





# Discrete and differential geometry applied to efficient numerical synthesis of spatially adaptive covariances

## R. James Purser SAIC, NOAA/NCEP/Environmental Modeling Center, Camp Springs, MD

CSCAMM, UMD March 26 2008 Acknowledgment: Much of the work described here benefitted from the collaboration with, and feedback from, Dave Parrish, Wan-Shu Wu, Manuel de Pondeca, John Derber, and Dezso Devenyi (CIRES/NOAA/ESRL).

•

# Introduction

The problem we address is how to efficiently construct and normalize covariance operators for meteorological data assimilation.

Covariances of what?

Spatial covariances of 'background' (forecast) error

Why are they needed?

The statistically optimal prescription of the assimilation is a generalized 'Least Squares' state in which the new observations and the background are weighted by their respective 'precisions', or inverse-covariances.

What properties do the covariances possess?

Smoothness, self-adjointness, negligible amplitude at large separation.

#### SUMMARY OF EQUATIONS FOR VARIATIONAL ANALYSIS

Using notation that, since Ide et al. (1997) has become rather standard, the variational statement of a meteorological optimal analysis is to seek the gridded state variable  $\mathbf{x}$  that minimizes the cost function

$$\begin{aligned} \mathcal{L}(\mathbf{x}) &= \frac{1}{2} (\mathbf{x}^b - \mathbf{x})^T \mathbf{B}^{-1} (\mathbf{x}^b - \mathbf{x}) \\ &+ \frac{1}{2} (\mathbf{y}^o - \mathbf{H}\mathbf{x})^T \mathbf{R}^{-1} (\mathbf{y}^o - \mathbf{H}\mathbf{x}). \end{aligned}$$

One of various ways of expressing the solution when the observation operator  ${\bf H}$  is linear is as follows:

$$\mathbf{x}^a = \mathbf{x}^b + \mathbf{B}\mathbf{H}^T\mathbf{f},$$

 $\mathbf{Q}\mathbf{f} = \mathbf{d},$ 

where,

$$\mathbf{d} \equiv \mathbf{y}^o - \mathbf{y}^b \equiv \mathbf{y}^o - \mathbf{H}\mathbf{x}^b,$$

is the "innovation" vector, where

$$\mathbf{Q} = \mathbf{H}\mathbf{B}\mathbf{H}^T + \mathbf{R} = \langle \mathbf{d}\mathbf{d}^T \rangle,$$

is the autocovariance of the innovation vector. It is rarely the case that the problem is small enough to allow a direct solution to be obtained. More commonly, an iterative approach is adopted, such as the method of conjugate gradients, or one of the many variants of this kind of procedure.

Then it is usually found that the computational cost is dominated by the matrix-times vector operations involving the background covariance B. It is important to note that, in an iterative approach, it is generally NOT necessary to construct the full covariance matrix B component by component. Rather, it is sufficient to be able to execute efficiently an operation of the form:

$$u_i = \sum_j C_{i,j}^T v_j$$

where index j is summed over grid values, and C is an operator satisfying

$$\mathbf{C}\mathbf{C}^T \equiv \mathbf{B}$$

By constructing the B operator as the product of a C and its adjoint we ensure that B is self-adjoint and that it possesses no negative eigenvalues (it is 'positive semi-definite') even when it is difficult or impossible to impose these properties on C itself. Therefore, we ALWAYS construct B in this way.

By making C smooth, we ensure B is also smooth.

If C is a Gaussian, or the result of a quasi-diffusive process, then B inherits this property also.

But why would we want the result to be Gaussian?

Gaussians make convenient building blocks from which more general shapes can be synthesized by a procedure analogous to (inverse) Laplace transformation.

Also, Gaussians in n dimensions are efficiently generated by convolving n transversally-oriented one-dimensional Gaussians, making their synthesis very efficient. When the intention is to produce a covariance shape that is isotropic (at least, in the horizontal directions) the Gaussian profile is the unique choice that factors into simple one-dimensional contributions.

Conversely, 1-D factors with NON-Gaussian profiles would lead to products, C (and hence B), whose anisotropy would betray the orientation of the computational lattice in an undesirable way.

