, \mathbf{y}_1, \ldots, \mathbf{r}_n, \mathbf{x}_n, \mathbf{y}_n) \in M_{state} \ \Big| \ \Lambda_i = -1, i = 1, 2, \ldots, n\Big\}$$




$$\Lambda_i = R(\alpha_i)\mathbf{x}_i \cdot \frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|}$$

$$= \boxed{\cos(\alpha_i)\beta_i + \sin(\alpha_i)\delta_i}$$

**Proposition**: $M_{Joint\ CB^-(\alpha)}$ is invariant under the CB closed-loop dynamics.
(Proof follows from the fact that $\dot{\Lambda}_i = -\mu_i\left(1 - \Lambda_i^2\right)$ under CB closed-loop dynamics.)

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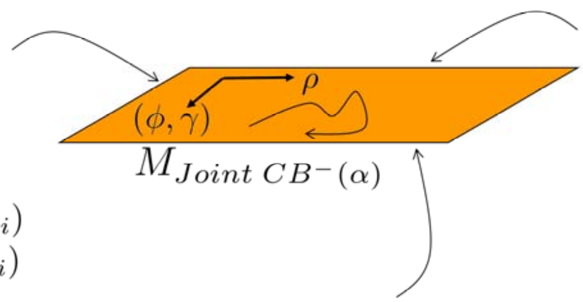
# Reduced dynamics on $M_{Joint\ CB^{-}}(\alpha)$

$$\dot{\phi}_i = -\gamma_i \left[ \frac{1}{\rho_i} \Big( (1-\phi_i)\sin(\alpha_i) - \gamma_i \cos(\alpha_i) \Big) - \frac{1}{\rho_{i+1}} \Big( (1-\phi_{i+1})\sin(\alpha_{i+1}) - \gamma_{i+1}\cos(\alpha_{i+1}) \Big) \right],$$
$$\dot{\gamma}_i = \phi_i \left[ \frac{1}{\rho_i} \Big( (1-\phi_i)\sin(\alpha_i) - \gamma_i \cos(\alpha_i) \Big) - \frac{1}{\rho_{i+1}} \Big( (1-\phi_{i+1})\sin(\alpha_{i+1}) - \gamma_{i+1}\cos(\alpha_{i+1}) \Big) \right],$$
$$\dot{\beta}_i = 0,$$
$$\dot{\delta}_i = 0,$$
$$\dot{\rho}_i = -(1-\phi_i)\cos(\alpha_i) - \gamma_i \sin(\alpha_i), \quad i = 1, 2, \ldots, n.$$




$$\beta_i \equiv -\cos(\alpha_i)$$
$$\delta_i \equiv -\sin(\alpha_i)$$

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Richardson’s dynamics are confined to the CP manifold defined by setting all the alpha’s = 0.

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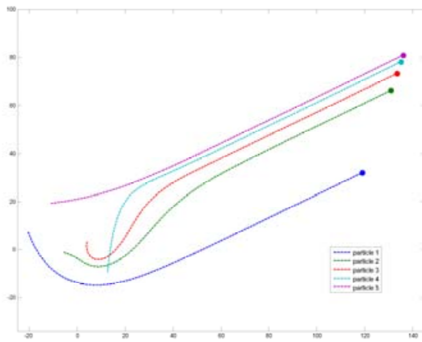
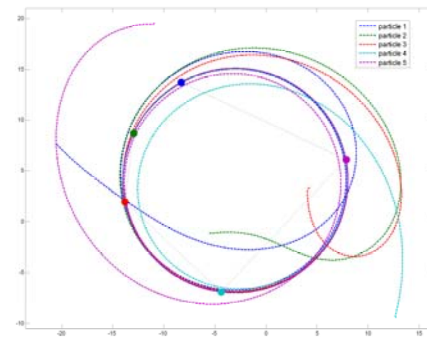
# Relative equilibria

| Rectilinear | Circling |
|---|---|
|  |  |
| $\alpha_1 = \pi/4$ | $\alpha_1 = \pi/10$ |
| $\alpha_2 = \pi/4$ | $\alpha_2 = \pi/10$ |
| $\alpha_3 = \pi/4$ | $\alpha_3 = \pi/5$ |
| $\alpha_4 = \pi/4$ | $\alpha_4 = 3\pi/10$ |
| $\alpha_5 = 5\pi/4$ | $\alpha_5 = 3\pi/10$ |

