

# On the System of Conservation Laws and its Perturbation in the Besov Spaces

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## Abstract

We prove that a system of conservation laws on  $\mathbb{R}^N$  is locally well posed in the ‘critical’ Besov space,  $B_{2,1}^{\frac{N}{2}+1}$ . The time of local existence depends only on the size of the inhomogeneous part of the initial data. We also obtain a blow-up criterion of the local solution. For the conservation system with a dissipation term added we prove global existence of solution under the assumption of smallness of the homogeneous part of the arbitrary sized  $B_{2,1}^{\frac{N}{2}-1}$  norm of the initial data. For the proof of these results we essentially use the Littlewood-Paley decomposition of functions to derive the energy type of estimate.

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# 1 Introduction and Main Results

We are concerned on the following system of conservation law:

$$A_0(u) \frac{\partial u}{\partial t} + \sum_{k=1}^N A_k(u) \frac{\partial u}{\partial x_k} = 0, \quad (x, t) \in \mathbb{R}^N \times (0, \infty) \quad (1.1)$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N \quad (1.2)$$

where  $u = (u_1, \dots, u_N)$ ,  $u_j = u_j(x, t)$ ,  $j = 1, 2, \dots, N$ . We assume that  $A_0(u)$  is a positive definite symmetric matrix satisfying the following uniform bounds

$$C^{-1}I \leq A_0(u) \leq CI \quad \forall u \in \mathbb{R}^N \quad (1.3)$$

for some constant  $C > 1$ , and  $A_k(u)$ 's are symmetric matrices. Additional technical assumptions on  $A_0(u)$  and  $A_k(u)$ 's will be specified in the statement of the main theorems below. There are many examples of partial differential equations that can be written in the form (1.1)(see e.g. [21]). For the Cauchy problem (1.1)-(1.2) with the initial data  $u_0$  given in  $H^m(\mathbb{R}^N)$ ,  $m > \frac{N}{2} + 1$ , the local unique existence of solution is proved by Kato [18] and Lax [20] independently. More specifically, they proved unique existence of solution belonging to  $C([0, T]; H^m(\mathbb{R}^N))$  for some  $T = T(\|u_0\|_{H^m})$ . Recently there are many studies on the extension of the regularity class of the initial data for local existence/global existence for small data using the Besov, or Triebel-Lizorkin spaces(e.g. [4]-[10],[12]-[16], [19], [24] and references therein). Most of those results are concerned on the specific equations. In this paper one of our aims is to prove the local unique existence and the continuation principle in the Besov spaces,  $B_{2,1}^{\frac{N}{2}+1}$  for a rather general type of equations as in (1.1)-(1.2). The other of our aim is to prove global existence for small data in  $B_{2,1}^{\frac{N}{2}-1}$  on the system of the type (1.1)-(1.2) with a dissipation term added. The following is our main theorem concerning (1.1)-(1.2).

**Theorem 1.1** *Let us assume that  $A_0(\cdot), A_k(\cdot), k = 1, 2, \dots, N$  satisfy (1.3), and the following conditions.*

$$A_k(0) = A'_k(0) = 0, \quad \forall k = 1, 2, \dots, N. \quad (1.4)$$

$$A_0, A_0^{-1}, A_k \in W^{[\frac{N}{2}]+3, \infty}(\mathbb{R}^N), \quad \forall k = 1, 2, \dots, N, \quad (1.5)$$

where  $[\frac{N}{2}]$  denotes the smallest integer exceeding  $\frac{N}{2}$ .

- (i) **Local in time existence:** Suppose  $u_0 \in B_{2,1}^{\frac{N}{2}+1}$ . Then, there exists  $T = T(\|u_0\|_{B_{2,1}^{\frac{N}{2}+1}})$  such that a unique solution  $u(t)$  of the system (1.1)-(1.2) exists, which belongs to  $L^\infty([0, T]; B_{2,1}^{\frac{N}{2}+1}) \cap C([0, T]; B_{2,1}^s) \cap \text{Lip}([0, T]; B_{2,1}^{\frac{N}{2}})$  for all  $s \in [0, \frac{N}{2} + 1)$ .

(ii) *Blow-up criterion: The local in time solution blows up in  $B_{2,1}^{\frac{N}{2}+1}$  at  $T_* > T$ , namely*

$$\limsup_{t \nearrow T_*} \|u(t)\|_{B_{2,1}^{\frac{N}{2}+1}} = \infty \quad (1.6)$$

*if and only if*

$$\int_0^{T_*} \|\nabla u(t)\|_{L^\infty} dt = \infty. \quad (1.7)$$

**Remark 1.1:** Regarding the continuity in time, although we proved below that  $u(t) \in C([0, T]; B_{2,1}^s)$  for  $s \in [0, \frac{N}{2} + 1)$ , we could actually improve it showing  $u(t) \in C([0, T]; B_{2,1}^{\frac{N}{2}+1})$  adapting the argument in the Sobolev space theory in Chapter 3 of [21].

Next, we consider the following hyperbolic conservation system with a dissipation term added.

$$\frac{\partial u}{\partial t} + \sum_{k=1}^N A_k(u) \frac{\partial u}{\partial x_k} = \mu \Delta u, \quad (x, t) \in \mathbb{R}^N \times (0, \infty) \quad (1.8)$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N, \quad (1.9)$$

where  $A_k(u)$ ,  $k = 1, 2, \dots, N$ , satisfy the conditions specified in the statement of the theorem below. The following is our main theorem for the system (1.8)-(1.9).

**Theorem 1.2 (global unique existence for small initial data)** *Here we set  $N > 2$ . Let us assume that  $A_k(\cdot)$ ,  $k = 1, \dots, N$  in (1.8) satisfy (1.4)-(1.5). Suppose  $u_0 \in B_{2,1}^{\frac{N}{2}-1} \cap \dot{B}_{2,1}^{\frac{N}{2}}$ . Then, there exists  $\varepsilon > 0$  with the following property:*

*If  $\|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} < \varepsilon$ , then there exists a unique solution  $u \in C([0, \infty); B_{2,1}^{\frac{N}{2}-1}) \cap L^1([0, \infty); \dot{B}_{2,1}^{\frac{N}{2}+1})$  of the system (1.8)-(1.9), which satisfies the estimate*

$$\sup_{0 \leq t < \infty} \|u(t)\|_{B_{2,1}^{\frac{N}{2}-1}} + C_I \mu \int_0^\infty \|u(s)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} ds \leq \|u_0\|_{B_{2,1}^{\frac{N}{2}-1}} \exp \left( C_{II} \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \right), \quad (1.10)$$

*where  $C_I, C_{II}$  are absolute constants.*

**Remark 1.2:** We will find that the local well-posedness for large initial data,  $u_0 \in B_{2,1}^{\frac{N}{2}+1}$ , holds for the system (1.8)-(1.9), by obvious modification of the proof of Theorem 1.1. For the case of large data in the less regular space considered in Theorem 1.2,  $B_{2,1}^{\frac{N}{2}-1}$ , however, the local well-posedness of (1.8)-(1.9) is open to the the author's knowledge.

After this research was completed the author was informed the work by D. Iftimie[17], where the author also considered the hyperbolic system in the Besov space, but with less general settings than those covered by Theorem 1.1.