In modern data assimilation, it IS desirable to generate covariances that are spatially adaptive (and therefore inhomogeneous) and locally anisotropic – stretched and tilted in response to the local meteorological flow conditions -- but still we wish to ensure that the principal directions of the computational lattice continue to have negligible imprint on the covariances that are synthesized.





Semigeos+lsent Model and Hybrid Model



Black Contours are potential T in K Panels a) and b) are for the Semigeos+Isent Model Panels c) and d) are for the hybrid Model

(Figures prepared by Manuel de Pondeca)

The stretched, tilted and deformed covariances are clearly NOT exactly Gaussians anymore.

The nearest analogue to the Gaussian in such a scenario is the 'Heat Kernel' associated with an effective Diffusivity which is no longer separable into scalars associated with the respective coordinate directions, but is a more general TENSORIAL quantity with significant off-diagonal components.

Can we create covariance operators by explicit simulations of diffusion?

Yes – this was first proposed (for ocean data assimilation) by Derber and Rosati (1989, GFDL) and extended to fully adaptive covariances in the ocean by Weaver and Courtier (2001). But the process can be VERY expensive computationally, especially when using explicit integration.

Are there short-cuts?

Yes! We shall discuss some of them...

### Aspect tensors and their geometry

'Aspect tensor' is the term we use to denote the local second-moment measure of spatial dispersion of the covariance contribution – for the case where this contribution is generated by a diffusive process acting for a period, T, of time, the aspect tensor is just 2 times T times the diffusivity. In other word, if we choose our period of time to be one half, the aspect tensor becomes identical to the diffusivity.

Aspect tensors are symmetric – it is convenient to think of the n-dimensional aspect tensors as occupying part of a linear space of dimension N(N+1)/2

However, useful aspects tensors are also POSITIVE DEFINITE; therefore, they cannot occupy ALL of 'aspect space'.

#### 2-D Aspect-tensor geometry

Define the tensor of covariance spatial dispersion, the "aspect tensor", to be,

$$\mathbf{S} = \begin{bmatrix} S_{xx}, & S_{xy} \\ S_{yx}, & S_{yy} \end{bmatrix}$$

which has eigenvalues,

$$\lambda_{\pm} = \frac{S_{xx} + S_{yy}}{2} \pm \sqrt{\left(\frac{S_{xx} + S_{yy}}{2}\right)^2 + S_{xy}^2}$$

12

Define,

$$S_{zz} = \frac{S_{xx} + S_{yy}}{2},$$
  

$$S_{aa} = \frac{S_{xx} - S_{yy}}{2},$$
  

$$S_{bb} = S_{xy},$$
  

$$S_{rr}^2 = S_{aa}^2 + S_{bb}^2,$$
  

$$D = S_{xx}S_{yy} - S_{xy}^2.$$

I.e.,

$$\lambda_{\pm} = S_{zz} \pm S_{rr}$$
$$D = \lambda_{+}\lambda_{-} \equiv S_{zz}^{2} - S_{rr}^{2}$$

The condition that both eigenvalues are positive therefore defines a cone <sup>13</sup>



For n=2 dimensions the set of valid aspect tensors forms a cone in the classical sense; in n>2 dimensions, aspect tensors also occupy a 'cone' of sorts, but one having a more complicated cross-section and symmetry group.

What interpretation do we give a tensor on the boundary of the aspect cone?

In n=2 dimensions, a tensor occupying the surface of the cone is of rank 1. It corresponds to a distribution on a line at some orientation.

If the aspect space is a LINEAR space, in what sense can aspect tensors add?

A property of normalized homogeneous distributions is that their aspect tensors ADD when the distributions are CONVOLVED.

What aspect tensor results from addition (= line-filter convolution) of two non-collinear members of the cone's boundary?

In n=2 dimensions, the resulting aspect tensor is a proper (non-degenerate) one belonging to the interior of the aspect cone.

ANY 2-dimensional aspect tensor is obtainable as the result of the composition of 2 suitably chosen line filters. Any n-dimensional aspect tensor is obtainable as the result of the composition of n suitably chosen line filters.

