PCM QUANTIZATION ERRORS AND THE WHITE NOISE HYPOTHESIS

DAVID JIMENEZ, LONG WANG, AND YANG WANG

ABSTRACT. The White Noise Hypothesis (WNH), introduced by Bennett half century ago, assumes that in the pulse code modulation (PCM) quantization scheme the errors in individual channels behave like white noise, i.e. they are independent and identically distributed random variables. The WNH is key to estimating the means square quantization error (MSE). But is the WNH valid? In this paper we take a close look at the WNH. We show that in a redundant system the errors from individual channels can never be independent. Thus to an extend the WNH is invalid. Our numerical experients also indicate that with coarse quantization the WNH is far from being valid. However, as the main result of this paper we show that with fine quantizations the WNH is essentially valid, in which the errors from individual channels become asymptotically *pairwise* independent, each uniformly distributed in $[-\Delta/2, \Delta/2)$, where Δ denotes the stepsize of the quantization.

1. Introduction

In processing, analysing and storaging of analog signals it is often necessary to make atomic decompositions of the signal using a given set of *atoms*, or *basis* $\{\mathbf{v}_j\}$. With the basis, a signal $\mathbf{x}$ is represented as a linear combination of $\{\mathbf{v}_j\}$,

$$\mathbf{x} = \sum_j c_j \mathbf{v}_j.$$

In practice $\{\mathbf{v}_j\}$ is a finite set. Furthermore, for the purpose of error correction, recovery from data erasures or robustness, redundancy is built into $\{\mathbf{v}_j\}$, i.e. more elements than needed are in $\{\mathbf{v}_j\}$. Instead of a true basis, $\{\mathbf{v}_j\}$ is chosen to be a frame. Since $\{\mathbf{v}_j\}$ is a finite set, we may without loss of generality assume $\{\mathbf{v}_j\}_{j=1}^N$ are vectors in $\mathbb{R}^d$ with $N \geq d$.

Let $F = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N]$ be the $d \times N$ matrix whose columns are $\mathbf{v}_1, \ldots, \mathbf{v}_N$. We say $\{\mathbf{v}_j\}_{j=1}^N$ is a *frame* if F has rank d. Let $\lambda_{\max} \geq \lambda_{\min} > 0$ be the maximal and minimal

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eigenvalues of FF^T, respectively. It is easily checked that

$$\lambda_{\min}\|\mathbf{x}\|^2 \le \sum_{j=1}^{N} |\mathbf{x}\cdot\mathbf{v}_j|^2 \le \lambda_{\max}\|\mathbf{x}\|^2. \tag{1.1}$$

$\lambda_{\max}$ and $\lambda_{\min}$ are called the *upper and lower frame bounds* for the frame, respectively. If $\lambda_{\max} = \lambda_{\min} = \lambda$, in which case $FF^T = \lambda I_d$, we call $\{\mathbf{v}_j\}_{j=1}^N$ a *tight frame* with frame bound λ. Note that any signal $\mathbf{x} \in \mathbb{R}^d$ can be easily reconstructed using the data $\{\mathbf{x}\cdot\mathbf{v}_j\}_{j=1}^N$. Set $\mathbf{y} = [\mathbf{x}\cdot\mathbf{v}_1, \mathbf{x}\cdot\mathbf{v}_2, \cdots, \mathbf{x}\cdot\mathbf{v}_N]^T$. Then $\mathbf{y} = F^T\mathbf{x}$ and

$$(FF^T)^{-1}F\mathbf{y} = (FF^T)^{-1}FF^T\mathbf{x} = \mathbf{x}.$$

Let $G = (FF^T)^{-1}F = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N]$. The set of columns $\{\mathbf{u}_j\}_{j=1}^N$ of G is called the *canonical dual frame* of the frame $\{\mathbf{v}_j\}_{j=1}^N$. We have the reconstruction

$$\mathbf{x} = \sum_{j=1}^{N} (\mathbf{x}\cdot\mathbf{v}_j)\,\mathbf{u}_j. \tag{1.2}$$

If $\{\mathbf{v}_j\}_{j=1}^N$ is a tight frame with frame bound λ, then $G = \lambda^{-1}F$, and we have the reconstruction

$$\mathbf{x} = \frac{1}{\lambda}\sum_{j=1}^{N} (\mathbf{x}\cdot\mathbf{v}_j)\,\mathbf{v}_j. \tag{1.3}$$

In digital applications, quantizations will have to be performed. The simplest scheme is the Pulse Code Modulation (PCM) quantization scheme, in which the coefficients $\{\mathbf{x}\cdot\mathbf{v}_j\}_{j=1}^N$ are quantized. In this paper we consider exclusively *linear quantizations*. Let $\mathcal{A} = \Delta\mathbb{Z}$ where $\Delta > 0$ is the quantization step. With linear quantization a real value t is replaced with the value in $\mathcal{A}$ that is the closest to t. So, in our setting, t is replaced with $Q_\Delta(t)$ given by

$$Q_\Delta(t) := \left\lfloor \frac{t}{\Delta} + \frac{1}{2} \right\rfloor \Delta.$$

Thus, given a frame $\{\mathbf{v}_j\}_{j=1}^N$ and its canonical dual frame $\{\mathbf{u}_j\}_{j=1}^N$, instead of using the data $\{\mathbf{x}\cdot\mathbf{v}_j\}_{j=1}^N$ and (1.2) to obtain a perfect reconstruction, we use the data $\{Q_\Delta(\mathbf{x}\cdot\mathbf{v}_j)\}_{j=1}^N$ and obtain an imperfect reconstruction

$$\widehat{\mathbf{x}} = \sum_{j=1}^{N} Q_\Delta(\mathbf{x}\cdot\mathbf{v}_j)\,\mathbf{u}_j. \tag{1.4}$$

This raises the following question: How good is the reconstruction? This question has been studied in terms of both the worst case error and the mean square error (**MSE**), see e.g. [12].

Note that the error from the reconstruction is

$$\mathbf{x} - \widehat{\mathbf{x}} = \sum_{j=1}^{N} \tau_\Delta \left(\mathbf{x} \cdot \mathbf{v}_j\right) \mathbf{u}_j, \tag{1.5}$$

where $\tau_\Delta(t) := t - Q_\Delta(t) = \left(\left\{\frac{t}{\Delta} + \frac{1}{2}\right\} - \frac{1}{2}\right)\Delta$, with $\{\cdot\}$ denoting the fractional part. While the *a priori* error bound is relatively straightforward to obtain, the *mean square error* $\mathbf{MSE} := \mathcal{E}\left(\|\mathbf{x} - \widehat{\mathbf{x}}\|^2\right)$, assuming certain probability distribution for $\mathbf{x}$, is much harder. To simplify the problem, the so-called *White Noise Hypothesis* (**WNH**), originally introduced in [4], is employed by engineers and mathematicians in this area. The **WNH** asserts the following:

- Each $\tau_\Delta\left(\mathbf{x} \cdot \mathbf{v}_j\right)$ is uniformly distributed in $[-\Delta/2, \Delta/2)$; hence it has mean 0 and variance $\Delta^2/12$.
- $\{\tau_\Delta\left(\mathbf{x} \cdot \mathbf{v}_j\right)\}_{j=1}^N$ are independent random variables.

With the **WNH** the **MSE** is easily shown to be

$$\mathcal{E}\left(\|\mathbf{x} - \widehat{\mathbf{x}}\|^2\right) = \frac{\Delta^2}{12}\sum_{j=1}^{d} \lambda_j^{-1} = \frac{\Delta^2}{12}\sum_{j=1}^{N} \|\mathbf{u}_j\|^2. \tag{1.6}$$

where $\{\lambda_j\}$ are the eigenvalues of FF^T.