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Equilibria of the shape dynamics correspond to relative equilibria of the full system dynamics. Systems of the form discussed in this paper admit only two types of relative equilibria: rectilinear and circling. It is not automatic that relative equilibria of particular type exist under **CB (alpha)** pursuit. The choice of **alphas** dictates which if any relative equilibria are selected.

# Existence of relative equilibria

**Proposition**: Given $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, a relative equilibrium corresponding to **rectilinear** motion on $M_{Joint\ CB^-(\alpha)}$ exists for the given closed-loop dynamics under $u_{CB(\alpha)}$ *if and only if* there exists a set of constants $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ such that $\sigma_i > 0,\ i = 1, 2, \ldots, n$ and

$$\sum_{i=1}^{n} \sigma_i e^{j(\alpha_i)} = 0.$$

**Proposition**: Given $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, a relative equilibrium corresponding to **circling** motion on a common orbit on $M_{Joint\ CB^-(\alpha)}$ exists for the given closed-loop dynamics under $u_{CB(\alpha)}$ *if and only if*

i. $\sin(\alpha_i) > 0\ \ \forall i \in \{1, 2, \ldots, n\}$ or $\sin(\alpha_i) < 0\ \ \forall i \in \{1, 2, \ldots, n\}$,

ii. $\sin\left(\sum_{i=1}^{n} \alpha_i\right) = 0.$

Alpha = multibearing (alpha_1, alpha_2,…,alpha_n)

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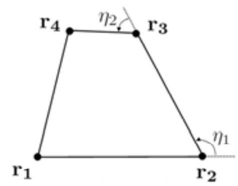
# Concepts of “shape”

1. Relative states (i.e. positions and velocities) of particles (Justh, PSK)
2. Relative configuration of particle positions with a concept of size (Jacobi)
3. Relative configuration of particle positions without a concept of absolute size (Kendall)




$$\bar{\rho}_i \triangleq \frac{|\mathbf{r}_i - \mathbf{r}_{i+1}|}{|\mathbf{r}_{i+1} - \mathbf{r}_{i+2}|} = \frac{\rho_i}{\rho_{i+1}},$$

Fumin Zhang, “Geometric Cooperative Control of Formations”, PhD thesis, 2004.

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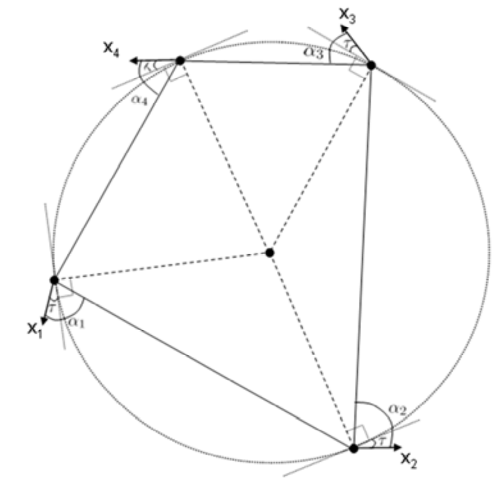
Here we wish to make contact with other notions of shape. The basic ideas in the context of the n-body problem go back to Jacobi, who gave us the **Jacobi vector parametrization**. Statisticians prefer to separate absolute size from shape. Recall congruent triangles (same triple of angles, same ratios of side lengths).