However, generic oblique line filters are inconvenient (though not impossible) to apply in a regular computational lattice.

What modification to the general rule would allow the general aspect tensor to be obtained from line filters restricted to oblique lines of the lattice? 15

The 'generalized grid lines' in a 2-D lattice are dense on the surface of the cone. In cross-section, they map to a dense covering of the bounding circle, so THREE line filters suffice, since any projected aspect can be placed inside a triangle whose vertices are each a projection of the aspect tensors of the generalized grid lines.

In n=3 dimensions, where the dimensionality of the aspect space is n(n+1)/2 = 6, the minimal number of generalized grid-line filters we need to convolve to form a product distribution with a given tensor is, in general, SIX.

But whether in 2, 3 or n dimensions, the choice of qualifying line filters is not unique.

How can we make the choice of qualifying line filters unique?

Along each generalized grid line, the 'GENERATOR' is the vector from one lattice point to the next and, regardless of its directional ambiguity, its self-outer-product tensor defines a unique image point on the boundary of the aspect cone. The CONVEX HULL of these aspect-space generator images is a polyhedral set inside the aspect cone and touching its boundary in such a way that EVERY proper aspect tensor projects onto a polygonal tile of the bounding shell of this new set. For n=2 each polygon is a triangle!

Generator	Generator image( times 2	<u>2)</u>		
[p, q]	[p <sup>2</sup> - q <sup>2</sup> , 2pq, p <sup>2</sup> + q <sup>2</sup> ]	Some of the 'triads' of this system of lattice generators are:		
[1,0]	[1, 0, 1]	{ [1,0], [0,1], [1,1] }, { [1,0], [0,1], [1,-1] },		
[0,1]	[-1, 0, 1]	{ [1,0], [1,1], [2,1] }.		
[1,1]	[0, 2, 2]	Aspect space images of the set of triads completely tile the cone of proper aspect tensors.		
[1, -1]	[0, -2, 2]			
[2, 1]	[3, 4, 5]	Is there a metric with respect to which all triads are congruent?		
[3, 2]	[5, 12, 13]			
	YES!	Let A and B be aspect tensors. Then		
$ A, B ^2 = Trace [ log A^{-1}B ]^2$ defines a Riemannian metric in which each triad's standard tile is equilateral, and is congruent to every other one.				



Two projections of the tiling of the aspect cone by triads. If each aspect tensor is assumed normalized to one of unit determinant, their metric geometry is of the 'hyperbolic' type. The gnomonic map projection (a) is usually called The 'Klein' representation and the stereographic map projection (b) is usually called the 'Poincare' representation (both were derived earlier by Beltrami!)



Two neighboring triads on the lattice. It is convenient to tag each generalized grid direction with a 'color' with no triad possessing the same color twice. Then the line filters of each color can be performed in sequence, avoiding the possibility of one line's filtering interrupting the application of the filter on a line that intersects the first.

The simplest system of chromatic triads involves just three colors.

Marking these colors on a lattice of generators, a periodic pattern emerges.

(a)



20

The triad-resolving algorithm loops over every geographical point of the computational lattice.

At each location, it starts with given intended aspect tensor and searches for a triad upon which the projection of the given aspect tensor leads to 'weights' associated with the corresponding three generators that are positive. These weights become the effective 1-D aspect tensors of the individual line filters (in grid units) at this location. If, at any stage of the search, the weight of one member of the triad remains negative, this member is discarded, being replaced by the unique alternative. This is a form of 'Simplex' algorithm, and it always converges in a finite number of steps.

In practice, the aspect tensor is never constant – it varies considerably with location, so the regions associated with each triad form a patchwork. Between adjacent patches one weight diminishes and becomes zero, the triad changes to a neighboring one, again, with an initially zero weight, but this weight then grows.

But is the resulting covariance field smooth enough at these transitions?



NO! Sample covariances taken across an idealized triad transition, using idealized filter numerics that exaggerate any defects, clearly show that there is a potential problem as one line-filter is retired, while another is recruited.