## 2 Preliminaries

In this section we set our notations, and recall definitions on the Besov spaces. We follow [22] and [23]. Let  $\mathcal{S}$  be the Schwartz class of rapidly decreasing functions. Given  $f \in \mathcal{S}$  its Fourier transform  $\mathcal{F}(f) = \hat{f}$  is defined by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-ix \cdot \xi} f(x) dx.$$

We consider  $\varphi \in \mathcal{S}$  satisfying  $\text{Supp } \hat{\varphi} \subset \{\xi \in \mathbb{R}^n \mid \frac{1}{2} \leq |\xi| \leq 2\}$ , and  $\hat{\varphi}(\xi) > 0$  if  $\frac{2}{3} < |\xi| < \frac{3}{2}$ . Setting  $\hat{\varphi}_j = \hat{\varphi}(2^{-j}\xi)$  (In other words,  $\varphi_j(x) = 2^{jN} \varphi(2^j x)$ ), we can adjust the normalization constant in front of  $\hat{\varphi}$  so that (See e.g. Lemma 6.1.7,[3])

$$\sum_{j \in \mathbb{Z}} \hat{\varphi}_j(\xi) = 1 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

Given  $k \in \mathbb{Z}$ , we define the function  $S_k \in \mathcal{S}$  by its Fourier transform

$$\hat{S}_k(\xi) = 1 - \sum_{j \geq k+1} \hat{\varphi}_j(\xi).$$

We observe

$$\text{Supp } \hat{\varphi}_j \cap \text{Supp } \hat{\varphi}_{j'} = \emptyset \text{ if } |j - j'| \geq 2. \quad (2.1)$$

Let  $s \in \mathbb{R}$ ,  $p, q \in [0, \infty]$ . Given  $f \in \mathcal{S}'$ , we denote  $\Delta_j f = \varphi_j * f$ , and then the homogeneous Besov norm  $\|f\|_{\dot{B}_{p,q}^s}$  is defined by

$$\|f\|_{\dot{B}_{p,q}^s} = \begin{cases} \left[ \sum_{j=-\infty}^{\infty} 2^{js} \|\varphi_j * f\|_{L^p}^q \right]^{\frac{1}{q}} & \text{if } q \in [1, \infty) \\ \sup_j [2^{js} \|\varphi_j * f\|_{L^p}] & \text{if } q = \infty \end{cases}.$$

The homogeneous Besov space  $\dot{B}_{p,q}^s$  is a semi-normed space with the semi-norm given by  $\|\cdot\|_{\dot{B}_{p,q}^s}$ . For  $s > 0, p, q \in [0, \infty]$  we define the inhomogeneous Besov space norm  $\|f\|_{B_{p,q}^s}$  of  $f \in \mathcal{S}'$  as

$$\|f\|_{B_{p,q}^s} = \|f\|_{L^p} + \|f\|_{\dot{B}_{p,q}^s}.$$

The inhomogeneous Besov space is a Banach space equipped with the norm,  $\|\cdot\|_{B_{p,q}^s}$ . Let  $f \in [L^p(\mathbb{R}^N)]^N$ , and  $A_0(g)$  be the symmetric positive definite matrix introduced in (1.1), then we introduce the modified  $L^p$  norm by

$$\|f\|_{L_{A_0(g)}^p} = \left( \int_{\mathbb{R}^N} (f, A_0(g)f)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}. \quad (2.2)$$

We will also use the following modified Besov space norm.

$$\|f\|_{\dot{\beta}_{p,q}^s(g)} = \begin{cases} \left[ \sum_{-\infty}^{\infty} 2^{jq_s} \|\varphi_j * f\|_{L_{A_0(g)}^p}^q \right]^{\frac{1}{q}} & \text{if } q \in [1, \infty) \\ \sup_j \left[ 2^{js} \|\varphi_j * f\|_{L_{A_0(g)}^p} \right] & \text{if } q = \infty \end{cases}.$$

for homogeneous spaces.

For  $s > 0, p, q \in [0, \infty]$  we also define the modified inhomogeneous Besov space norm  $\|f\|_{\beta_{p,q}^s(g)}$  as

$$\|f\|_{\beta_{p,q}^s(g)} = \|f\|_{L_{A_0(g)}^p} + \|f\|_{\dot{\beta}_{p,q}^s(g)}.$$

Thanks to the condition (1.3) on  $A_0$  we have the following equivalences

$$\|f\|_{L^p, g} \sim \|f\|_{L_{A_0(g)}^p}, \quad \|f\|_{\dot{B}_{p,q}^s} \sim \|f\|_{\dot{\beta}_{p,q}^s(g)}, \quad \|f\|_{B_{p,q}^s} \sim \|f\|_{\beta_{p,q}^s(g)}. \quad (2.3)$$

We now recall the following basic lemmas. *We note that we use the same constant  $C$  in the inequalities following, which might be differ in each context.*

**Lemma 2.1 (Bernstein's Lemma)** *Assume that  $f \in L^p$ ,  $1 \leq p \leq \infty$ , and  $\text{Supp } \hat{f} \subset \{2^{j-2} \leq |\xi| < 2^j\}$ , then there exists a constant  $C_k$  such that the following inequality holds*

$$C_k^{-1} 2^{jk} \|f\|_{L^p} \leq \|D^k f\|_{L^p} \leq C_k 2^{jk} \|f\|_{L^p}. \quad (2.4)$$

As an immediate corollary of the above lemma we have the equivalence of norms,

$$\|D^k f\|_{\dot{B}_{p,q}^s} \sim \|f\|_{\dot{B}_{p,q}^{s+k}}. \quad (2.5)$$

**Lemma 2.2** *Let  $s > 0, q \in [1, \infty]$ , then there exists a constant  $C$  such that the following inequalities hold.*

$$\|fg\|_{\dot{B}_{p,q}^s} \leq C(\|f\|_{L^{p_1}} \|g\|_{\dot{B}_{p_2,q}^s} + \|g\|_{L^{r_1}} \|f\|_{\dot{B}_{r_2,q}^s}), \quad (2.6)$$

where  $p_1, r_1 \in [1, \infty]$  such that  $1/p = 1/p_1 + 1/p_2 = 1/r_1 + 1/r_2$ . If  $s_1, s_2 \leq \frac{N}{p}$ ,  $p \geq 2$ ,  $s_1 + s_2 > 0$ ,  $f \in \dot{B}_{p,1}^{s_1}$ ,  $g \in \dot{B}_{p,1}^{s_2}$ , then

$$\|fg\|_{\dot{B}_{p,1}^{s_1+s_2-\frac{N}{p}}} \leq C \|f\|_{\dot{B}_{p,1}^{s_1}} \|g\|_{\dot{B}_{p,1}^{s_2}}. \quad (2.7)$$

**Lemma 2.3** (i) *Let  $s_1, s_2 \in \mathbb{R}$ ,  $\theta \in [0, 1]$ . Then, the following interpolation inequality holds.*

$$\|f\|_{\dot{B}_{2,1}^{\theta s_1 + (1-\theta)s_2}} \leq \|f\|_{\dot{B}_{2,1}^{s_1}}^\theta \|f\|_{\dot{B}_{2,1}^{s_2}}^{1-\theta}. \quad (2.8)$$

*Similar inequality holds also for inhomogeneous norms.*

(ii) Let  $s > 0$  and  $f \in \dot{B}_{p,1}^s \cap L^\infty$ . Suppose  $G \in W^{[s]+2,\infty}(\mathbb{R}^N)$  such that  $G(0) = 0$  and  $s > 0$ . Then we have

$$\|G(f)\|_{\dot{B}_{p,1}^s} \leq C \|f\|_{\dot{B}_{p,1}^s}. \quad (2.9)$$

(iii) Let  $f, g \in \dot{B}_{p,1}^{\frac{N}{p}}$ ,  $f - g \in \dot{B}_{p,1}^s$  for  $s \in (-\frac{N}{p}, \frac{N}{p}]$ , and  $G \in W^{[\frac{N}{2}]+3,\infty}$ ,  $G'(0) = 0$ , then  $G(f) - G(g) \in \dot{B}_{p,1}^s$ , and the following inequality holds

$$\|G(f) - G(g)\|_{\dot{B}_{p,1}^s} \leq C \left( \|f\|_{\dot{B}_{p,1}^{\frac{N}{p}}} + \|g\|_{\dot{B}_{p,1}^{\frac{N}{p}}} \right) \|f - g\|_{\dot{B}_{p,1}^s}. \quad (2.10)$$

**Lemma 2.4** If  $s$  satisfies  $s \in (-\frac{N}{p} - 1, \frac{N}{p}]$ , then we have

$$\|[u, \Delta_j]w\|_{L^p} \leq c_j 2^{-j(s+1)} \|u\|_{\dot{B}_{p,1}^{\frac{N}{p}+1}} \|w\|_{\dot{B}_{p,1}^s} \quad (2.11)$$

with  $\sum_{j \in \mathbb{Z}} c_j \leq 1$ .