# Other special solutions

-Invariant "pure shape", but varying size



# Invariant pure shape

**Proposition**: Assume $\exists$ a constant angle $\tau \in [0, 2\pi)$ such that $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ satisfy the following properties:

i. $\sin(\alpha_i - \tau) > 0 \;\; \forall i \in \{1, 2, \ldots, n\}$
or $\sin(\alpha_i - \tau) < 0 \;\; \forall i \in \{1, 2, \ldots, n\}$

ii. $\sin\left(-n\tau + \sum_{i=1}^{n} \alpha_i\right) = 0.$

Let a particular set of pure shape coordinates be specified by

$$\eta_i^* = \alpha_i + \alpha_{i+1} - 2\tau, \;\; \bar{\rho}_i^* = \frac{\sin(\alpha_i - \tau)}{\sin(\alpha_{i+1} - \tau)}, \; i = 1, 2, \ldots, n-2.$$

Then the submanifold given by $M^{\eta^*, \bar{\rho}^*} \bigcap M_{Joint\ CB^-(\alpha)}$ is invariant under the CB cyclic pursuit. Furthermore, the polygon shapes are cyclic.

# n=2 case (mutual CB pursuit)

By assignment

$$\phi = \phi_1 = \phi_2$$
$$\gamma = \gamma_1 = -\gamma_2$$
$$\rho = \rho_1 = \rho_2$$

$$\phi = -\beta_1\beta_2 - \delta_1\delta_2$$
$$\gamma = -\beta_1\delta_2 + \delta_1\beta_2$$

Derived using basis expansion

$$\dot{\beta}_1 = \mu_1\left[\sin(\alpha_1)\beta_1\delta_1 - \cos(\alpha_1)\delta_1^2\right]$$
$$\dot{\delta}_1 = \mu_1\left[\cos(\alpha_1)\beta_1\delta_1 - \sin(\alpha_1)\beta_1^2\right]$$
$$\dot{\beta}_2 = \mu_2\left[\sin(\alpha_2)\beta_2\delta_2 - \cos(\alpha_2)\delta_2^2\right]$$
$$\dot{\delta}_2 = \mu_2\left[\cos(\alpha_2)\beta_2\delta_2 - \sin(\alpha_2)\beta_2^2\right]$$
$$\dot{\rho} = \beta_1 + \beta_2$$

**Simplified two-particle dynamics**

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# Stability analysis for n=2 case

**Proposition**: For $i = 1, 2$, the point $(\beta_i, \delta_i) = -\Big(\cos(\alpha_i), \sin(\alpha_i)\Big)$ is an asymptotically stable equilibrium point for the subsystem $\Big(\dot{\beta}_i, \dot{\delta}_i\Big)$, with region of convergence given by $\Big\{(\beta_i, \delta_i) \in S^1 \Big| (\beta_i, \delta_i) \neq \Big(\cos(\alpha_i), \sin(\alpha_i)\Big)\Big\}$.

⇒ Implies asymptotic convergence to $M_{Joint\ CB^-(\alpha)}$

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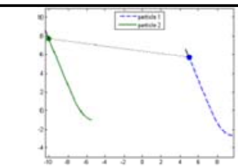
# Asymptotic system behavior for n=2

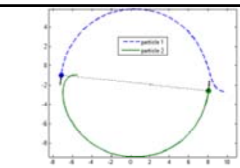
On $M_{Joint\ CB^-(\alpha)}$, we have

$$\dot\rho(t) = -\left[\cos(\alpha_1) + \cos(\alpha_2)\right]$$
$$u(t) = \frac{1}{\rho(t)}\left[\sin(\alpha_1) + \sin(\alpha_2)\right].$$