But surprisingly there has not been any study on the legitimacy of the **WNH**, especially considering the fact that it is made under very general settings, where both the frame $\{\mathbf{v}_j\}_{j=1}^N$ and the probability distribution of $\mathbf{x} \in \mathbb{R}^d$ can take on numerous possibilities. Thus, the **WNH** deserves a closer scrutiny, which is what this paper intends to do.

We prove in this paper that under the assumption that the distribution of $\mathbf{x}$ has a density (absolutely continuous), the components of the quantization errors $\{\tau_\Delta\left(\mathbf{x} \cdot \mathbf{v}_j\right)\}_{j=1}^N$ can *never* be independent if $N > d$, and thus the **WNH** can never hold. However, our main result is that of a vindication of the **WNH**. We show that as $\Delta \to 0^+$, $\{\tau_\Delta\left(\mathbf{x} \cdot \mathbf{v}_j\right)\}_{j=1}^N$ becomes asymptotically *pairwise independent* and thus *pairwise uncorrelated*, as long as $\mathbf{v}_i$ is not parrallel to $\mathbf{v}_j$ for any $i \neq j$. Additionally each $\tau_\Delta\left(\mathbf{x} \cdot \mathbf{v}_j\right)$ indeed becomes asymptotically uniformly distributed on $[-\Delta/2, \Delta/2]$. These slightly weaker properties are sufficient to lead to the **MSE** given by (1.6) asymptotically. We also characterize the asymptotic behavior of the **MSE** if some vectors are parallel. These and other results are stated and proved in subsequent sections.

2. *A Priori* Error Bound and MSE under the WNH

In this section we derive *a priori* error bounds and the formula for the **MSE** under the **WNH**. These results are not new. We include them for self-containment. We use the following settings throughout this section: Let $\{\mathbf{v}_j\}_{j=1}^N$ be a frame in $\mathbb{R}^d$ with corresponding frame matrix $F = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N]$. The eigenvalues of FF^T are $\lambda_{\max} = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d = \lambda_{\min} > 0$. Let $\{\mathbf{u}_j\}_{j=1}^N$ be the canonical dual frame with corresponding matrix $G = (FF^T)^{-1}F$. For any $\mathbf{x} = \sum_{j=1}^N (\mathbf{x}\cdot\mathbf{v}_j)\,\mathbf{u}_j$, using the quantization alphabet $\mathcal{A} = \Delta\mathbb{Z}$ we have the PCM quantized reconstruction

$$\widehat{\mathbf{x}} = \sum_{j=1}^N Q_\Delta(\mathbf{x}\cdot\mathbf{v}_j)\,\mathbf{u}_j.$$

Proposition 2.1. *For any* $\mathbf{x} \in \mathbb{R}^d$ *we have*

$$\|\mathbf{x} - \widehat{\mathbf{x}}\| \leq \frac{1}{2}\sqrt{\frac{N}{\lambda_{\min}}}\Delta. \tag{2.1}$$

If in addition $\{\mathbf{v}_j\}_{j=1}^N$ *is a tight frame with frame bound* λ*, then*

$$\|\mathbf{x} - \widehat{\mathbf{x}}\| \leq \frac{1}{2}\sqrt{\frac{N}{\lambda}}\Delta. \tag{2.2}$$

Proof. We have $\mathbf{x}-\widehat{\mathbf{x}} = \sum_{j=1}^N \tau_\Delta(\mathbf{x}\cdot\mathbf{v}_j)\,\mathbf{u}_j = G\mathbf{y}$, where $\mathbf{y} = [\tau_\Delta(\mathbf{x}\cdot\mathbf{v}_1), \ldots, \tau_\Delta(\mathbf{x}\cdot\mathbf{v}_N)]^T$. Thus $\|\mathbf{x}-\widehat{\mathbf{x}}\|^2 = \mathbf{y}^T G^T G\mathbf{y} \leq \rho(G^T G)\,\|\mathbf{y}\|^2$ where $\rho(\cdot)$ denotes the spectral radius. Now $\rho(G^TG) = \rho(GG^T) = \rho((FF^T)^{-1}) = \lambda_{\min}^{-1}$. Observe that $|\tau_\Delta(\mathbf{x}\cdot\mathbf{v}_j)| \leq \Delta/2$. Thus $\|\mathbf{y}\|^2 \leq N(\Delta/2)^2$. This yields the *a priori* error bound (2.1). The bound (2.2) is an immediate corollary. ■

Proposition 2.2. *Under the **WNH**, the* **MSE** *is*

$$\mathcal{E}\left(\|\mathbf{x}-\widehat{\mathbf{x}}\|^2\right) = \frac{\Delta^2}{12}\sum_{j=1}^d \lambda_j^{-1} = \frac{\Delta^2}{12}\sum_{j=1}^N \|\mathbf{u}_j\|^2. \tag{2.3}$$

In particular, if $\{\mathbf{v}_j\}_{j=1}^N$ *is a tight frame with frame bound* λ*, then*

$$\mathcal{E}\left(\|\mathbf{x}-\widehat{\mathbf{x}}\|^2\right) = \frac{d}{12\lambda}\Delta^2. \tag{2.4}$$

Proof. Denote $G^TG = [b_{ij}]_{i,j=1}^N$ and again let $\mathbf{y} = [\tau_\Delta(\mathbf{x}\cdot\mathbf{v}_1), \ldots, \tau_\Delta(\mathbf{x}\cdot\mathbf{v}_N)]^T$. Note that with the **WNH**, $\mathcal{E}(y_iy_j) = \mathcal{E}(\tau_\Delta(\mathbf{x}\cdot\mathbf{v}_i)\tau_\Delta(\mathbf{x}\cdot\mathbf{v}_j)) = (\Delta^2/12)\delta_{ij}$. Now $\mathbf{x}-\widehat{\mathbf{x}} = G\mathbf{y}$ and hence

$$\mathcal{E}\left(\|\mathbf{x}-\widehat{\mathbf{x}}\|^2\right) = \mathcal{E}\left(\mathbf{y}^T G^T G \mathbf{y}\right) = \sum_{i,j=1}^{N} b_{ij}\mathcal{E}\left(y_i y_j\right) = \sum_{i=1}^{N} b_{ii}\frac{\Delta^2}{12} = \frac{\Delta^2}{12}\mathrm{tr}(G^T G).$$

Finally, $\mathrm{tr}(G^T G) = \sum_{j=1}^{N} \|\mathbf{u}_j\|^2$, and $\mathrm{tr}(G^T G) = \mathrm{tr}(GG^T) = \mathrm{tr}((FF^T)^{-1}) = \sum_{j=1}^{d} \lambda_j^{-1}$. ■

Remark: The **MSE** formulae (2.2-2.4) still hold if the independence of $\{\tau_\Delta\left(\mathbf{x}\cdot\mathbf{v}_j\right)\}_{j=1}^{N}$ in the **WNH** is replaced with the weaker condition that $\{\tau_\Delta\left(\mathbf{x}\cdot\mathbf{v}_j\right)\}_{j=1}^{N}$ are uncorrelated.

3. A Closer Look at the WNH

The **WNH** asserts that the error components $\{\tau_\Delta\left(\mathbf{x}\cdot\mathbf{v}_j\right)\}_{j=1}^{N}$ are independent and identically distributed random variables. We show that in general this is not true.