Lemma 2.1 is classical, and proved e.g. in [11]. The inequality (2.5) and (2.6) of Lemma 2.2 is proved e.g. in [8] and [10] respectively. In Lemma 2.3 (i) follows immediately from the definition of norms; (ii) is proved in [10], and (iii) is an immediate consequence of (ii), using the mean value theorem. Lemma 2.4 is proved in the appendix of [15].

### 3 Proof of Theorem 1.1

We define a sequence of functions  $\{u^{(m)}\}_{m=1}^\infty$  as solutions of the sequence of linear system recurrently defined by

$$A_0(u^{(m)}) \frac{\partial u^{(m+1)}}{\partial t} + \sum_{k=1}^N A_k(u^{(m)}) \frac{\partial u^{(m+1)}}{\partial x_k} = 0 \quad (x, t) \in \mathbb{R}^n \times (0, \infty) \quad (3.1)$$

$$u^{(m+1)}(x, 0) = S_{m+1} u_0(x), \quad x \in \mathbb{R}^N, \quad (3.2)$$

where  $m = 0, 1, 2, \dots$ . In particular we set  $u^{(0)} = 0$ .

(i) Uniform bounds in  $X_T := C([0, T]; B_{2,1}^{\frac{N}{2}+1})$ : We claim that there exist constants  $T_1 > 0$ ,  $M_1 > 0$  such that the solutions  $\{u^{(m)}\}$  defined by (3.1) for  $m = 1, \dots$  satisfy

$$\sup_{0 \leq t \leq T_1} \left( \|u^{(m+1)}\|_{B_{2,1}^{\frac{N}{2}+1}} + \left\| \frac{\partial u^{(m+1)}}{\partial t} \right\|_{B_{2,1}^{\frac{N}{2}}} \right) \leq M_1. \quad (3.3)$$

Taking operation  $A_0^{-1}(u^{(m)})$ , and then  $\Delta_j$  on the both sides of (3.1), we have

$$\frac{\partial}{\partial t} \Delta_j u^{(m+1)} + \sum_{k=1}^N \Delta_j \left\{ A_0^{-1}(u^{(m)}) A_k(u^{(m)}) \frac{\partial u^{(m+1)}}{\partial x_k} \right\} = 0 \quad (3.4)$$

Applying  $A_0(u^{(m)})$ , and then taking scalar product (3.4) with  $\Delta_j u$ , we obtain after elementary computations

$$\begin{aligned} \frac{\partial}{\partial t} (\Delta_j u^{(m+1)}, A_0(u^{(m)}) \Delta_j u^{(m+1)}) &= (\Delta_j u^{(m+1)}, \operatorname{div} \mathbb{A}(u^{(m)}) \Delta_j u^{(m+1)}) \\ &\quad - \sum_{k=1}^N \frac{\partial}{\partial x_k} (\Delta_j u^{(m+1)}, A_k(u^{(m)}) \Delta_j u^{(m+1)}) \\ &\quad + 2 \sum_{k=1}^N \left( \Delta_j u^{(m+1)}, A_0(u^{(m)}) [\Delta_j, A_0^{-1}(u^{(m)}) A_k(u^{(m)})] \frac{\partial u^{(m+1)}}{\partial x_k} \right), \end{aligned} \quad (3.5)$$

whereafter we use the following notations:

$$\mathbb{A} = (A_0, A_1, \dots, A_N), \quad \operatorname{div} \mathbb{A}(u^{(m)}) = \sum_{k=1}^N \frac{\partial A_k(u^{(m)})}{\partial x_k},$$

and

$$\|f\|_{L_{A_0}^2} = \left( \int_{\mathbb{R}^N} (f, A_0(u^{(m)}) f) dx \right)^{\frac{1}{2}}, \quad \|f\|_{\dot{\beta}_{2,1}^s} = \|f\|_{\dot{\beta}_{2,1}^s(u^{(m)})}.$$

Integrating (3.5) over  $\mathbb{R}^N$ , we have the estimates

$$\begin{aligned} \frac{d}{dt} \|\Delta_j u^{(m+1)}\|_{L_{A_0}^2}^2 &= \int_{\mathbb{R}^N} (\Delta_j u^{(m+1)}, \operatorname{div} \mathbb{A}(u^{(m)}) \Delta_j u^{(m+1)}) dx \\ &\quad + 2 \sum_{k=1}^N \int_{\mathbb{R}^N} \left( \Delta_j u^{(m+1)}, A_0(u^{(m)}) [\Delta_j, A_0^{-1}(u^{(m)}) A_k(u^{(m)})] \frac{\partial u^{(m+1)}}{\partial x_k} \right) dx \\ &\leq C \|\operatorname{div} \mathbb{A}(u^{(m)})\|_{L^\infty} \|\Delta_j u^{(m+1)}\|_{L_{A_0}^2}^2 \\ &\quad + C c_j 2^{-j(\frac{N}{2}+1)} \sum_{k=1}^N \|\Delta_j u^{(m+1)}\|_{L_{A_0}^2} \|A_0^{-1}(u^{(m)}) A_k(u^{(m)})\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|Du^{(m+1)}\|_{L^\infty} \\ &\leq C \|\mathbb{A}(u^{(m)})\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|\Delta_j u^{(m+1)}\|_{L_{A_0}^2}^2 \\ &\quad + C c_j 2^{-j(\frac{N}{2}+1)} \|\Delta_j u^{(m+1)}\|_{L_{A_0}^2} \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \\ &\leq C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|\Delta_j u^{(m+1)}\|_{L_{A_0}^2}^2 \\ &\quad + C c_j 2^{-j(\frac{N}{2}+1)} \|\Delta_j u^{(m+1)}\|_{L_{A_0}^2} \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)}\|_{\dot{\beta}_{2,1}^{\frac{N}{2}+1}}, \end{aligned} \quad (3.6)$$

where we used (2.7) and (2.9) for  $s = \frac{N}{2} + 1$  in the second estimate. Dividing the both sides by  $\|\Delta_j u^{(m+1)}\|_{L_{A_0}^2}$ , we obtain

$$\begin{aligned} \frac{d}{dt} \|\Delta_j u^{(m+1)}\|_{L_{A_0}^2} &\leq C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|\Delta_j u^{(m+1)}\|_{L_{A_0}^2} \\ &\quad + C c_j 2^{-j(\frac{N}{2}+1)} \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}}. \end{aligned} \quad (3.7)$$

Multiplying both sides of (3.7) by  $2^{j(\frac{N}{2}+1)}$ , and summing over  $j \in \mathbb{Z}$ , we have

$$\frac{d}{dt} \|u^{(m+1)}(t)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \leq C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}}. \quad (3.8)$$

Gronwall's lemma combined with the equivalence of norms in (2.2) implies

$$\begin{aligned} \|u^{(m+1)}(t)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} &\leq C \|S_{m+1} u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \exp \left( C \int_0^t \|u\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} ds \right) \\ &\leq C_1 \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \exp \left( C_1 T \sup_{0 \leq t \leq T} \|u^{(m)}(t)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \right) \end{aligned} \quad (3.9)$$

for some absolute constants  $C_1$ , where we used the fact

$$\begin{aligned} \|S_{m+1} u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} &\leq \sum_{j \in \mathbb{Z}} \sum_{j' \leq m+1} 2^{j(\frac{N}{2}+1)} \|\Delta_j \Delta_{j'} u_0\|_{L^2} \\ &\leq C \sum_{j' \leq m+1} 2^{j'(\frac{N}{2}+1)} \sum_{|j-j'| \leq 3} 2^{(j-j')(\frac{N}{2}+1)} \|\Delta_{j'} u_0\|_{L^2} \\ &\leq C \sum_{j' \leq m+1} 2^{j'(\frac{N}{2}+1)} \|\Delta_{j'} u_0\|_{L^2} \leq C \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}}. \end{aligned}$$

We note parenthetically that the procedure from (3.7) to (3.9) is rather formal, but can be easily justifiable by integration over  $[0, t]$  of (3.7), and then using the Gronwall lemma in the version of integral inequality instead of one of differential inequality. By the standard induction argument applied to, recalling  $u^{(0)} = 0$ , we find that for

$$T_1 = \frac{\ln 2}{2C_1 \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}}} \quad (3.10)$$

we have

$$\sup_{0 \leq t \leq T_1} \|u^{(m+1)}(t)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \leq 2C_1 \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \quad \forall m \geq 0. \quad (3.11)$$