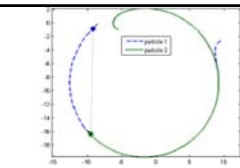
| **Case** | $\dot\rho(t);\ u(t)$ | $e^{j\alpha_1} + e^{j\alpha_2}$ | **Description** |
|---|---|---|---|
| I. | $\dot\rho(t) \equiv 0;\ u(t) \equiv 0$ | 0 | rectilinear equilibrium |
| II. | $\dot\rho(t) \equiv 0;\ u(t) > 0$ | positive imaginary axis | CCW circling equilibrium |
| III. | $\dot\rho(t) \equiv 0;\ u(t) < 0$ | negative imaginary axis | CW circling equilibrium |
| IV. | $\dot\rho(t) < 0;\ u(t) \equiv 0$ | positive real axis | "straight flight rendezvous" |
| V. | $\dot\rho(t) > 0;\ u(t) \equiv 0$ | negative real axis | "straight flight retreat" |
| VI. | $\dot\rho(t) < 0;\ u(t) > 0$ | quadrant I | CCW "inward sweep" |
| VII. | $\dot\rho(t) > 0;\ u(t) > 0$ | quadrant II | CCW "outward sweep" |
| VIII. | $\dot\rho(t) > 0;\ u(t) < 0$ | quadrant III | CW "outward sweep" |
| IX. | $\dot\rho(t) < 0;\ u(t) < 0$ | quadrant IV | CW "inward sweep" |



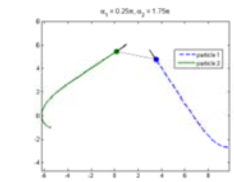
(a) Case I: Rectilinear equilibrium $(\pi/3,\ 4\pi/3)$



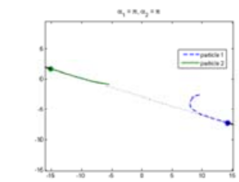
(b) Case II: CCW circling equilibrium $(\pi/2,\ \pi/2)$



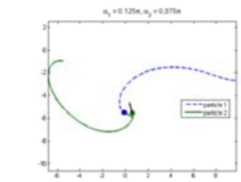
(c) Case III: CW circling equilibrium $(5\pi/4,\ 7\pi/4)$



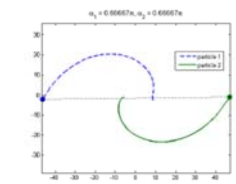
(d) Case IV: "Straight flight rendezvous" $(\pi/4,\ 7\pi/4)$



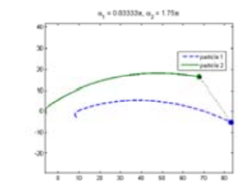
(e) Case V: "Straight flight retreat" $(\pi,\ \pi)$



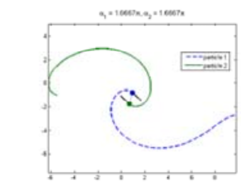
(f) Case VI: CCW "inward sweep" $(\pi/8,\ 3\pi/8)$



(g) Case VII: CCW "outward sweep" $(2\pi/3,\ 2\pi/3)$



(h) Case VIII: CW "outward sweep" $(5\pi/6,\ 7\pi/4)$



(i) Case IX: CW "inward sweep" $(5\pi/3,\ 5\pi/3)$

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# Remark

- Stability of many particle version of CB pursuit and the 3D setting are being investigated. In the 3D setting CB pursuit is parametrised by cone angles. Double helices and conchospirals arise in the 2 particle case.

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# Outline

- Modeling interactions
- Strategies and steering laws
- Symmetry, shape, and reduction
- Special solutions
- Mutual CB pursuit
- Mutual MC Pursuit

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# Dynamics in Mutual MC Pursuit

$$\dot{r} = g$$

$$\dot{g} = uh$$

$$\dot{h} = -ug$$

$Here \quad g = x_p - \nu x_e \quad and \quad h = y_p - \nu y_e.$

$Let \quad \lambda = \frac{r}{|r|} \bullet h \quad \gamma = \frac{r}{|r|} \bullet g \quad and \quad \rho = |r|.$

$Note \quad u = -\mu\lambda.$

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# Symmetries and Reduction

$$\dot{\rho} = \gamma$$

$$\dot{\gamma} = (\frac{1}{\rho} - \mu)(\delta^2 - \gamma^2)$$

*Here we have used the conservation law* $\gamma^2 + \lambda^2 \equiv \delta^2$.