Theorem 3.1. *Let $\mathbf{X}\in\mathbb{R}^d$ be an absolutely continuous random vector. Let $\{\mathbf{v}_j\}_{j=1}^{N}$ be a frame in $\mathbb{R}^d$ with $N > d$ and $\mathbf{v}_j \neq 0$. Then the random variables $\{\tau_\Delta\left(\mathbf{X}\cdot\mathbf{v}_j\right)\}_{j=1}^{N}$ are not independent.*

Proof. Let F be the frame matrix for the frame $\{\mathbf{v}_j\}$. Then $\dim(\mathrm{range}(F^T)) = d$, and therefore $\mathcal{L}(\mathrm{range}(F^T)) = 0$ where $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}^N$. Let $\mathbf{Y} = [Y_1,\ldots,Y_N]^T := F^T\mathbf{X}$, and let $\widehat{\mathbf{Y}} = [Q_\Delta(Y_1),\ldots,Q_\Delta(Y_N)]^T$ be the quantized $\mathbf{Y}$. Denote $\mathbf{Z} = \mathbf{Y}-\widehat{\mathbf{Y}} = [Z_1,\ldots,Z_N]^T$. Note that $Y_j = \mathbf{v}_j\cdot\mathbf{X}$, so each Y_j is absolutely continuous, and therefore so is each Z_j. Assume Z_j has density $\psi_j(\cdot)$. If $\{Z_j\}$ are independent, then $\mathbf{Z}$ has density

$$\psi(z_1,\ldots,z_N) = \prod_{j=1}^{N}\psi_j(z_j).$$

Now, for $\mathbf{y}\in\mathbb{R}^N$ denote $\widehat{\mathbf{y}} = [Q_\Delta(y_1),\ldots,Q_\Delta(y_N)]^T$. Set $\Omega := \{\mathbf{y}-\widehat{\mathbf{y}} : \mathbf{y}\in\mathrm{Img}(F^T)\}$, $K_{\mathbf{w}} := \{\mathbf{y}-\mathbf{w} : \mathbf{y}\in\mathrm{Img}(F^T),\ \widehat{\mathbf{y}}=\mathbf{w}\}$ and $L_{\mathbf{w}} := \{\mathbf{y} : \mathbf{y}\in\mathrm{Img}(F^T),\ \widehat{\mathbf{y}}=\mathbf{w}\}$. Note that $K_{\mathbf{w}}$ and $L_{\mathbf{w}}$ are just translations of one another. Therefore $\mathcal{L}(K_{\mathbf{w}}) = \mathcal{L}(L_{\mathbf{w}})$. If $\Lambda = \{\widehat{\mathbf{y}} : \mathbf{y}\in\mathrm{Img}(F)\}$, then, Λ is countable, and therefore $\Omega = \bigcup_{\mathbf{w}\in\Lambda} K_{\mathbf{w}}$, while $\mathrm{Img}(F) = \bigcup_{\mathbf{w}\in\Lambda} L_{\mathbf{w}}$. Therefore $\mathcal{L}(K_{\mathbf{w}}) \leq \mathcal{L}(\mathrm{Img}(F)) = 0$, and hence $\mathcal{L}(\Omega) = 0$. Nevertheless, note that $\psi(x) = 0$ for $x\in\Omega^c$. It follows,

$$P(\mathbf{Z}\in\mathbb{R}^n) = \int_\Omega \psi(x)\,dx = 1,$$

which is a contradition. ■

Theorem 3.2. *Let* $\mathbf{X} = [X_1, \ldots, X_m]^T$ *be a random vector in* $\mathbb{R}^m$ *whose distribution has density function* $g(x_1, \ldots, x_m)$.

(1) *The error components* $\{\tau_\Delta(X_j)\}_{j=1}^m$ *are independent if and only if there exist complex numbers* $\{\beta_j(n) : 1 \le j \le m, n \in \mathbb{Z}\}$ *such that*

$$\widehat{g}\left(\frac{a_1}{\Delta}, \ldots, \frac{a_m}{\Delta}\right) = \beta_1(a_1) \cdots \beta_m(a_m) \tag{3.1}$$

for all $[a_1, \ldots, a_m]^T \in \mathbb{Z}^m$.

(2) *Let* $h_j(t)$ *be the marginal density of* X_j. *Then* $\{\tau_\Delta(X_j)\}_{j=1}^m$ *are identically distributed if and only if*

$$\sum_{n \in \mathbb{Z}} h_j(t - n\Delta) = H(t) \quad \text{a.e.}$$

for some $H(t)$ *independent of* j. *They are uniformly distributed on* $[-\Delta/2, \Delta/2]$ *if and only if* $H(t) = 1/\Delta$ *a.e..*

Proof. To prove (1) we first prove that $\mathbf{Y} = [\tau_\Delta(X_1), \ldots, \tau_\Delta(X_m)]^T$ has a density function. Let $\mathcal{I}_\Delta = [-\Delta/2, \Delta/2]$. Set

$$h(\mathbf{y}) := \sum_{\mathbf{a} \in \mathbb{Z}^m} g(\mathbf{y} - \Delta\mathbf{a}) \tag{3.2}$$

for $\mathbf{y} \in \mathcal{I}_\Delta^m$. For any $\Omega \subseteq \mathcal{I}_\Delta^m$ we have

$$\int_\Omega h(\mathbf{y}) = P(\mathbf{Y} \in \Omega) = P(\mathbf{X} \in \Omega + \Delta\mathbb{Z}^m) = \sum_{\mathbf{a} \in \mathbb{Z}^m} \int_\Omega g(\mathbf{y} - \Delta\mathbf{a}) = \int_\Omega \sum_{\mathbf{a} \in \mathbb{Z}^m} g(\mathbf{y} - \Delta\mathbf{a}).$$

Thus, the density of $\mathbf{Y}$ is given by (3.2) for $\mathbf{y} \in \mathcal{I}_\Delta^m$. Now, on $\mathcal{I}_\Delta$ the Fourier series of $h(\mathbf{y})$ is $h(\mathbf{y}) = \sum_{\mathbf{a}\in\mathbb{Z}^m} c_{\mathbf{a}} e^{2i\pi\frac{\mathbf{a}}{\Delta}\cdot\mathbf{y}}$, where

$$\begin{aligned}
c_{\mathbf{a}} &= \left\langle g(\mathbf{y}), e^{2i\pi\frac{\mathbf{a}}{\Delta}\cdot\mathbf{y}} \right\rangle_{L^2(\mathcal{I}_\Delta^m)} \\
&= \int_{\mathcal{I}_\Delta^m} \sum_{\mathbf{b}\in\mathbb{Z}^m} g(\mathbf{y}-\Delta\mathbf{b}) e^{-2i\pi\frac{\mathbf{a}}{\Delta}\cdot\mathbf{y}}\, d\mathbf{y} \\
&= \sum_{\mathbf{b}\in\mathbb{Z}^m} \int_{\mathcal{I}_\Delta^m} g(\mathbf{y}-\Delta\mathbf{b}) e^{-2i\pi\frac{\mathbf{a}}{\Delta}\cdot\mathbf{y}}\, d\mathbf{y} \\
&= \sum_{\mathbf{b}\in\mathbb{Z}^m} \int_{\mathcal{I}_\Delta^m+\Delta\mathbf{b}} g(\mathbf{y}) e^{-2i\pi\frac{\mathbf{a}}{\Delta}\cdot(\mathbf{y}+\Delta\mathbf{b})}\, d\mathbf{y} \\
&= \sum_{\mathbf{b}\in\mathbb{Z}^m} \int_{\mathcal{I}_\Delta^m+\Delta\mathbf{b}} g(\mathbf{y}) e^{-2i\pi\frac{\mathbf{a}}{\Delta}\cdot\mathbf{y}}\, d\mathbf{y} \\
&= \int_{\mathbb{R}^m} g(\mathbf{y}) e^{-2i\pi\frac{\mathbf{a}}{\Delta}\cdot\mathbf{y}}\, d\mathbf{y} \\
&= \widehat{g}\left(\frac{\mathbf{a}}{\Delta}\right).
\end{aligned}$$