The linearity of aspect tensors can be invoked to solve the problem.

Imagine the intended aspect tensor as the centroid of a symmetric 'cloud' of nearby aspect-points contained within a sphere of radius small enough to ensure that the cloud only overlaps, at most, one pair of adjacent triads. Each point of the 'cloud' resolves into either one triad or the other and is characterized by three weights in a proportion that depends upon its relative location within that triad. Similarly, all the other points contribute with their own weights for one or other triad in a characteristic proportion. But, by linearity, the contributing weights may be integrated over the whole of the cloud and, if appropriately normalized, will supply weights for the four generators associated with the adjacent pair of involved triads that guarantee that the final composition of line filters produces the intended aspect tensor.

In this case, the weights associated with any given line direction involved in the recruitment/retirement progression make the transition from or to zero in a sufficiently gradual way to ensure the desired smoothness of the synthesized covariance field.



Can we still 'color' the grid lines to avoid numerical conflicts?

Yes! And a new periodic pattern emerges:

(a)



25

## 3 Dimensions

Is there a corresponding 'HEXAD' method in 3D?

YES. The convex-hull method shows that the six-dimensional aspect space is completely tiled with congruent tiles, each with cross-section of a six-vertex 'simplex'. (This is NOT obvious; in n=4 spatial dimensions the tiles for the ten-dimensional aspect space are of two kinds, of which only one is a simplex, while the other is a 12-vertex polytope).

Is there a 'Blended Hexads' method?

YES

Do these 3D methods have associated 'colorings'?

YES.

So – what does the pre-image of the generic hexad tile's vertices look like?



The six generators (and their six opposites) of each hexad form a lattice CUBOCTAHEDRON (when their convex-hull is considered).

Above, we show generators (and their cuboctahedra) of a neighboring pair of hexads, sharing generators  $g_2 - g_6$ , but exchanging  $g_7$  for  $g_1$ .

Blending hexads requires precise knowledge of how they tile the aspect space and how they join in the interior of the aspect cone.

For triads, their junctions involve only pairs; for any proper aspect tensor there is always some finite radius such that neighborhoods no larger than this radius are guaranteed to overlap no more than two triads.

For any proper aspect tensor in n=3 dimensions we wish to be able to assert that there is always some finite radius such that neighborhoods no larger than this radius are guaranteed to overlap no more than M hexads. What is the smallest M?

How many generalized grid line directions are then involved?

The aspect cone's interior junctions among its hexads involve, at most, M = 16 hexads in mutual contact.

13 line orientations are involved in each such junction.

Although the generic hexad junction slices through six dimensions of aspect space, in three of those dimensions the configuration exhibits no structure. A three-dimensional 'cross-section' through the junction suffices to reveal its geometrical structure. It looks like this:



The radial distribution of the 'cloud' that we average to form the blending of Hexads can be chosen to make the integration involved relatively straightforward. (Members of the symmetric-parameter 'Beta' distribution serve admirably.)

In practice the blended hexads method pre-calculates a table of standardized weights in a table sufficient to cover (via interpolation) any point of the generic Hexad.

What general principle allows us to 'color', (a) the lines of the basic hexad?

(b) The blended hexads?

It is the generalization of the pattern of periodic 2\*2 or 3\*3 squares (2D) to periodic 2\*2\*2 and 3\*3\*3 cubes (3D) that provides the necessary generalization. The sides need to be PRIME NUMBERS. Structures of this special kind occur in the abstract algebraic theory of GALOIS FIELDS.

Construction of Galois Field,  $GF\{2^3\}$ Primitive polynomial:  $1 + \lambda + \lambda^3$ Primitive element :  $\lambda$ 

 $\begin{array}{c|c} \lambda^{0} & 1 \\ \lambda^{1} & 0 + \lambda \\ \lambda^{2} & 0 + 0 + \lambda^{2} \\ \lambda^{3} & 1 + \lambda \\ \lambda^{4} & 0 + \lambda + \lambda^{2} \\ \lambda^{5} & 1 + \lambda + \lambda^{2} \\ \lambda^{6} & 1 + 0 + \lambda^{2} \end{array}$ 

Repeated exponentiation of lambda is done, modulo-2 and modulothe primitive polynomial.