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# Discrete Symmetry

The system is reversible under the involution

$$(\rho, \gamma) \mapsto (\rho, -\gamma)$$

In fact, Birkhoff's theorem applies, and all orbits are periodic.
The "energy integral"

$$E(\rho, \gamma) = \rho^2 (\delta^2 - \gamma^2) \exp(-2\mu\rho)$$

implies a Poisson bracket

$$\{\rho, \gamma\} = -\frac{\exp(2\mu\rho)}{2\rho^2}$$

G. D. Birkhoff (1915), The restricted problem of three bodies, *Rend. Circ. Mat. Palermo*, vol. 39, 265-334.

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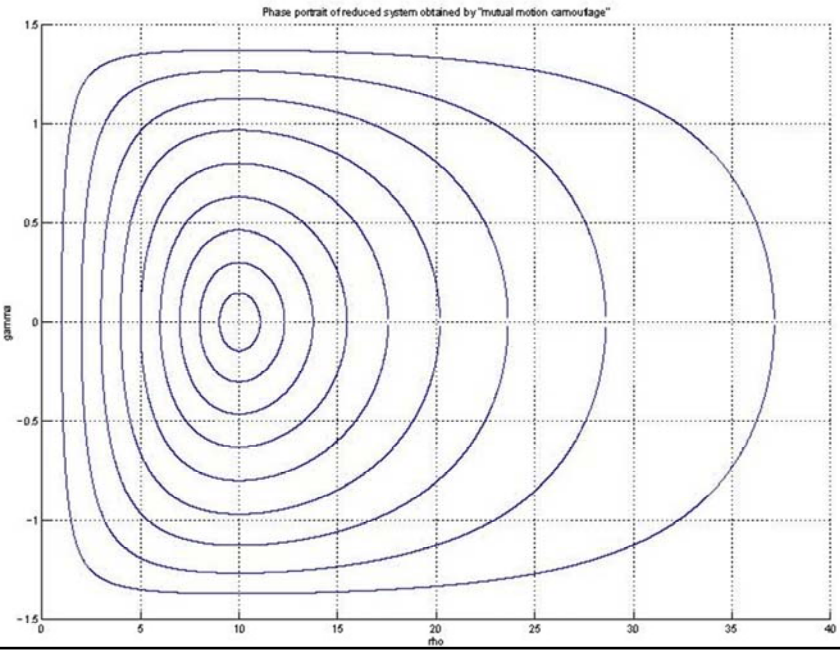
Birkhoff : Given a system reversible under an involution (i.e. the push forward of the dynamics under the involution is the time reversal of the dynamics), and S is the fixed point set of the involution, if an orbit through a point of the set S has the property that it intersects S at another point , then it is a periodic orbit.

# Phase Portrait of Reduced System

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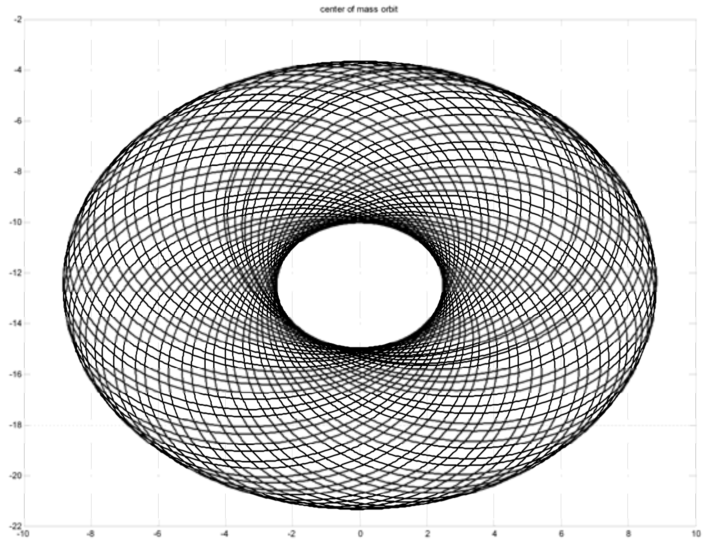
# Center of Mass Orbit



# References & Support

K. Galloway, E. W. Justh and P. S. Krishnaprasad (2009), "Geometry of Cyclic Pursuit," *Proc. 48th IEEE Conference on Decision and Control*, 7485-7490.

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