But $\{Y_j\}_{j=1}^m$ are independent if and only if on $\mathcal{I}_\Delta^m$ we have $g(y_1,\ldots,y_m) = g_1(y_1)\cdots g_m(y_m)$. It is easily checked that this happens if and only if

$$\widehat{g}\left(\frac{a_1}{\Delta}, \frac{a_2}{\Delta}, \ldots, \frac{a_m}{\Delta}\right) = h_1\left(\frac{a_1}{\Delta}\right) h_2\left(\frac{a_2}{\Delta}\right)\cdots h_m\left(\frac{a_m}{\Delta}\right)$$

for all $\mathbf{a} = [a_1,\ldots,a_m]^T \in \mathbb{Z}^m$, with $h_j(\xi) = \widehat{g}_i(\xi)$. This part of the theorem is proved by setting $\beta_j(n) = h_j(n)$.

To prove (2), we only have to observe that the density of $\tau_\Delta(X_j)$ is precisely $\sum_{n\in\mathbb{Z}} h_j(t-\Delta n)$ for $t \in \mathcal{I}_\Delta$. This immediately yields $\sum_{n\in\mathbb{Z}} h_j(t-\Delta n) = H(t)$ for some $H(t)$ on $\mathcal{I}_\Delta$. But each $\sum_{n\in\mathbb{Z}} h_j(t-\Delta n)$ is Δ-periodic. Furthermore, if $\tau_\Delta(X_j)$ is uniformly distributed on $\mathcal{I}_\Delta$ then $H(t) = 1/\Delta$. This completes the proof of the theorem. ∎

Theorem 3.2 puts strong constraints on the distribution of $\mathbf{x}$ for the **WNH** to hold. Let $\mathbf{X} \in \mathbb{R}^d$ be a random vector with joint density $f(\mathbf{x})$. Let $\{\mathbf{v}_j\}_{j=1}^d$ be linearly independent, and let $\mathbf{Y} = [\mathbf{X}\cdot\mathbf{v}_1, \mathbf{X}\cdot\mathbf{v}_2, \ldots, \mathbf{X}\cdot\mathbf{v}_d]^T$. Then the joint density of $\mathbf{Y}$ is $g(\mathbf{y}) = |\det(F)|^{-1} f\left((F^T)^{-1}\mathbf{y}\right)$ where $F = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d]$. Thus, both the independence and the identical distribution assumptions in the **WNH**, even for $N = d$, will not be true unless very exact conditions are met.

Corollary 3.3. *Let $\mathbf{X} \in \mathbb{R}^d$ be a random vector with joint density $f(\mathbf{x})$ and $\{\mathbf{v}_j\}_{j=1}^d$ be linearly independent vectors in $\mathbb{R}^d$. Let $\mathbf{Y} = F^T\mathbf{X} = [\mathbf{X}\cdot\mathbf{v}_1, \ldots, \mathbf{X}\cdot\mathbf{v}_N]^T$ and $g(\mathbf{y}) = |\det(F)|^{-1} f\left((F^T)^{-1}\mathbf{y}\right)$ where $F = [\mathbf{v}_1, \ldots, \mathbf{v}_d]$.*

(1) *$\{\tau_\Delta(Y_j)\}_{j=1}^d$ are independent random variables if and only if there exist complex numbers $\{\beta_j(n) : 1 \leq j \leq d, n \in \mathbb{Z}\}$ such that*

$$\widehat{g}\left(\frac{a_1}{\Delta}, \ldots, \frac{a_d}{\Delta}\right) = \beta_1(a_1)\cdots\beta_d(a_d) \tag{3.3}$$

for all $[a_1, \ldots, a_d]^T \in \mathbb{Z}^d$.

(2) *Let $h_j(t) = \int_{\mathbb{R}^{d-1}} g(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_d)\, dx_1 \cdots\, dx_{j-1}\, dx_{j+1} \ldots\, dx_d$. Then $\{\tau_\Delta(X_j)\}_{j=1}^d$ are identically distributed if and only if $\sum_{n\in\mathbb{Z}} h_j(t - n\Delta) = H(t)$ a.e. for some $H(t)$ independent of j. They are uniformly distributed on $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$ if and only if $H(t) = 1/\Delta$ a.e..*

Proof. We only have to observe that $g(\mathbf{y})$ is the density of $\mathbf{Y}$ and that h_j is the marginal density of Y_j. The corollary now follows directly from the theorem. ∎

4. Asymptotic Behavior of Errors: Linear Independence Case

In many practical applications such as music CD, fine quantizations with 16 bits or more have been adopted. Although the **WNH** is not valid in general, with fine quantizations we prove here that a weaker version of the **WNH** is close to being valid, which yields an asymptotic formula for the PCM quantized **MSE**.

We again consider the same setup as before. Let $\{\mathbf{v}_j\}_{j=1}^N$ be a frame in $\mathbb{R}^d$ with corresponding frame matrix $F = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N]$. The eigenvalues of FF^T are $\lambda_{\max} = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d = \lambda_{\min} > 0$. Let $\{\mathbf{u}_j\}_{j=1}^N$ be the canonical dual frame with corresponding matrix $G = (FF^T)^{-1}F$. For any $\mathbf{x} \in \mathbb{R}^d$ we have $\mathbf{x} = \sum_{j=1}^N (\mathbf{x}\cdot\mathbf{v}_j)\,\mathbf{u}_j$. Using the quantization alphabet $\mathcal{A} = \Delta\mathbb{Z}$ we have the PCM reconstruction (1.4). Note that $\widehat{\mathbf{x}} = \widehat{\mathbf{x}}(\Delta)$ as it depends on Δ. With the **WNH** we obtain the **MSE**

$$\mathbf{MSE} = \mathcal{E}\left(\|\mathbf{x} - \widehat{\mathbf{x}}\|^2\right) = \frac{\Delta^2}{12}\sum_{j=1}^N \lambda_j^{-1}.$$

To study the asymptotic behavior of the error components, we study as $\Delta \to 0^+$ the normalized quantization error

$$\frac{1}{\Delta}(\mathbf{x}-\widehat{\mathbf{x}}) = \sum_{j=1}^{N} \frac{1}{\Delta}\tau_\Delta\left(\mathbf{x}\cdot\mathbf{v}_j\right)\mathbf{u}_j. \tag{4.1}$$

Theorem 4.1. *Let $\mathbf{X} \in \mathbb{R}^d$ be an absolutely continuous random vector. Let $\mathbf{w}_1, \ldots, \mathbf{w}_m$ be linearly independent vectors in $\mathbb{R}^d$. Then*

$$\left[\frac{1}{\Delta}\tau_\Delta\left(\mathbf{X}\cdot\mathbf{w}_1\right), \ldots, \frac{1}{\Delta}\tau_\Delta\left(\mathbf{X}\cdot\mathbf{w}_m\right)\right]^T$$

converges in distribution as $\Delta \to 0^+$ to a random vector unformly distributed in $[-1/2, 1/2]^m$.