By repeating the pattern of "colors" (the **non-zero** elements of GF(8)) in 2\*2\*2 blocks that cover the entire grid of lattice line generators, we find that each generator takes one of seven colors and the six distinct line generators of the hexad, corresponding to diameters of the associated cuboctahedron, are each assigned a <u>different</u> color.

In the extension of the hexad method to the "**blended**" form, the aspect tensor is, in effect, mapped symmetrically to a spherical "cloud" (in aspect space) before being resolved into hexad weights. Coded carefully, this can be shown to result in a configuration of smoothing operations that separate naturally into the 13 "colors" that are implied by the Galois field associated with a repetition of 3\*3\*3 blocks – the field, GF(27).



	Construction of Galois Field, GF{3 <sup>3</sup> } [GF			
	Primitive polynomial:	1	$1+2\lambda+\lambda^3$	
	Primitive element :	λ		
$\lambda^{0}$	1	$\lambda^{13}$	2	
$\lambda^{1}$	$0 + \lambda$	$\lambda^{14}$	$0 + 2\lambda$	
$\lambda^2$	$0+0+\lambda^2$	$\lambda^{15}$	$0+0+2\lambda^2$	
$\lambda^3$	$2 + \lambda$	$\lambda^{16}$	$1 + 2\lambda$	
$\lambda^4$	$0 + 2\lambda + \lambda^2$	$\lambda^{17}$	$0 + \lambda + 2\lambda^2$	
$\lambda^{5}$	$2 + \lambda + 2\lambda^2$	$\lambda^{18}$	$1 + 2\lambda + \lambda^2$	
$\lambda^{6}$	$1 + \lambda + \lambda^2$	$\lambda^{19}$	$2+2\lambda+2\lambda^2$	
$\lambda^7$	$2 + 2\lambda + \lambda^2$	$\lambda^{20}$	$1 + \lambda + 2\lambda^2$	
$\lambda^8$	$2 + 0 + 2\lambda^2$	$\lambda^{21}$	$1 + 0 + \lambda^2$	
$\lambda^9$	$1 + \lambda$	$\lambda^{22}$	$2 + 2\lambda$	
$\lambda^{10}$	$0 + \lambda + \lambda^2$	$\lambda^{23}$	$0 + 2\lambda + 2\lambda^2$	
$\lambda^{11}$	$2 + \lambda + \lambda^2$	$\lambda^{24}$	$1 + 2\lambda + 2\lambda^2$	
$\lambda^{12}$	$2+0+\lambda^2$	$\lambda^{25}$	$1+0+2\lambda^2$	
			NCEP	

[(27)] Repeated exponentiation of lambda is done modulo-3 and modulo- the primitive polynomial.







An algebraic field possesses the operation of multiplication and, for any non-null element, inversion, as well as addition. The triad and hexad methods clearly exploit the **additive** group property.

Does the **multiplicative** property of a Galois Field play a role?

In the construction of an efficient Hexad-resolving algorithm the multiplicative aspects of GF(8) played a valuable role.

The triads link to form a tree (simply-connected); hexads link to form a network allowing multiple paths from one hexad to any other (and hence, circuits). There are clearly many equally symmetric ways to label the generators of a hexad – the elements of the octahedral group applied to the cuboctahedron switch among these equivalent labelings. For each of the SIX possible transitions to a neighboring hexad, this choice must be resolved. It is desirable if, from this multitude of combinations, a systematic and symmetrical prescription for the transition re-labelings can be chosen to ensure that the same final labeling is given to any hexad, regardless of the route taken to reach it. The key to achieving this, is to re-label the hexad generators so that the 'colors' in the new labeling pattern Are just MULTIPLES, with respect to GF(8), of the replaced tableau of labels.



The rule for the replacement of g(1), -g(1):



We have described how methods of (mostly) discrete geometry can be applied, together with important ingredients from abstract algebra, to formulate efficient algorithms for synthesizing spatially inhomogeneous and anisotropic covariance operators used for statistical data assimilation.