In order to have uniform bound in  $\dot{B}_{2,1}^{\frac{N}{2}+1}$  norm of  $\{u^{(m)}\}$  we start with estimate in  $L_{A_0}^2$  norm. We take  $L^2$  inner product (3.1) with  $u^{(m+1)}$ . Then, after



integration by parts we have

$$\begin{aligned}
\frac{d}{dt} \|u^{(m+1)}\|_{L^2_{A_0}}^2 &= \int_{\mathbb{R}^N} (u^{(m+1)}, \operatorname{div} \mathbb{A}(u^{(m)}) u^{(m+1)}) dx \\
&\leq \| \operatorname{div} \mathbb{A}(u^{(m)}) \|_{L^\infty} \|u^{(m+1)}\|_{L^2}^2 \leq C \| \mathbb{A}(u^{(m)}) \|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)}\|_{L^2_{A_0}}^2 \\
&\leq C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)}\|_{L^2_{A_0}}^2.
\end{aligned} \tag{3.12}$$

Adding (3.12) to (3.8), we have

$$\frac{d}{dt} \|u^{(m+1)}(t)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \leq C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}}. \tag{3.13}$$

The same remark as the one below (3.9) holds here. By Gronwall's lemma we have

$$\|u^{(m+1)}(t)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \leq \|u_0^{(m+1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \exp \left( C \int_0^t \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} ds \right), \tag{3.14}$$

and due to the equivalence of norms in (2.2),

$$\begin{aligned}
\|u^{(m+1)}(t)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} &\leq C \|u_0^{(m+1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \exp \left( C \int_0^t \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} ds \right) \\
&\leq C \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \exp \left( CT \sup_{0 \leq t \leq T} \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \right).
\end{aligned} \tag{3.15}$$

This, combined with (3.11), implies

$$\sup_{0 \leq t \leq T_1} \|u^{(m+1)}(t)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \leq 2C_1 \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \exp(C_2 T_1 \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}}) \quad \forall m \geq 0, \tag{3.16}$$

where  $C_2$  is an absolute constant. From (3.4) we have

$$\left\| \Delta_j \frac{\partial u^{(m+1)}}{\partial t} \right\|_{L^2} \leq \sum_{k=1}^N \left\| \Delta_j \left\{ A_0^{-1}(u^{(m)}) A_k(u^{(m)}) \frac{\partial u^{(m+1)}}{\partial x_k} \right\} \right\|_{L^2}. \tag{3.17}$$

Multiplying  $2^{\frac{N}{2}j}$ , and summing over  $j \in \mathbb{Z}$  of (3.17), we find

$$\begin{aligned}
\left\| \frac{\partial u^{(m+1)}}{\partial t} \right\|_{\dot{B}_{2,1}^{\frac{N}{2}}} &\leq \sum_{k=1}^N \left\| A_0^{-1}(u^{(m)}) A_k(u^{(m)}) \frac{\partial u^{(m+1)}}{\partial x_k} \right\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \\
&\leq C \sum_{k=1}^N \|A_0^{-1}(u^{(m)}) A_k(u^{(m)})\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \left\| \frac{\partial u^{(m+1)}}{\partial x_k} \right\|_{L^\infty} \\
&\quad + C \sum_{k=1}^N \|A_0^{-1}(u^{(m)}) A_k(u^{(m)})\|_{L^\infty} \left\| \frac{\partial u^{(m+1)}}{\partial x_k} \right\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \\
&\leq C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \|u^{(m+1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}},
\end{aligned} \tag{3.18}$$

where we used (2.8), and the imbedding  $\dot{B}_{2,1}^{\frac{N}{2}} \hookrightarrow L^\infty(\mathbb{R}^N)$ . On the other hand, from (3.1) we have immediately

$$\begin{aligned}
\left\| \frac{\partial u^{(m+1)}}{\partial t} \right\|_{L^2} &\leq \sum_{k=1}^N \left\| A_0^{-1}(u^{(m)}) A_k(u^{(m)}) \frac{\partial u^{(m+1)}}{\partial x_k} \right\|_{L^2} \\
&\leq C \sum_{k=1}^N \|A_0^{-1}(u^{(m)}) A_k(u^{(m)})\|_{L^\infty} \left\| \frac{\partial u^{(m+1)}}{\partial x_k} \right\|_{L^2} \\
&\leq C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \|Du^{(m+1)}\|_{L^2}.
\end{aligned} \tag{3.19}$$

Adding (3.19) to (3.18) we find

$$\left\| \frac{\partial u^{(m+1)}}{\partial t} \right\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \leq C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \|u^{(m+1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}}. \tag{3.20}$$

From (3.16), we obtain

$$\sup_{0 \leq t \leq T_1} \left\| \frac{\partial u^{(m+1)}}{\partial t} \right\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \leq C. \tag{3.21}$$

Combining (3.21) with (3.16), we completes the proof of (3.3).

(ii) *Contraction in  $Y_T := C([0, T]; B_{2,1}^s, s \in [0, \frac{N}{2} + 1])$ :* We will show that  $\|u^{(m+1)} - u^{(m)}\|_{Y_{T_0}} \rightarrow 0$  as  $m \rightarrow \infty$  for some  $T_0 \in (0, T_1]$ . Taking difference between the  $(m+1)$ th equation and the  $(m)$ th equation, we obtain after arrangement of terms

$$\begin{aligned}
&\frac{\partial}{\partial t} \Delta_j(u^{(m+1)} - u^{(m)}) + \sum_{k=1}^N \Delta_j \left\{ (A_0^{-1} A_k)(u^{(m)}) \frac{\partial(u^{(m+1)} - u^{(m)})}{\partial x_k} \right\} \\
&= - \sum_{k=1}^N \Delta_j \left\{ ((A_0^{-1} A_k)(u^{(m)}) - (A_0^{-1} A_k)(u^{(m-1)})) \frac{\partial u^{(m)}}{\partial x_k} \right\}.
\end{aligned} \tag{3.22}$$

Applying  $A_0(u^{(m)})$ , and then taking scalar product in  $\mathbb{R}^N$  of (3.22) with  $\Delta_j(u^{(m+1)} - u^{(m)})$  we obtain by the computations similar to the above

$$\begin{aligned}
&\frac{\partial}{\partial t} (\Delta_j(u^{(m+1)} - u^{(m)}), A_0(u^{(m)}) \Delta_j(u^{(m+1)} - u^{(m)})) \\
&= (\Delta_j(u^{(m+1)} - u^{(m)}), \operatorname{div} \mathbb{A}(u^{(m)}) \Delta_j(u^{(m+1)} - u^{(m)})) \\
&\quad - \sum_{k=1}^N \frac{\partial}{\partial x_k} (\Delta_j(u^{(m+1)} - u^{(m)}), A_k(u^{(m)}) \Delta_j(u^{(m+1)} - u^{(m)}))
\end{aligned}$$

$$\begin{aligned}
& +2 \sum_{k=1}^N \left( \Delta_j(u^{(m+1)} - u^{(m)}), A_0(u^{(m)}) [\Delta_j, A_0^{-1}(u^{(m)}) A_k(u^{(m)})] \frac{\partial}{\partial x_k} (u^{(m+1)} - u^{(m)}) \right) \\
& - \sum_{k=1}^N \left( \Delta_j(u^{(m+1)} - u^{(m)}), A_0(u^{(m)}) \Delta_j \left\{ ((A_0^{-1} A_k)(u^{(m)}) - (A_0^{-1} A_k)(u^{(m-1)})) \frac{\partial u^{(m)}}{\partial x_k} \right\} \right).
\end{aligned}$$