Proof. Denote $Y_j = \mathbf{X}\cdot\mathbf{w}_j$. Since $\{\mathbf{w}_j\}$ are linearly independent, $\mathbf{Y} = [Y_1, \ldots, Y_m]^T$ is absolutely continuous with some joint density $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^m$. As a consequence of (3.2) one has that the distribution of $\mathbf{Z} = [Z_1, \ldots, Z_m]^T$, where $Z_j = \frac{1}{\Delta}\tau_\Delta(Y_j) = \left\{\frac{Y_j}{\Delta} + \frac{1}{2}\right\} - \frac{1}{2}$, is

$$f_\Delta(\mathbf{x}) := \Delta^m \sum_{\mathbf{a}\in\mathbb{Z}^m} f(\Delta\mathbf{x} - \Delta\mathbf{a}). \tag{4.2}$$

for $\mathbf{x} \in [-1/2, 1/2]^m$. Again denote $\mathcal{I}_1 := [-1/2, 1/2]$. Observe that

$$\begin{aligned}
\|f_\Delta\|_{L^1(\mathcal{I}_1^m)} &= \int_{\mathcal{I}_1^m} |f_\Delta(\mathbf{x})|\, d\mathbf{x} \\
&= \int_{\mathcal{I}_1^m} \Delta^m \left|\sum_{\mathbf{a}\in\mathbb{Z}^m} f(\Delta\mathbf{x} - \Delta\mathbf{a})\right| d\mathbf{x} \\
&\le \sum_{\mathbf{a}\in\mathbb{Z}^m} \int_{\mathcal{I}_1^m} \Delta^m \left|f(\Delta\mathbf{x} - \Delta\mathbf{a})\right|\, d\mathbf{x} \\
&= \sum_{\mathbf{a}\in\mathbb{Z}^m} \int_{\mathcal{I}_\Delta^m + \Delta\mathbf{a}} |f(\mathbf{y})|\, d\mathbf{y} \\
&= \int_{\bigcup_{\mathbf{a}\in\mathbb{Z}^m}(\mathcal{I}_\Delta^m + \Delta\mathbf{a})} |f(\mathbf{y})|\, d\mathbf{y} \\
&= \int_{\mathbb{R}^m} |f(\mathbf{y})|\, d\mathbf{y} \\
&= \|f\|_{L^1(\mathbb{R}^m)}.
\end{aligned}$$

Now, if $\Omega = [a_1, b_1] \times \cdots \times [a_m, b_m]$ and $f(\mathbf{x}) = \mathbf{1}_\Omega(\mathbf{x})$, then for $\mathbf{x} \in \mathcal{I}_1^m$ observe that $f_\Delta(\mathbf{x}) = \Delta^m K_\Delta$ where $K_\Delta(\mathbf{x}) = \#\{\mathbf{a} \in \mathbb{Z}^m : \Delta\mathbf{x} + \Delta\mathbf{a} \in \Omega\}$. Obviously, $K_\Delta(\mathbf{x}) = s/\Delta^m + O(\Delta^{-m+1})$ where $s = \mathcal{L}(\Omega)$ is the Lebesgue measure of Ω. Then $f_\Delta \to s\mathbf{1}_{\mathcal{I}_1^m}$ in $L^1(\mathcal{I}_1^m)$ as $\Delta \to 0^+$.

Coming back to the case when $f(\mathbf{x})$ is the density of $\mathbf{Y}$. For any $\varepsilon > 0$ it is possible to choose a $g(\mathbf{x}) \in L^1(\mathbb{R}^m)$ such that $\|f - g\|_{L^1} < \frac{\varepsilon}{3}$, and furthermore, $g(\mathbf{x}) = \sum_{j=1}^N c_j \mathbf{1}_{E_j}(\mathbf{x})$ is a simple function where $c_j \in \mathbb{R}$ and each E_j is a product of finite intervals. Observe that $\int_{\mathbb{R}^m} g = \sum_{j=1}^N c_j \mathcal{L}(E_j)$. Since $(\mathbf{1}_{E_j})_\Delta \to \mathcal{L}(E_j)\mathbf{1}_{\mathcal{I}_1^m}$ in L^1 we have $g_\Delta \to \left(\int_{\mathbb{R}^m} g\right) \mathbf{1}_{\mathcal{I}_1^m}$ as $\Delta \to 0$. Hence there exists a $\delta > 0$ such that $\left\|g_\Delta - \left(\int_{\mathbb{R}^m} g\right) \mathbf{1}_{\mathcal{I}_1^m}\right\|_{L^1} < \varepsilon/3$ whenever $\Delta < \delta$. Now, for $\Delta < \delta$,

$$
\begin{aligned}
\left\|f_\Delta - \mathbf{1}_{\mathcal{I}_1^m}\right\|_{L^1(\mathcal{I}_1^m)} &= \|f_\Delta - g_\Delta\|_{L^1(\mathcal{I}_1^m)} + \left\|g_\Delta - \left(\textstyle\int_{\mathbb{R}^m} g\right) \mathbf{1}_{\mathcal{I}_1^m}\right\|_{L^1(\mathcal{I}_1^m)} \\
&\quad + \left|1 - \left(\textstyle\int_{\mathbb{R}^m} g\right)\right| \|\mathbf{1}_{\mathcal{I}_1^m}\|_{L^1(\mathcal{I}_1^m)} \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \left|1 - \left(\textstyle\int_{\mathbb{R}^m} g\right)\right| \\
&= \frac{2\varepsilon}{3} + \left|\left(\textstyle\int_{\mathbb{R}^m} g\right) - \left(\textstyle\int_{\mathbb{R}^m} g\right)\right| \\
&< \varepsilon.
\end{aligned}
$$

■

Applying the above theorem to the **MSE**, if $\{\mathbf{v}_j\}_{j=1}^N$ are pairwise linearly independent then the error components $\{\tau_\Delta(\mathbf{X} \cdot \mathbf{v}_j)\}_{j=1}^N$ become asymptotically pairwise independent and each uniformly distributed in $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$. Therefore we have the following:

Corollary 4.2. *Let $\mathbf{X} \in \mathbb{R}^d$ be an absolutely continuous random vector. If $\{\mathbf{v}_j\}_{j=1}^N$ are pairwise linearly independent, then as $\Delta \to 0^+$ we have*

$$
\mathcal{E}\left(\|\mathbf{X} - \widehat{\mathbf{X}}\|^2\right) = \frac{\Delta^2}{12} \sum_{j=1}^d \lambda_j^{-1} + o(\Delta^2) = \frac{\Delta^2}{12} \sum_{j=1}^N \|\mathbf{u}_j\|^2 + o(\Delta^2). \tag{4.3}
$$

Proof. Denote F the frame matrix associated to $\{\mathbf{v}_j\}_{j=1}^N$, $H = (FF^T)^{-1}$, $Y_j = \mathbf{X} \cdot \mathbf{v}_j$, $Z_j = \left\{\frac{Y_j}{\Delta} + \frac{1}{2}\right\} - \frac{1}{2}$, and $\mathbf{Z} = [Z_1, \ldots, Z_m]^T$. By Theorem 4.1, $\mathcal{E}(Z_i) \to 0$ and $\mathcal{E}(Z_i Z_j) \to$