An important, and rather difficult technical problem plaguing such adaptive covariance algorithms has been the problem of amplitude normalization.

When the shape and spatial extent of the quasi-Gaussian covariance contributions vary with location, and especially when this rate of variation is relatively large in comparison with the intrinsic scale coherence of the covariances themselves, it becomes quite difficult to make a good approximation to the central amplitude of the final result of set period of simulated inhomogeneous diffusion. Monte Carlo techniques, proposed by Weaver and Courtier, estimate this amplitude by a method of multiple costly trials, but the convergence of estimates by this approach is very slow.

Is there a non-iterative asymptotic estimation method?

YES; it is known as the 'Parametrix Expansion Method' and it works assuming ISOTROPIC diffusion in a Riemannian geometry.





### Differential Geometry and the "Parametrix" Method

In the general case, though, the transformation to a Riemannian geometry has not made the amplitude estimation problem an easy one. However, there does exist a literature on the estimation of solutions to the "Heat Kernel" in curved spaces (for example, Rosenberg 1997: The Laplacian on a Riemannian manifold, C.U.P.) and one approach, although it can only lead to an asymptotic approximation at each point, looks promising.

This approach is the "Parametrix method". Normal coordinates are constructed to be "as Cartesian as possible locally"; axes are orthogonal and the radials through the local origin are geodesics on which distances measure true. The idea behind the parametrix method is to represent the evolving solution of the diffusion problem as the corresponding Euclidean solution in the normal coordinates, multiplied by a modulating function that is smooth in "time" and space.





Then, for sufficiently small "times" we should be able to approximate the modulating function in a series of powers of the normal coordinates and "time". The only part of the solution we shall end up being interested in is the solution at the origin and at "time" =  $\frac{1}{2}$  (when the aspect tensor corresponds with the diffusivity). There is a systematic recursive algorithm for generating successive approximations in finite powers where one order of solution, fed back into original diffusion equation, provides the next order of correction.

While, in principle, this process can be continued without limit, in practice it rapidly leads to extremely complicated algebra and the successive approximations, typical of many asymptotic series, do not converge to the true solution for any finite time. Nevertheless, provided the original aspect tensor varies sufficiently smoothly and gradually, the approximate solutions obtained by this approach should be adequate. Most of the algebraic operations involved can actually be reduced to forms that lend themselves to mechanization. We give two idealized examples of the results obtained from the asymptotic method but only give an outline here of some of the steps required for the general case.



These graphs show the 2D results comparing the asymptotic expansion for the amplitude quotient with the true solution in the special case where the Gaussian curvature K is uniform. Even out to a curvature of +/- 5 non-dimensional units, the asymptotic method with a few terms should give a very good approximation, as shown. However, the expansion is formally divergent. The true amplitude quotient is denoted "A"; other graphs show asymptotic expansions truncated to the degrees indicated by the superscript. <sup>40</sup>



Corresponding results in 3D, with uniform sectional curvature, K. But now, the asymptotic expansion converges to exp(K/2). For negative curvature (hyperbolic geometry) this is the exact solution; for positive curvature, there is an error that grows with K.





An outline of the treatment in the case of general Riemannian geometry follows.

First, we observe that, in normal coordinates we can express the covariant metric tensor as a Taylor series (starting with the identity, and with vanishing first-order term). The construction of the normal coordinate system, in which radials are geodesics measuring true at all distances out, implies that the radial vector is an eigenvector, with eigenvalue=one, of both metric tensors at every point. This imposes important constraints on all the quadratic, and higher, Taylor series coefficients of the metric; these become the celebrated "Bianchi identities" (algebraic and differential) when translated into the implications for the Riemann curvature tensor. (In fact, it is possible to express the array of quadratic Taylor series coefficients for the metric directly as simple linear multiples of the Riemann tensor.)

The Bianchi, and other symmetries, restrict the actual number of degrees of freedom at each degree of the Taylor series. For example, in 3D, we might expect the Taylor series for a symmetric metric tensor (6 components) to require 6\*6=36 independent coefficients at second degree, but only 6 are actually needed. At 3<sup>rd</sup> degree, the Bianchi differential identities come into play to keep the independent coefficients at only 15.