Integrating both sides of this equation over  $\mathbb{R}^n$ , we have the following estimates

$$\begin{aligned}
& \frac{d}{dt} \|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L_{A_0}^2}^2 \\
& = \int_{\mathbb{R}^N} (\Delta_j(u^{(m+1)} - u^{(m)}), \operatorname{div} \mathbb{A}(u^{(m)}) \Delta_j(u^{(m+1)} - u^{(m)})) dx \\
& + 2 \sum_{k=1}^N \int_{\mathbb{R}^N} \left( \Delta_j(u^{(m+1)} - u^{(m)}), A_0(u^{(m)}) [\Delta_j, A_0^{-1}(u^{(m)}) A_k(u^{(m)})] \frac{\partial}{\partial x_k} (u^{(m+1)} - u^{(m)}) \right) dx \\
& - \sum_{k=1}^N \int_{\mathbb{R}^N} \left( \Delta_j(u^{(m+1)} - u^{(m)}), A_0(u^{(m)}) \Delta_j \left\{ ((A_0^{-1} A_k)(u^{(m)}) - (A_0^{-1} A_k)(u^{(m-1)})) \frac{\partial u^{(m)}}{\partial x_k} \right\} \right) dx \\
& \leq C \|\operatorname{div} \mathbb{A}(u^{(m)})\|_{L^\infty} \|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L_{A_0}^2}^2 \\
& + C c_j 2^{-\frac{N}{2}j} \sum_{k=1}^N \|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L_{A_0}^2} \|A_0^{-1}(u^{(m)}) A_k(u^{(m)})\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \\
& + C \|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L_{A_0}^2} \sum_{k=1}^N \left\| \Delta_j \left[ \left\{ (A_0^{-1} A_k)(u^{(m)}) - (A_0^{-1} A_k)(u^{(m-1)}) \right\} \frac{\partial u^{(m)}}{\partial x_k} \right] \right\|_{L^2} \\
& \leq C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L_{A_0}^2}^2 \\
& + C c_j 2^{-\frac{N}{2}j} \|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L_{A_0}^2} \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \\
& + C \|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L_{A_0}^2} \sum_{k=1}^N \left\| \Delta_j \left[ \left\{ (A_0^{-1} A_k)(u^{(m)}) - (A_0^{-1} A_k)(u^{(m-1)}) \right\} \frac{\partial u^{(m)}}{\partial x_k} \right] \right\|_{L^2}.
\end{aligned}$$

Dividing both sides by  $\|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L_{A_0}^2}$ , we obtain

$$\begin{aligned}
& \frac{d}{dt} \|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L_{A_0}^2} \leq C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L_{A_0}^2} \\
& + C c_j 2^{-\frac{N}{2}j} \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \\
& + C \sum_{k=1}^N \left\| \Delta_j \left[ \left\{ (A_0^{-1} A_k)(u^{(m)}) - (A_0^{-1} A_k)(u^{(m-1)}) \right\} \frac{\partial u^{(m)}}{\partial x_k} \right] \right\|_{L^2}.
\end{aligned}$$

Multiplying  $2^{\frac{N}{2}j}$  and, then summing over  $j \in \mathbb{Z}$  both sides, we obtain

$$\begin{aligned}
& \frac{d}{dt} \|u^{(m+1)}(t) - u^{(m)}(t)\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \leq C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \\
& \quad + C \sum_{k=1}^N \left\| \{ (A_0^{-1} A_k)(u^{(m)}) - (A_0^{-1} A_k)(u^{(m-1)}) \} \frac{\partial u^{(m)}}{\partial x_k} \right\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \\
& \leq C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \\
& \quad + C \sum_{k=1}^N \| (A_0^{-1} A_k)(u^{(m)}) - (A_0^{-1} A_k)(u^{(m-1)}) \|_{\dot{B}_{2,1}^{\frac{N}{2}}} \left\| \frac{\partial u^{(m)}}{\partial x_k} \right\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \\
& \leq C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \\
& \quad + C (\|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}} + \|u^{(m-1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}}) \|u^{(m)} - u^{(m-1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \\
& \leq C \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}} + C \|u^{(m)} - u^{(m-1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \quad (3.23)
\end{aligned}$$

for  $t \in [0, T_1]$ , where we used (2.10) combined with the uniform estimate (3.11). The corresponding estimate of  $\|u^{(m+1)} - u^{(m)}\|_{L_{A_0}^2}$  is similar the above, and simpler. Taking  $L^2$  inner product with the equation of  $A_0(u^{(m)}) \frac{\partial}{\partial t}(u^{(m+1)} - u^{(m)})$  by  $u^{(m+1)} - u^{(m)}$ , and then estimating using the Hölder inequality, and integrating by part, and then dividing the both sides by  $\|u^{(m+1)} - u^{(m)}\|_{L_{A_0}^2}$ , we have

$$\begin{aligned}
& \frac{d}{dt} \|u^{(m+1)}(t) - u^{(m)}(t)\|_{L_{A_0}^2} \leq C \|\operatorname{div} \mathbb{A}(u^{(m)})\|_{L^\infty} \|u^{(m+1)} - u^{(m)}\|_{L_{A_0}^2} \\
& \quad + C \sum_{k=1}^N \| [A_k(u^{(m)}) - A_k(u^{(m-1)})] \|_{L^2} \left\| \frac{\partial u^{(m)}}{\partial x_k} \right\|_{L^\infty} \\
& \leq C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)} - u^{(m)}\|_{L_{A_0}^2} + C \|u^{(m)} - u^{(m-1)}\|_{L^2} \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \\
& \leq C \|u^{(m+1)} - u^{(m)}\|_{L_{A_0}^2} + C \|u^{(m)} - u^{(m-1)}\|_{L_{A_0}^2} \quad (3.24)
\end{aligned}$$

for  $t \in [0, T_1]$ . Adding (3.24) to (3.23), we have

$$\begin{aligned}
& \frac{d}{dt} \|u^{(m+1)}(t) - u^{(m)}(t)\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \leq C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \\
& \leq C \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}} + C \|u^{(m)} - u^{(m-1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \quad (3.25)
\end{aligned}$$

on  $[0, T_1]$ . Using Gronwall's inequality, and the equivalence of norms in (2.2), we obtain

$$\begin{aligned}
\|u^{(m+1)} - u^{(m)}\|_{Y_T} & \leq C e^{CT} (\|u_0^{(m+1)} - u_0^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}} + T \|u^{(m)} - u^{(m-1)}\|_{Y_T}) \\
& \leq C e^{CT} (\|\Delta_m u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}}} + T \|u^{(m)} - u^{(m-1)}\|_{Y_T}). \quad (3.26)
\end{aligned}$$

for  $T \leq T_1$  From this combined with the fact

$$\|\Delta_m u_0\|_{B_{2,1}^{\frac{N}{2}}} \leq C2^{-m} \|\Delta_m u_0\|_{B_{2,1}^{\frac{N}{2}+1}} \leq C2^{-m} \|u_0\|_{B_{2,1}^{\frac{N}{2}+1}} \leq C2^{-m}, \quad (3.27)$$

which follows from Bernstein's lemma, we can easily deduce that there exists  $u \in L^\infty([0, T_1]; B_{2,1}^{\frac{N}{2}+1})$  such that the sequence,  $\{u^{(m)}\}$  converges to  $u$  in  $Y_T$ . Due to interpolation inequality (2.7) for inhomogeneous space we have for all  $s \in [\frac{N}{2}, \frac{N}{2} + 1)$

$$\begin{aligned} & \sup_{0 \leq t \leq T} \|u(t) - u^{(m)}(t)\|_{B_{2,1}^s} \\ & \leq \sup_{0 \leq t \leq T} \left\{ \|u(t) - u^{(m)}(t)\|_{B_{2,1}^{\frac{N}{2}}}^{1+\frac{N}{2}-s} (\|u(t)\|_{B_{2,1}^{\frac{N}{2}+1}} + \|u^{(m)}(t)\|_{B_{2,1}^{\frac{N}{2}+1}})^{s-\frac{N}{2}} \right\} \\ & \leq C \sup_{0 \leq t \leq T} \|u - u^{(m)}\|_{B_{2,1}^{\frac{N}{2}}}^{1+\frac{N}{2}-s} \rightarrow 0 \quad \text{as } m \rightarrow \infty, \end{aligned}$$

and by the three  $\varepsilon$ -argument we find  $u \in C([0, T]; B_{2,1}^s)$  for all  $s \in [\frac{N}{2}, \frac{N}{2} + 1)$ . Now, from the uniform estimate (3.16) we have

$$\|u^{(m)}(t_1) - u^{(m)}(t_2)\|_{B_{2,1}^{\frac{N}{2}}} \leq \int_{t_2}^{t_1} \left\| \frac{\partial u^{(m+1)}}{\partial t} \right\|_{B_{2,1}^{\frac{N}{2}}} dt \leq C|t_1 - t_2|$$

for all  $t_1, t_2 \in [0, T_1]$ . Thus, we have

$$\|u(t_1) - u(t_2)\|_{B_{2,1}^{\frac{N}{2}}} \leq \|u(t_1) - u^{(m)}(t_1)\|_{B_{2,1}^{\frac{N}{2}}} + C|t_1 - t_2| + \|u(t_2) - u^{(m)}(t_2)\|_{B_{2,1}^{\frac{N}{2}}}. \quad (3.28)$$