$\frac{1}{12}\delta_{ij}$ as $\Delta \to 0^+$. It follows from the proof of Proposition 2.2 that

$$\begin{aligned}
\frac{1}{\Delta^2}\mathcal{E}(\|\mathbf{X}-\widehat{\mathbf{X}}\|^2) &= \mathcal{E}(\mathbf{Z}^T H\mathbf{Z}) \\
&= \mathcal{E}\left(\sum_{i,j=1}^{N} Z_i Z_j h_{ij}\right) \\
&= \sum_{i,j+1}^{N} h_{ij}\mathcal{E}\left(Z_i Z_j\right) \\
&= \frac{1}{12}\sum_{i=1}^{N} h_{ii} + o(1) \\
&= \frac{1}{12}\sum_{j=1}^{d} \lambda_j^{-1} + o(1),
\end{aligned}$$

and hence

$$\mathcal{E}\left(\|\mathbf{X}-\widehat{\mathbf{X}}\|^2\right) = \frac{\Delta^2}{12}\sum_{j=1}^{N}\lambda_j^{-1} + o(\Delta^2) = \frac{\Delta^2}{12}\sum_{j=1}^{N}\|\mathbf{u}_j\|^2 + o(\Delta^2).$$

■

5. Asymptotic Behavior of Errors: Linear Dependence Case

Mathematically it is very interesting to understand what happens if $\{\mathbf{v}_j\}_{j=1}^N$ are not pairwise linearly independent, and how the **MSE** behaves as $\Delta \to 0^+$. We return to previous calculations and note that

$$\mathcal{E}(\|\mathbf{X}-\widehat{\mathbf{X}}\|^2) = \sum_{i,j=1}^{N} h_{ij}\mathcal{E}\left(\tau_\Delta(\mathbf{X}\cdot\mathbf{v}_i)\tau_\Delta(\mathbf{X}\cdot\mathbf{v}_j)\right).$$

Our main result in this section is:

Theorem 5.1. *Let X be an absolutely continuous real random variable. Let $\alpha \in \mathbb{R}\setminus\{0\}$. Then*

$$\lim_{\Delta\to 0^+}\frac{1}{\Delta^2}\mathcal{E}\left(\tau_\Delta(X)\tau_\Delta(\alpha X)\right) = \begin{cases} 0, & \alpha \neq \mathbb{Q}, \\ \frac{1}{12pq}, & \alpha = \frac{p}{q} \text{ and } p+q \text{ is even}, \\ -\frac{1}{24pq}, & \alpha = \frac{p}{q} \text{ and } p+q \text{ is odd}, \end{cases} \tag{5.1}$$

where p, q are coprime integers.

Proof. Denote $g(x) := \{x + \frac{1}{2}\} - \frac{1}{2}$, and let $g_n(x)$ be a small perturbation of $g(x)$ such that

(a) $|g_n(x)| \le 1/2$;
(b) $\mathrm{supp}(g(x) - g_n(x)) \subseteq [\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}] + \mathbb{Z}$;
(c) $g_n(x) \in C^\infty$, and is $\mathbb{Z}$-periodic.

Now, set

$$\begin{aligned} E(\Delta) &:= \mathcal{E}\left(\tau_\Delta(X)\tau_\Delta(\alpha X)\right) \\ &= \mathcal{E}\left(g\left(\frac{X}{\Delta}\right) g\left(\frac{\alpha X}{\Delta}\right)\right) \\ &= \int_{\mathbb{R}} g\left(\frac{x}{\Delta}\right) g\left(\frac{\alpha x}{\Delta}\right) f(x)\, dx, \end{aligned}$$

and

$$E_n(\Delta) := \int_{\mathbb{R}} g_n\left(\frac{x}{\Delta}\right) g_n\left(\frac{\alpha x}{\Delta}\right) f(x)\, dx.$$

Claim: *$E_n(\Delta) \to E(\Delta)$ as $n \to \infty$ uniformly for all $\Delta > 0$.*

Proof of the Claim. For any $\varepsilon > 0$,

$$\begin{aligned} |E_n(\Delta) - E(\Delta)| &= \left|\int_{\mathbb{R}} \left[g_n\left(\frac{x}{\Delta}\right) g_n\left(\frac{\alpha x}{\Delta}\right) - g\left(\frac{x}{\Delta}\right) g\left(\frac{\alpha x}{\Delta}\right)\right] f(x)\, dx\right| \\ &\le \frac{1}{2}\int_{\mathbb{R}} \left|g_n\left(\frac{x}{\Delta}\right) - g\left(\frac{x}{\Delta}\right)\right| f(x)\, dx + \frac{1}{2}\int_{\mathbb{R}} \left|g_n\left(\frac{\alpha x}{\Delta}\right) - g\left(\frac{\alpha x}{\Delta}\right)\right| f(x)\, dx. \end{aligned}$$

Now there exists an $M > 0$ such that $\int_{[-M,M]^c} f(x)\, dx < \frac{\varepsilon}{2}$. So

$$\int_{\mathbb{R}} \left|g_n\left(\frac{x}{\Delta}\right) - g\left(\frac{x}{\Delta}\right)\right| f(x)\, dx \le \int_{-M}^{M} \left|g_n\left(\frac{x}{\Delta}\right) - g\left(\frac{x}{\Delta}\right)\right| f(x)\, dx + \frac{\varepsilon}{2}.$$

Furthermore, let $A_n(\Delta, M) := \mathrm{supp}(g_n(x/\Delta) - g(x/\Delta)) \cap [-M, M]$. Then we have

$$A_n(\Delta, M) \subseteq ([\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}] + \mathbb{Z}) \cap [-M, M].$$

Hence $\mathcal{L}(A_n(\Delta, M)) \le \frac{2M}{\Delta} \cdot \frac{2\Delta}{n} = \frac{4M}{n}$, and thus

$$\int_{-M}^{M} \left|g_n\left(\frac{x}{\Delta}\right) - g\left(\frac{x}{\Delta}\right)\right| f(x)\, dx \le \int_{A_n(\Delta,M)} f(x)\, dx < \frac{\varepsilon}{2}$$

by choosing n sufficiently large (independent of Δ), which yields

$$\int_{\mathbb{R}} \left|g_n\left(\frac{x}{\Delta}\right) - g\left(\frac{x}{\Delta}\right)\right| f(x)\, dx < \varepsilon.$$

Similarly we have

$$\int_{\mathbb{R}} \left| g_n\left(\frac{\alpha x}{\Delta}\right) - g\left(\frac{\alpha x}{\Delta}\right)\right| f(x)\, dx < \varepsilon$$

for sufficiently large n, proving the Claim. □

Now consider the Fourier Series of $g_n(t)$,

$$g_n(t) = \sum_{k\in\mathbb{Z}} c_k^{(n)} e^{2\pi ikt}.$$

It is well known that the Fourier series converges to $g_n(t)$ uniformly for all t, see e.g. [18]. Furthermore, since $g_n(t)$ is C^∞ we have $|c_k^{(n)}| = o\big((|k|+1)^{-L}\big)$ for all $L > 0$, giving absolute convergence of the Fourier series. Thus

$$\begin{aligned} E_n(\Delta) &= \lim_{K\to\infty} \int_{\mathbb{R}} \Big(\sum_{|k|\le K} c_k^{(n)} e^{2\pi ikt\Delta^{-1}}\Big)\Big(\sum_{|k|\le K} c_k^{(n)} e^{2\pi ik\alpha t\Delta^{-1}}\Big) f(t)\, dt \\ &= \lim_{K\to\infty} \sum_{|k|,|\ell|\le K} c_k^{(n)} c_\ell^{(n)} \widehat{f}\Big(-\frac{k+\alpha\ell}{\Delta}\Big). \end{aligned}$$