The second derivatives of the covariant metric (in normal coordinate) provide the Riemann tensor. Successive covariant differentiations of the Riemann tensor produce new tensors which can be equated to tensorial expressions that involve only the Taylor coefficient arrays of the metric tensor up to a finite degree. However, it is the origin-evaluated derivatives of the metric tensor – i.e., these same Taylor coefficients, that force the successive terms in the parametrix method for approximating the modulating function relating the non-Euclidean solution of the diffusion problem to its standard Euclidean counterpart.

It therefore becomes possible to express the successive terms in the series expansion of the amplitude adjustment quotient directly in terms of the Riemann tensor and its first few covariant derivatives. These latter quantities are straightforward to evaluate on the original grid and, being tensorial, are therefore relatively easy to convert to normal coordinate representations if needed. However, contracted versions of the curvature: the "**Gaussian curvature**" in 2D; the "**Ricci curvature**" in 3D; should make it more convenient to express the approximations to the amplitude quotients directly in terms of these simpler quantities. The parametrix expansion for the amplitude quotient to second order:

In n dimensions:

$$A \approx 1 + \frac{R}{12} + \frac{1}{1440} \left( 12\nabla^2 R + 5R^2 - 2R_{ij}R^{ij} + 2R_{ijkl}R^{ijkl} \right).$$

When n = 3:

$$A \approx 1 + \frac{R}{12} + \frac{1}{480} \left( 2R_{ij}R^{ij} + R^2 + 4\nabla^2 R \right).$$

When n = 2, and  $\kappa$  is the Gaussian curvature:

$$A\approx 1+\frac{\kappa}{6}+\frac{1}{60}(\kappa^2+\nabla^2\kappa).$$





#### Conclusions

Using the "Triad" and "Hexad" methods, we can now generate smooth quasi-Gaussian covariance contributions with arbitrary degrees of anisotropy. Asymptotic methods based on differential geometric ideas are being developed and tested in the Gridpoint Statistical Interpolation at NCEP to make the normalizaton of these filters in inhomogeneous cases more accurate and efficient.

We have illustrated, by the choice of topics highlighted in this talk, how areas of what might once have been regarded as mostly "Pure" mathematics are making increasingly profound and essential contributions to the way Applied computational mathematics is put into operational practice in a context where timeliness of the final result, and therefore the numerical efficiency of the procedure employed to achieve it, is such an overriding constraint.

#### REFERENCES

- Derber, J. C., and A. Rosati, 1989: A global ocean data assimilation system. J. Phys. Ocean., 19, 1333–1347.
- Dickson, L. E., 1958: Linear Groups (With an Exposition of the Galois Field theory). Dover, New York, 336 pp.
- Gilkey, P. B., 1984: Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem. Publish or Perish, Wilmington, DE, 345 pp.
- Purser, R. J., 2005: A geometrical approach to the synthesis of smooth anisotropic covariance operators for data assimilation. NOAA/NCEP Office Note 447, 60 pp.
- Purser, R. J., 2008: Amplitude normalization of inhomogeneous diffusive filters using asymptotic expansions and techniques of non-Euclidean geometry; Part I: Analytic solutions for symmetric cases and the validation of practical algorithms; Part II: Riemannian geometry and the generic parametrix expansion method. NOAA/NCEP Office Notes (In preparation).
- Purser, R. J., W.-S. Wu, D. F. Parrish, and N. M. Roberts, 2003: Numerical aspects of the application of recursive filters to variational statistical analysis. Part II: Spatially inhomogeneous and anisotropic general covariances. *Mon. Wea. Rev.*, 131, 1536–1548.
- Rosenberg, S., 1997: The Laplacian on a Riemannian Manifold; London Mathematical Society Student Texts 31. Cambridge, 174 pp.
- Weaver, A., and P. Courtier, 2001: Correlation modelling on the sphere using a generalized diffusion equation. *Quart. J. Roy. Meteor. Soc.*, **127**, 1815–1846.