Passing to limit  $m \rightarrow \infty$  in (3.28), we obtain  $u \in \text{Lip}([0, T_1]; B_{2,1}^{\frac{N}{2}})$ . The proof of uniqueness of solution is similar to the contraction part of the proof above, and we will be very brief. If  $u_1, u_2$  are the two solutions corresponding to initial datum,  $u_{1,0}, u_{2,0}$  respectively, then Following the same procedure leading to (3.26), we obtain

$$\|u_1 - u_2\|_{Y_T} \leq Ce^{CT} (\|u_{1,0} - u_{2,0}\|_{B_{2,1}^{\frac{N}{2}}} + T\|u_1 - u_2\|_{Y_T}),$$

from which we deduce that if  $u_{1,0} = u_{2,0}$ , then  $u_1(t) = u_2(t)$  for all  $t \in [0, T_1]$  if  $Ce^{CT_2 T_2} < 1$ . Thus the uniqueness of solution is proved on  $[0, T]$  for  $T \leq \min\{T_1, T_2\}$ . In order to prove the continuation principle we just observe that starting from (1.1), and following the procedure leading to (3.15), we find that

$$\sup_{0 \leq t \leq T} \|u(t)\|_{B_{2,1}^{\frac{N}{2}+1}} \leq C\|u_0\|_{B_{2,1}^{\frac{N}{2}+1}} \exp \left( C \int_0^T \|\nabla u\|_{L^\infty} ds \right). \quad (3.29)$$

On the other hand, we have the obvious inequalities,

$$\int_0^T \|\nabla u\|_{L^\infty} dt \leq C \int_0^T \|u(s)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} ds \leq CT \sup_{0 \leq t \leq T} \|u(t)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}}, \quad (3.30)$$

thanks to the embedding  $\dot{B}_{2,1}^{\frac{N}{2}+1} \hookrightarrow Lip(\mathbb{R}^N)$ . The blow-up criterion follows from (3.29) and (3.30) immediately. This completes the proof of Theorem 1.1.  $\square$

## 4 Proof of Theorem 1.2

We define a sequence  $\{u^{(m)}\}$  by solving the linear system iteratively

$$\frac{\partial u^{(m+1)}}{\partial t} + \sum_{k=1}^N A_k(u^{(m)}) \frac{\partial u^{(m+1)}}{\partial x_k} = \mu \Delta u^{(m+1)}, \quad (x, t) \in \mathbb{R}^N \times (0, \infty) \quad (4.1)$$

$$u^{(m+1)}(x, 0) = S_m u_0(x), \quad x \in \mathbb{R}^N, \quad (4.2)$$

where  $m = 0, 1, 2, \dots$ , and we set  $u^{(0)} = 0$ .

*(i) Uniform bounds in  $C([0, \infty); B_{2,1}^{\frac{N}{2}-1}) \cap C([0, \infty); \dot{B}_{2,1}^{\frac{N}{2}}) \cap L^1([0, \infty); \dot{B}_{2,1}^{\frac{N}{2}+1})$  for small  $\|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}}$ :*

Taking operation  $\Delta_j$  on (4.1), we obtain

$$\frac{\partial}{\partial t} \Delta_j u^{(m+1)} + \sum_{k=1}^N \Delta_j \left\{ A_k(u^{(m)}) \frac{\partial u^{(m+1)}}{\partial x_k} \right\} = \mu \Delta \Delta_j u^{(m+1)}. \quad (4.3)$$

Taking  $L^2$  inner product of (4.3) with  $\Delta_j u^{(m+1)}$ , we have after integration by parts

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Delta_j u^{(m+1)}\|_{L^2}^2 + \mu \|\nabla \Delta_j u^{(m+1)}\|_{L^2}^2 \\ &= - \sum_{k=1}^N \left( [\Delta_j, A_k(u^{(m)})] \frac{\partial u^{(m+1)}}{\partial x_k}, \Delta_j u^{(m+1)} \right)_{L^2} \\ & \quad - \frac{1}{2} \sum_{k=1}^N \left( \Delta_j u^{(m+1)} \frac{\partial A_k(u^{(m)})}{\partial x_k}, \Delta_j u^{(m+1)} \right)_{L^2}. \end{aligned} \quad (4.4)$$

Thus, using (2.3), (2.11), we estimate

$$\begin{aligned}
& \frac{d}{dt} \|\Delta_j u^{(m+1)}\|_{L^2}^2 + C\mu 2^{2j} \|\Delta_j u^{(m+1)}\|_{L^2}^2 \\
& \leq 2 \sum_{k=1}^N \left\| [\Delta_j, A_k(u^{(m)})] \frac{\partial u^{(m+1)}}{\partial x_k} \right\|_{L^2} \|\Delta_j u^{(m+1)}\|_{L^2} \\
& \quad + \sum_{k=1}^N \left\| \frac{\partial A_k(u^{(m)})}{\partial x_k} \right\|_{L^\infty} \|\Delta_j u^{(m+1)}\|_{L^2}^2 \\
& \leq Cc_j 2^{-j(\frac{N}{2}-1)} \sum_{k=1}^N \|A_k(u^{(m)})\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \|\Delta_j u^{(m+1)}\|_{L^2} \\
& \quad + C \sum_{k=1}^N \|A_k(u^{(m)})\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|\Delta_j u^{(m+1)}\|_{L^2}^2. \tag{4.5}
\end{aligned}$$

Dividing the both sides by  $\|\Delta_j u^{(m+1)}\|_{L^2}$ , and using (2.9), we obtain

$$\begin{aligned}
& \frac{d}{dt} \|\Delta_j u^{(m+1)}\|_{L^2} + C\mu 2^{2j} \|\Delta_j u^{(m+1)}\|_{L^2} \\
& \leq Cc_j 2^{-j(\frac{N}{2}-1)} \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \\
& \quad + C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|\Delta_j u^{(m+1)}\|_{L^2}. \tag{4.6}
\end{aligned}$$

Multiplying  $2^{j(\frac{N}{2}-1)}$ , and summing over  $j \in \mathbb{Z}$ , we find

$$\frac{d}{dt} \|u^{(m+1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} + C_1 \mu \|u^{(m+1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \leq C_2 \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}}. \tag{4.7}$$

By Gronwall's lemma we have

$$\begin{aligned}
& \sup_{0 \leq t < \infty} \|u^{(m+1)}(t)\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} + C_1 \mu \int_0^\infty \|u^{(m+1)}(s)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} ds \\
& \leq \|u_0^{(m+1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \exp \left( C \int_0^\infty \|u^{(m)}(s)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} ds \right) \\
& \leq C_2 \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \exp \left( C_2 \int_0^\infty \|u^{(m)}(s)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} ds \right) \tag{4.8}
\end{aligned}$$

for  $m = 1, 2, \dots$ . We set  $K = \max\{C_1 \mu, C_2\}$ . Let the initial data  $u_0$  satisfy  $\|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \leq \frac{\ln(2/K)}{2K}$ , namely

$$K \exp(2K \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}}) \leq 2. \tag{4.9}$$

Then, from our setting  $u^{(0)} = 0$ , and using (4.8)-(4.9), we can see easily

$$\sup_{0 \leq t < \infty} \|u^{(m+1)}(t)\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} + C_1 \mu \int_0^\infty \|u^{(m+1)}(s)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} ds \leq 2 \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \tag{4.10}$$

for all  $m = 1, 2, \dots$  by an induction argument.