Observe that $|\widehat{f}(\xi)| \le \|f\|_{L^1} = 1$, and $|c_k^{(n)}| = o\big((|k|+1)^{-L}\big)$ for any $L > 0$. So the series converges absolutely and uniformly in Δ. Thus

$$E_n(\Delta) = \sum_{k,\ell\in\mathbb{Z}} c_k^{(n)} c_\ell^{(n)} \widehat{f}\Big(-\frac{k+\alpha\ell}{\Delta}\Big). \tag{5.2}$$

For any $n > 0$ we have

$$\lim_{\Delta\to 0^+} E_n(\Delta) = \sum_{k,\ell\in\mathbb{Z}} c_k^{(n)} c_\ell^{(n)} \lim_{\Delta\to 0^+} \widehat{f}\Big(-\frac{k+\alpha\ell}{\Delta}\Big)$$

because the series converges absolutely and uniformly. Suppose $\alpha \notin \mathbb{Q}$. Then $k + \alpha\ell \ne 0$ if either $k \ne 0$ or $\ell \ne 0$. Thus $\big|-\frac{k+\alpha\ell}{\Delta}\big| \to \infty$, and hence $\lim_{\Delta\to 0^+} \widehat{f}\big(-\frac{k+\alpha\ell}{\Delta}\big) = 0$ as $f \in L^1(\mathbb{R})$. It follows that

$$\lim_{\Delta\to 0^+} E_n(\Delta) = 0.$$

But $E_n(\Delta) \to E(\Delta)$ as $n \to \infty$ uniformly in Δ, which yields $E(\Delta) \to 0$ as $\Delta \to 0^+$.

Next, suppose $\alpha = \frac{p}{q}$ where $p, q \in \mathbb{Z}$, $(p,q) = 1$. We observe that $k + \alpha\ell = 0$ if and only if $k = qm$ and $\ell = -pm$ for some $m \in \mathbb{Z}$. In such a case

$$\widehat{f}\Big(-\frac{k+\alpha\ell}{\Delta}\Big) = \widehat{f}(0) = \int_{\mathbb{R}} f = 1.$$

It follows that

$$\lim_{\Delta\to 0^+} E_n(\Delta) = \sum_{m\in\mathbb{Z}} c_{qm}^{(n)} c_{-pm}^{(n)} \widehat{f}(0) = \sum_{m\in\mathbb{Z}} c_{qm}^{(n)} c_{-pm}^{(n)} = \sum_{m\in\mathbb{Z}} c_{qm}^{(n)} \overline{c_{pm}^{(n)}}.$$

For $r \in \mathbb{Z}, r \neq 0$ set

$$G_r^{(n)}(x) := \sum_{m\in\mathbb{Z}} c_{rm}^{(n)} e^{2\pi imx}$$

and

$$G_r(x) := \sum_{m\in\mathbb{Z}} c_{rm} e^{2\pi imx}.$$

By Parseval we have

$$E_n(\Delta) = \left\langle G_q^{(n)}, G_p^{(n)} \right\rangle_{L^2([0,1])}.$$

It is easy to check that

$$G_r^{(n)} = \frac{1}{|r|} \sum_{j=0}^{|r|-1} g_n\Big(\frac{x+j}{r}\Big).$$

Hence $G_r^{(n)}$ converges in $L^2([0,1])$ to $G_r(x) = \frac{1}{|r|}\sum_{j=0}^{|r|-1} g\big(\frac{x+j}{r}\big)$, which has Fourier series $G_r(x) = \sum_{m\in\mathbb{Z}} c_{rm} e^{2\pi imx}$ with $c_0 = 0$ and $c_k = (-1)^{k-1}(2\pi ik)$ for $k \neq 0$. This yields

$$\lim_{n\to\infty} \lim_{\Delta\to 0^+} E_n(\Delta) = \lim_{n\to\infty} \left\langle G_q^{(n)}, G_p^{(n)} \right\rangle = \langle G_q, G_p \rangle = \sum_{m\in\mathbb{Z}} c_{qm}\overline{c_{pm}}.$$

Finally

$$\begin{aligned} \sum_{m\in\mathbb{Z}} c_{qm}\overline{c_{pm}} &= \sum_{m\in\mathbb{Z}\setminus\{0\}} \Big(\frac{(-1)^{qm-1}}{2\pi imq}\Big)\overline{\Big(\frac{(-1)^{pm-1}}{2\pi imp}\Big)} \\ &= \frac{1}{2pq\pi^2}\sum_{m=1}^{\infty} \frac{(-1)^{(p+q)m}}{m^2}. \end{aligned}$$

Note that if $p+q$ is even then $\sum_{m=1}^{\infty} \frac{(-1)^{(p+q)m}}{m^2} = \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}$. On the other hand, if $p+q$ is odd then $\sum_{m=1}^{\infty} \frac{(-1)^{(p+q)m}}{m^2} = \sum_{m=1}^{\infty} \frac{(-1)^m}{m^2} = -\frac{\pi^2}{12}$. The theorem follows. ■

Corollary 5.2. *Let* $\mathbf{X}$ *be an absolutely continuous random vector in* $\mathbb{R}^d$, $\mathbf{w} \neq 0$, $\mathbf{w} \in \mathbb{R}^d$ *and* $\alpha \in \mathbb{R} \setminus \{0\}$. *Then*

$$\lim_{\Delta\to 0^+} \frac{1}{\Delta^2}\mathcal{E}\left(\tau_\Delta(\mathbf{w}\cdot\mathbf{X})\tau_\Delta(\alpha\mathbf{w}\cdot\mathbf{X})\right) = \begin{cases} 0, & \alpha \neq \mathbb{Q}, \\ \dfrac{1}{12pq}, & \alpha = \dfrac{p}{q} \text{ and } p+q \text{ is even,} \\ -\dfrac{1}{24pq}, & \alpha = \dfrac{p}{q} \text{ and } p+q \text{ is odd,} \end{cases} \tag{5.3}$$

where p, q *are coprime integers.*

Proof. We only need to note that $\mathbf{w}\cdot\mathbf{X}$ is an absolutely continuous random variable. The corollary follows immediately from Theorem 4.1. ■

We can now characterize completely the asymptotic bahavior of the MSE in all cases. For any two vectors $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{R}^d$ define $r(\mathbf{w}_1, \mathbf{w}_2)$ by

$$r(\mathbf{w}_1, \mathbf{w}_2) = \begin{cases} \frac{1}{pq}\mathbf{w}_1 \cdot \mathbf{w}_2, & \mathbf{w}_1 = \frac{p}{q}\mathbf{w}_2, \text{ and } p+q \text{ is even}, \\ -\frac{1}{2pq}\mathbf{w}_1 \cdot \mathbf{w}_2, & \mathbf{w}_1 = \frac{p}{q}\mathbf{w}_2, \text{ and } p+q \text{ is odd}, \\ 0, & \text{otherwise}, \end{cases}$$

where p, q are coprime integers.

Corollary 5.3. *Let $\mathbf{X} \in \mathbb{R}^d$ be an absolutely continuous random vector. Then as $\Delta \longrightarrow 0^+$ the* **MSE** *satisfies*

$$\mathcal{E}\left(\|\mathbf{x} - \widehat{\mathbf{x}}\|^2\right) = \frac{\Delta^2}{12}\sum_{j=1}^{d}\lambda_j^{-1} + \frac{\Delta^2}{6}\sum_{1\le i<j\le N} r(\mathbf{u}_i, \mathbf{u}_j) + o(\Delta^2), \tag{5.4}$$

Proof. In the proof of (4.2) we showed that

$$\lim_{\Delta\to 0^+} \frac{1}{\Delta^2}\mathcal{E}\left(\|\mathbf{x} - \widehat{\mathbf{x}}\|^2\right) = \Delta^2 \sum_{i,j} h_{ij}\mathcal{E}\left(Z_i Z_j\right)$$

with the notations there. The result is immediate from Theorem 5.2. ■

For fixed quantization step $\Delta > 0$ we shall denote

$$\mathbf{MSE}_{ideal} = \frac{\Delta^2}{12}\sum_{j=1}^{d}\lambda_j^{-1} + \frac{\Delta^2}{6}\sum_{1\le i<j\le N} r(\mathbf{u}_i, \mathbf{u}_j), \tag{5.5}$$

and call it the *ideal* ***MSE***. If $\{\mathbf{v}_j\}_{j=1}^N$ are pairwise linearly independent, then the $\mathbf{MSE}_{ideal}$ is simply $\frac{\Delta^2}{12}\sum_{j=1}^d \lambda_j^{-1}$, the **MSE** under the **WNH**.

In the next section we shall show some numerical data, comparing the actual **MSE** with the ideal **MSE**.

Appendix. Numerical Results

Here we present data from our computer experiments comparing the ideal **MSE** to the actual **MSE**. We have performed simulations for several different sets of frames. We also experimented with various distributions for $\mathbf{x} \in \mathbb{R}^d$. As it turns out, we get very similar results for the distributions we used for most of the frames we tried. In the examples shown, the random vectors X are all chosen to be uniformly distributed in $[-5, 5]^d$.

Example 5.1. *Let* $\{\mathbf{v}_j\}_{j=1}^N$ *be the harmonic frame in* $\mathbb{R}^2$*, with* $\mathbf{v}_j = \left[\cos\frac{2\pi j}{N}, \sin\frac{2\pi j}{N}\right]^T$. *This is a tight frame with frame bound* $\lambda = \frac{N}{2}$. *The ideal* ***MSE*** *is* $\frac{\Delta^2}{3N}$ *for* N *odd. Taking* $\Delta = \frac{1}{2}$, *Table 1 displays the actual* ***MSE****, the ideal* ***MSE*** *and the ratio between them. It shows that as* N *gets larger than 129, the actual* ***MSE*** *does not improve, which shows that the WNH is invalid for large* Δ.

N	Actual **MSE**	Ideal **MSE**	*Ratio*
9	0.00934342	0.00925926	1.009090
17	0.00479521	0.00252525	0.976808
33	0.00246669	0.00490196	0.978223
65	0.00122499	0.00128205	0.955496
129	0.00065858	0.00645995	1.019480
257	0.00057971	0.00032425	1.787810
513	0.00056039	0.00016244	3.449740
1025	0.00052914	0.00008130	6.508450
2049	0.00053895	0.00004067	13.25180
4097	0.00058846	0.00002034	28.93090

TABLE 1. The Harmonic frame in $\mathbb{R}^2$

Example 5.2. *Let* $\{\mathbf{v}_j\}_{j=1}^N$ *be* N *independently and randomly generated vectors uniformly distributed on the unit sphere in* $\mathbb{R}^4$. *Table 2 shows the ratio between the actual* ***MSE*** *and the ideal* ***MSE****, where* $\mathbf{MSE}_{ideal} = \frac{\Delta^2}{12}(\sum_{j=1}^d \lambda_j^{-1})$, *with* $\Delta = 2^{-k}$.

Example 5.3. *Let* $\{\mathbf{v}_j\}_{j=0}^{N-1}$ *be the harmonic frame in* $\mathbb{R}^4$*, with*

$$\mathbf{v}_j = \sqrt{\frac{1}{2}}\left[\cos\frac{2\pi j}{N}, \sin\frac{2\pi j}{N}, \cos\frac{4\pi j}{N}, \sin\frac{4\pi j}{N}\right]^T.$$

This is a tight frame with frame bound $\lambda = \frac{N}{4}$*, and the ideal* ***MSE*** *is* $\frac{4\Delta^2}{3N}$. *Table 3 shows the ratio between the actual* ***MSE*** *and the ideal* ***MSE*** *where* $\Delta = 2^{-k}$.

Example 5.4. *Let* $\{\mathbf{v}_j\}_{j=0}^5$ *be a frame in* $\mathbb{R}^3$*, with the corresponding matrix*

$$F = \begin{pmatrix} 1 & 1 & 1 & 2 & -3 \\ 1 & -1 & -1 & 2 & -3 \\ 1 & 0 & -1 & 2 & -3 \end{pmatrix}$$

k/N	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
k= 0	1.581960	2.232260	3.697160	6.497800	12.20670
k= 1	1.076590	1.130510	1.397840	1.649530	2.480920
k= 2	1.003680	0.995214	1.008370	1.033280	1.196680
k= 3	0.967138	0.990876	0.999648	0.981633	1.010090
k= 4	0.989295	1.009840	1.032110	1.002630	1.002260
k= 5	1.011720	1.035590	1.020870	1.002350	1.022250
k= 6	0.978712	1.006760	0.992207	1.001490	0.979342
k= 7	0.997524	1.017840	0.995852	0.972120	0.976273
k= 8	0.998725	1.011380	1.040270	0.978204	0.973284
k= 9	0.982450	1.038580	0.994463	1.021580	1.037800
k=10	0.993099	1.002340	1.009930	1.009870	0.974017
k=11	0.981428	0.998280	0.975881	1.049010	1.009570

TABLE 2. The randomly generated frame in $\mathbb{R}^4$

k/N	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
k= 0	0.997218	0.928318	1.287990	2.312710	4.497050
k= 1	1.005460	1.004720	0.950783	1.339810	2.395180
k= 2	0.990253	1.001070	0.977474	0.960994	1.354320
k= 3	0.995848	0.993963	0.981683	0.992655	0.955345
k= 4	0.987371	1.007310	1.028120	1.016760	1.002570
k= 5	0.993840	1.015230	1.026680	1.003770	1.023820
k= 6	1.012230	1.012280	0.996363	0.999742	1.004120
k= 7	1.020450	1.025820	1.031120	1.003770	1.004770
k= 8	1.004710	1.010820	0.999289	0.973596	0.970415
k= 9	0.993542	1.003380	0.981550	0.984594	0.981001
k=10	1.015610	1.008740	0.997469	0.986705	1.004360
k=11	1.010690	1.009080	0.994975	1.010510	0.998485

TABLE 3. The Harmonic frame in $\mathbb{R}^4$

Note that the set contains many parallel vectors. The ***MSE*** *under the WNH is* $0.2946\Delta^2$ *and our result, the ideal* ***MSE*** *is* $0.2959\Delta^2$*, due to the fact that* $\{\mathbf{v}_j\}_{j=0}^5$ *contains parallel vectors. Table 4 shows the actual* ***MSE****, the ideal* ***MSE****, and the ratio between the actual* ***MSE*** *and the ideal* ***MSE****, where* $\Delta = 2^{-k}$.

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k	Actual **MSE**	Ideal **MSE**	*Ratio*
2	0.00466163000	0.004623720000	1.008200
3	0.00116029000	0.001155930000	1.003770
4	0.00029280000	0.000288983000	1.013220
5	0.00007111000	0.000072246000	0.984317
6	0.00001799100	0.000001806000	0.996100
7	0.00000438600	0.000004515000	0.971450
8	0.00000109200	0.000011288000	0.967129
9	0.00000028070	0.000000280000	0.994956
10	0.00000007063	0.000000070550	1.001090
11	0.00000001776	0.000000017638	1.006860

TABLE 4. The frame of Example 5.4 in $\mathbb{R}^3$

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