Taking  $L^2$  inner product of (4.1) with  $u^{(m+1)}$ , we have after integration by parts

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u^{(m+1)}\|_{L^2}^2 + \mu \|\nabla u^{(m+1)}\|_{L^2}^2 &= -\frac{1}{2} \sum_{k=1}^N \left( u^{(m+1)} \frac{\partial A_k(u^{(m)})}{\partial x_k}, u^{(m+1)} \right)_{L^2} \\ &\leq \frac{1}{2} \sum_{k=1}^N \left\| \frac{\partial A_k(u^{(m)})}{\partial x_k} \right\|_{L^\infty} \|u^{(m+1)}\|_{L^2}^2 \leq \frac{1}{2} \sum_{k=1}^N \|A_k(u^{(m)})\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)}\|_{L^2}^2 \\ &\leq C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)}\|_{L^2}^2. \end{aligned} \quad (4.11)$$

By Gronwall's lemma, we obtain

$$\begin{aligned} \sup_{0 \leq t < \infty} \|u^{(m+1)}(t)\|_{L^2} &\leq \|u_0^{(m+1)}\|_{L^2} \exp \left( C_2 \int_0^\infty \|u^{(m)}(t)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} dt \right) \\ &\leq C_2 \|u_0\|_{L^2} \exp \left( C_2 \int_0^\infty \|u^{(m)}(t)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} dt \right) \end{aligned} \quad (4.12)$$

Combining (4.11) and (4.12), we have

$$\begin{aligned} \sup_{0 \leq t < \infty} \|u^{(m+1)}(t)\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} + C_1 \mu \int_0^\infty \|u^{(m+1)}(s)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} ds \\ \leq C_2 \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \exp \left( C_2 \int_0^\infty \|u^{(m)}(s)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} ds \right). \end{aligned} \quad (4.13)$$

for  $m = 1, 2, \dots$ . From an uniform estimate (4.13), and (4.9) we have

$$\sup_{0 \leq t < \infty} \|u^{(m+1)}(t)\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} + C_1 \mu \int_0^\infty \|u^{(m+1)}(s)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} ds \leq 2 \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \quad (4.14)$$

for all  $m = 1, 2, \dots$ , if  $u_0 \in \dot{B}_{2,1}^{\frac{N}{2}-1}$  satisfies (4.9).

Next, we derive the uniform estimate of  $u^{(m)}$  in  $C([0, \infty); \dot{B}_{2,1}^{\frac{N}{2}})$ . Following the same procedure as in (4.1) to (4.5), but using (2.11) with  $s = \frac{N}{2} - 1$  instead of  $s = \frac{N}{2} - 2$ , we obtain

$$\begin{aligned} \frac{d}{dt} \|\Delta_j u^{(m+1)}\|_{L^2}^2 + C \mu 2^{2j} \|\Delta_j u^{(m+1)}\|_{L^2}^2 \\ \leq C c_j 2^{-j \frac{N}{2}} \sum_{k=1}^N \|A_k(u^{(m)})\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \|\Delta_j u^{(m+1)}\|_{L^2} \\ + C \sum_{k=1}^N \|A_k(u^{(m)})\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|\Delta_j u^{(m+1)}\|_{L^2}^2. \end{aligned}$$



Dividing the both sides by  $\|\Delta_j u^{(m+1)}\|_{L^2}$ , and using (2.9) again, we obtain

$$\begin{aligned} \frac{d}{dt} \|\Delta_j u^{(m+1)}\|_{L^2} &\leq C c_j 2^{-j \frac{N}{2}} \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \\ &\quad + C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|\Delta_j u^{(m+1)}\|_{L^2}. \end{aligned}$$

Multiplying  $2^{j \frac{N}{2}}$ , and summing over  $j \in \mathbb{Z}$ , we find

$$\frac{d}{dt} \|u^{(m+1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \leq C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}}.$$

By Gronwall's lemma we have

$$\begin{aligned} \sup_{0 \leq t < \infty} \|u^{(m+1)}(t)\|_{\dot{B}_{2,1}^{\frac{N}{2}}} &\leq \|u_0^{(m+1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \exp \left( C \int_0^\infty \|u^{(m)}(s)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} ds \right) \\ &\leq C \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \exp \left( C \int_0^\infty \|u^{(m)}(s)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} ds \right) \end{aligned}$$

for  $m = 1, 2, \dots$ . Since we have the uniform bound,

$$\sup_{m \in \mathbb{N}} \int_0^\infty \|u^{(m)}(s)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} ds < \infty$$

by (4.10), we have the desired uniform estimate in  $C([0, \infty); \dot{B}_{2,1}^{\frac{N}{2}})$ .

(ii) *Contraction in  $C([0, \infty); B_{2,1}^{\frac{N}{2}-1}) \cap L^1([0, \infty); \dot{B}_{2,1}^{\frac{N}{2}+1})$ :* Taking difference between  $(m)$ th equation and  $(m-1)$ th equations of (4.1), we have

$$\begin{aligned} &\frac{\partial(u^{(m+1)} - u^{(m)})}{\partial t} + \sum_{k=1}^N A_k(u^{(m)}) \frac{\partial(u^{(m+1)} - u^{(m)})}{\partial x_k} \\ &= - \sum_{k=1}^N (A_k(u^{(m)}) - A_k(u^{(m-1)})) \frac{\partial u^{(m)}}{\partial x_k} + \mu \Delta(u^{(m+1)} - u^{(m)}), \end{aligned} \quad (4.15)$$

$$(u^{(m+1)} - u^{(m)})(0, x) = \Delta_m u_0(x). \quad (4.16)$$

Operating  $\Delta_j$  on (4.15), and then taking  $L^2$  inner product with  $\Delta_j(u^{(m+1)} -$

$u^{(m)}$ ), we find after integration by part,

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L^2}^2 + \mu \|\nabla \Delta_j(u^{(m+1)} - u^{(m)})\|_{L^2}^2 \\
&= - \sum_{k=1}^N \left( [\Delta_j, A_k(u^{(m)})] \frac{\partial(u^{(m+1)} - u^{(m)})}{\partial x_k}, \Delta_j(u^{(m+1)} - u^{(m)}) \right)_{L^2} \\
&\quad + \frac{1}{2} \sum_{k=1}^N \left( \Delta_j(u^{(m+1)} - u^{(m)}) \frac{\partial A_k(u^{(m)})}{\partial x_k}, \Delta_j(u^{(m+1)} - u^{(m)}) \right)_{L^2} \\
&\quad - \sum_{k=1}^N \left( \Delta_j \left\{ [A_k(u^{(m)}) - A_k(u^{(m-1)})] \frac{\partial u^{(m)}}{\partial x_k} \right\}, \Delta_j(u^{(m+1)} - u^{(m)}) \right)_{L^2} \\
&\leq \sum_{k=1}^N \left\| [\Delta_j, A_k(u^{(m)})] \frac{\partial u^{(m+1)} - u^{(m)}}{\partial x_k} \right\|_{L^2} \|\Delta_j u^{(m+1)} - u^{(m)}\|_{L^2} \\
&\quad + \frac{1}{2} \sum_{k=1}^N \left\| \frac{\partial A_k(u^{(m)})}{\partial x_k} \right\|_{L^\infty} \|\Delta_j u^{(m+1)} - u^{(m)}\|_{L^2}^2 \\
&\quad + \sum_{k=1}^N \left\| \Delta_j \left\{ [A_k(u^{(m)}) - A_k(u^{(m-1)})] \frac{\partial u^{(m)}}{\partial x_k} \right\} \right\|_{L^2} \|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L^2} \\
&\leq C c_j 2^{-j(\frac{N}{2}-1)} \sum_{k=1}^N \|A_k(u^{(m)})\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L^2} \\
&\quad + C \sum_{k=1}^N \|A_k(u^{(m)})\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L^2}^2 \\
&\quad + \sum_{k=1}^N \left\| \Delta_j \left\{ [A_k(u^{(m)}) - A_k(u^{(m-1)})] \frac{\partial u^{(m)}}{\partial x_k} \right\} \right\|_{L^2} \|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L^2} \\
&\leq C c_j 2^{-j(\frac{N}{2}-1)} \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L^2} \\
&\quad + C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L^2}^2 \\
&\quad + \sum_{k=1}^N \left\| \Delta_j \left\{ [A_k(u^{(m)}) - A_k(u^{(m-1)})] \frac{\partial u^{(m)}}{\partial x_k} \right\} \right\|_{L^2} \|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L^2}.
\end{aligned} \tag{4.17}$$

As previously, we divide by  $\|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L^2}$ , and multiply  $2^{j(\frac{N}{2}-1)}$ , and

sum over  $j \in \mathbb{Z}$ . Then we have

$$\begin{aligned}
& \frac{d}{dt} \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} + C\mu \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \\
& \leq C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \\
& \quad + C \sum_{k=1}^N \left\| [A_k(u^{(m)}) - A_k(u^{(m-1)})] \frac{\partial u^{(m)}}{\partial x_k} \right\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \\
& \leq C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \\
& \quad + C \|A_k(u^{(m)}) - A_k(u^{(m-1)})\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \\
& \leq C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \\
& \quad + C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m)} - u^{(m-1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}}. \tag{4.18}
\end{aligned}$$

where we used (2.10) and the uniform estimate of  $\{u^{(m)}\}$  in  $C([0, \infty); \dot{B}_{2,1}^{\frac{N}{2}})$  proved in the step (i) above. Taking  $L^2$  inner product (4.15) with  $u^{(m+1)} - u^{(m)}$ , we have after integration by part

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|u^{(m+1)} - u^{(m)}\|_{L^2}^2 + \mu \|\nabla(u^{(m+1)} - u^{(m)})\|_{L^2}^2 \\
& = -\frac{1}{2} \sum_{k=1}^N \left( (u^{(m+1)} - u^{(m)}) \frac{\partial A_k(u^{(m)})}{\partial x_k}, (u^{(m+1)} - u^{(m)}) \right)_{L^2} \\
& \quad + \sum_{k=1}^N \left( \left\{ [A_k(u^{(m)}) - A_k(u^{(m-1)})] \frac{\partial u^{(m)}}{\partial x_k} \right\}, u^{(m+1)} - u^{(m)} \right)_{L^2} \\
& \leq \frac{1}{2} \sum_{k=1}^N \left\| \frac{\partial A_k(u^{(m)})}{\partial x_k} \right\|_{L^\infty} \|u^{(m+1)} - u^{(m)}\|_{L^2}^2 \\
& \quad + \sum_{k=1}^N \|A_k(u^{(m)}) - A_k(u^{(m-1)})\|_{L^2} \left\| \frac{\partial u^{(m)}}{\partial x_k} \right\|_{L^\infty} \|u^{(m+1)} - u^{(m)}\|_{L^2} \\
& \leq C \sum_{k=1}^N \|A_k(u^{(m)})\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)} - u^{(m)}\|_{L^2}^2 \\
& \quad + C \sum_{k=1}^N \|A_k(u^{(m)}) - A_k(u^{(m-1)})\|_{L^2} \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)} - u^{(m)}\|_{L^2} \\
& \leq C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)} - u^{(m)}\|_{L^2}^2 \\
& \quad + C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m)} - u^{(m-1)}\|_{L^2} \|u^{(m+1)} - u^{(m)}\|_{L^2}. \tag{4.19}
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|u^{(m+1)} - u^{(m)}\|_{L^2}^2 + \mu \|\nabla(u^{(m+1)} - u^{(m)})\|_{L^2} \\
& \leq C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)} - u^{(m)}\|_{L^2} \\
& \quad + C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m)} - u^{(m-1)}\|_{L^2}.
\end{aligned} \tag{4.20}$$

Adding (4.20) to (4.18), we have

$$\begin{aligned}
& \frac{d}{dt} \|u^{(m+1)} - u^{(m)}\|_{B_{2,1}^{\frac{N}{2}-1}} + C\mu \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \\
& \leq C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)} - u^{(m)}\|_{B_{2,1}^{\frac{N}{2}-1}} \\
& \quad + C \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m)} - u^{(m-1)}\|_{B_{2,1}^{\frac{N}{2}-1}}.
\end{aligned} \tag{4.21}$$

By Gronwall's lemma we obtain

$$\begin{aligned}
& \sup_{0 \leq t < \infty} \|u^{(m+1)}(t) - u^{(m)}(t)\|_{B_{2,1}^{\frac{N}{2}-1}} + C_3\mu \int_0^\infty \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} dt \\
& \leq \|u_0^{(m+1)} - u_0^{(m)}\|_{B_{2,1}^{\frac{N}{2}-1}} \exp \left( C \int_0^\infty \|u^{(m)}(t)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} dt \right) \\
& \quad + C \sup_{0 \leq t < \infty} \|u^{(m)}(t) - u^{(m-1)}(t)\|_{B_{2,1}^{\frac{N}{2}-1}} \int_0^\infty \|u^{(m)}(t)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} dt \times \\
& \quad \times \exp \left( C \int_0^\infty \|u^{(m)}(t)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} dt \right) \\
& \leq C 2^{(\frac{N}{2}-1)m} \|\Delta_m u_0\|_{L^2} \exp \left( C_4 \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \right) \\
& \quad + C_5 \left[ \sup_{0 \leq t < \infty} \|u^{(m)}(t) - u^{(m-1)}(t)\|_{B_{2,1}^{\frac{N}{2}-1}} + C_3\mu \int_0^\infty \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} dt \right] \times \\
& \quad \times \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \exp \left( C_4 \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \right).
\end{aligned} \tag{4.22}$$

Thus, if  $\|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}}$  is so small that

$$C_5 \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \exp(C_4 \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}}) \leq \frac{1}{2} \tag{4.23}$$

then we have

$$\delta_{m+1} \leq C\alpha_m + \frac{1}{2}\delta_m, \tag{4.24}$$

where we set

$$\delta_{m+1} = \sup_{0 \leq t < \infty} \|u^{(m+1)}(t) - u^{(m)}(t)\|_{B_{2,1}^{\frac{N}{2}-1}} + C_3\mu \int_0^\infty \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} dt,$$

and

$$\alpha_m = 2^{(\frac{N}{2}-1)m} \|\Delta_m u_0\|_{L^2}.$$

Since

$$\sum_{m=1}^{\infty} \alpha_m \leq \sum_{m=-\infty}^{\infty} 2^{(\frac{N}{2}-1)m} \|\Delta_m u_0\|_{L^2} = \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} < \infty,$$

we deduce easily that  $\delta_m \rightarrow 0$  as  $m \rightarrow \infty$ , and thus  $\{u^{(m)}\}$  is a Cauchy sequence, and there exists  $u$  in  $L^\infty([0, \infty); B_{2,1}^{\frac{N}{2}-1}) \cap L^1([0, \infty); \dot{B}_{2,1}^{\frac{N}{2}+1})$ , which is the limit if the sequence  $\{u^{(m)}\}$ , and a solution of the system (1.12)-(1.13). The fact  $u \in C([0, \infty); B_{2,1}^{\frac{N}{2}-1})$  follows by the convergence,  $u^{(m)} \rightarrow u$  in  $L^\infty([0, \infty); B_{2,1}^{\frac{N}{2}-1})$ , and applying the three  $\varepsilon$  argument. For the proof of uniqueness we suppose two solutions  $u, v \in L^\infty([0, \infty); B_{2,1}^{\frac{N}{2}-1}) \cap L^1([0, \infty); \dot{B}_{2,1}^{\frac{N}{2}+1})$  correspond to the same initial data  $u_0 = v_0$ . We set

$$\delta = \sup_{0 \leq t < \infty} \|u(t) - v(t)\|_{B_{2,1}^{\frac{N}{2}-1}} + C_3 \mu \int_0^\infty \|u(t) - v(t)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} dt.$$

Following the same procedure as the proof of contraction part of  $\{u^{(m)}\}$  above, we obtain  $\delta \leq \frac{1}{2}\delta$ , instead of (4.24), if  $\|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}}$  is so small that

$$C_5 \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \exp(C_4 \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}}) \leq \frac{1}{2}.$$

Thus we have  $\delta = 0$ , and  $u_1 = u_2$ . This completes the proof of Theorem 1.2.  $\square$

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